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A note on a paper of S. Watanabe

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In his paper WATANABE [1] asked (p. 38) if every closed positive linear map $\Phi: A_0 \to B$ (A_0 is a star subalgebra of a unital C^*-algebra containing the unit, B is a C^*-algebra and $\Phi(x^*x) \geqq 0$ for $x \in A_0$) is automatically continuous. He proved that when Φ is 2-positive. In the general case, too, the answer is "yes". The proof (similar to that of Theorem 1 in [1]) is based on the lemma of [1] on p. 37 and on a corollary from the following theorem of Palmer.

Theorem (T. PALMER [2]). *Let A be a complex unital Banach *-algebra with continuous involution and $H=\{x: x \in A, x=x^*\}$, $E=\{e^{ih}: h \in H\}$. If the set E is bounded, the algebra A is C^*-equivalent.*

Corollary. *In the above notations, if the set $K=\{u^2: u \in H, u^2+v^2=\mathbf{1}$ for some $v \in H\}$ is bounded, A is C^*-equivalent.*

Proof. We have $e^{ih}=\cos(h)+i\sin(h)$ $(h \in H)$ where $\cos(h)=(e^{ih}+e^{-ih})/2$, $\sin(h)=(e^{ih}-e^{-ih})/(2i) \in H$ and $\cos(h)^2+\sin(h)^2=\mathbf{1}$. If $\|\sin(h)^2\| \leqq N$ (a constant) for every $h \in H$, we obtain that $\|\cos(h)\|=\|\mathbf{1}-2\sin(h/2)^2\| \leqq 2N+1$, $\|\sin(h)\|=$ $=\|\cos(\pi\mathbf{1}/2-h)\| \leqq 2N+1$. Hence $\|e^{ih}\| \leqq 2(2N+1)$ and A is C^*-equivalent.

Now following the lines of the proof of Theorem 1 in [1] we obtain the modification

Theorem 1'. *Let Φ be a closed linear map of A_0 into a Banach space B. If Φ is norm bounded on the set K (defined for A_0, see the corollary above and the lemma in* [1]*), then A_0 is a C^*-algebra (the original C^*-norm in A_0 turns out to be equivalent to the graph norm in it) and Φ is bounded.*

When B is a C^*-algebra and Φ is positive (we need only $\Phi(x^2) \geqq 0$ when $x=x^* \in A_0$), this is fulfilled: if $u^2+v^2=\mathbf{1}$ (u, v are hermitian in A_0), it follows that $\Phi(u^2)+\Phi(v^2)=\Phi(\mathbf{1})$, hence $\|\Phi(u^2)\| \leqq \|\Phi(\mathbf{1})\|$, i.e., Φ is bounded on K.

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References

[1] S. Watanabe, The boundedness of closed linear maps in C*-algebras, *Acta Sci. Math.*, **43** (1981), 37—39.

[2] T. W. Palmer, The Gelfand—Naimark pseudonorm on Banach *-algebras, *J. London Math. Soc.*, 3 (1971), 59—66.

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