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Infinite-dimensional Jordan models and Smith McMillan forms. II

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1. Introduction

This paper is a continuation of [3]. Throughout we follow the notation and terminology established there and in [11]. The k-dimensional space of complex k-tuples is denoted by $\mathscr{E}^k$ and $z=e^{it}$ for $t\in[0, 2\pi]$. The orthogonal projection onto a subspace $\mathscr{X}$ is denoted by $P_{\mathscr{X}}$. The greatest common inner divisor of the functions α, β in H^∞ is $\alpha\wedge\beta$. A *bounded analytic function* $\{\mathscr{E}^m, \mathscr{E}^n, \Omega\}$ is a Lebesgue measurable operator valued function such that $\Omega(z)$ maps $\mathscr{E}^m$ into $\mathscr{E}^n$ for all z, $\Omega(z)$ has analytic continuation into the open unit disc and $\|\Omega(z)\|\leqq M<\infty$ a.e. The Hardy H^2-space of analytic functions with values in $\mathscr{E}$ is denoted by $H^2(\mathscr{E})$. The forward shift U_+ on $H^2(\mathscr{E})$ is defined by $U_+f:=zf$ where f is in $H^2(\mathscr{E})$. Let $\{\mathscr{E}^k, \mathscr{E}^n, \Phi\}$ be an inner function. Then $\mathscr{H}(\Phi):=H^2(\mathscr{E}^n)\ominus\Phi H^2(\mathscr{E}^k)$ and $S(\Phi)$ is the compression of U_+ to $\mathscr{H}(\Phi)$. Recall [11] that $S(\Phi)$ is a C_0 contraction if and only if Φ is inner from both sides, i.e., $k=n$. Finally, let $\{\mathscr{E}^m, \mathscr{E}^n, \Omega\}$ be a bounded analytic function then $\{\mathscr{E}^k, \mathscr{E}^n, C(\Omega)\}$ is the inner function uniquely defined by

(1)
$$\mathscr{H}(C(\Omega)) := \bigvee_{j\geqq 0} U_+^{*j}\Omega\mathscr{E}^m$$

Note $C(\Omega)$ is well defined by the Beurling—Lax theorem [11].

Throughout $N(z)$ is a Lebesgue measurable function in $[0, 2\pi]$ whose values are a.e. nonnegative self adjoint operators mapping $\mathscr{E}^m$ into $\mathscr{E}^m$ and $\|N(z)\|\leqq\leqq M<\infty$ a.e. It is also assumed that N admits a factorization of the form $N(z)==\theta^*(z)\theta(z)$ a.e., where $\{\mathscr{E}^m, \mathscr{E}^n, \theta\}$ is a bounded analytic outer function; such a θ will be called an outer factor of N. In the previous paper [3] we gave a simple procedure to compute the Jordan model for $S(C(\theta))$ by means of θ. Here this is done without computing θ or the inner function $C(\theta)$ generated by θ. That is,

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our present procedure calculates this Jordan model directly from N, by using a generalized Smith—McMillan procedure. Our procedure, given in Theorem 1, plays an important role in infinite-dimensional stochastic realization theory [4]. The following is needed.

Lemma 1. [2] *Let* $\{\mathscr{E}^m, \mathscr{E}^n, \theta\}$ *be the outer factor for* N. *Then* $S(C(\theta))$ *is a* C_0 *contraction if and only if there exists an inner function* c *in* H^∞ *such that* cN *is a bounded analytic function.*

Remark 1. The above lemma allows us to determine if $S(C(\theta))$ is a C_0 contraction directly from N without obtaining θ or $C(\theta)$. Finally, if cN is a bounded analytic function for some c in H^∞ then N always admits an outer spectral factor [2]. (In this situation our factorization assumption on N is redundant.)

2. Main result

For convenience we recall some terminology in [9], [10]. Let $\{\mathscr{E}^n, \mathscr{E}^m, H\}$ and $\{\mathscr{E}^n, \mathscr{E}^m, H_1\}$ be two bounded analytic functions. H is *quasi-equivalent* to H_1 if for every scalar valued inner function c there exists two bounded analytic functions $\{\mathscr{E}^m, \mathscr{E}^m, A\}$, $\{\mathscr{E}^n, \mathscr{E}^n, B\}$ such that $\det(A)$ and $\det(B)$ are prime to c and $HB=AH_1$. Quasi-equivalence is an equivalence relation. It can be shown that $\{\mathscr{E}^n, \mathscr{E}^m, H\}$ is quasi-equivalent to $\{\mathscr{E}^n, \mathscr{E}^m, D\}$ where D is a diagonal analytic function of the form

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

and $D_1 = \operatorname{diag}[d_1, d_2, \ldots, d_k]$. The d_i's are scalar valued inner functions such that d_i divides d_{i+1} for $i=1, \ldots, k-1$. Furthermore, this representation is unique and called the *normal form of* H. The normal form D can be obtained from the invariant factors of H [9], [10]. Define $\mathscr{D}_r$ as the greatest common inner divisor of all minors in H of order r, with $\mathscr{D}_0=1$. The *invariant factors* for H are $\mathscr{E}_i(H) := \mathscr{D}_i/\mathscr{D}_{i-1}$ for $i=1, \ldots, \min(m, n)$. By convention $\mathscr{E}_j(H)=0$ for all $j \geqq i-1$ if $\mathscr{D}_{i-1}=0$. If $\mathscr{E}_i(H)$ is nonzero then $\mathscr{E}_{i-1}(H)$ divides $\mathscr{E}_i(H)$. It can be shown that the normal form for H is given by (2) where $D_1 = \operatorname{diag}[\mathscr{E}_1(H), \ldots, \mathscr{E}_k(H)]$ and k is the number of nonzero invariant factors for H.

A Jordan model is an operator of the form $S(m_1) \oplus S(m_2) \oplus \ldots \oplus S(m_k)$ where the m_i's are inner functions in H^∞, see [1], [12], [13], [14] for further details. Finally we need

Lemma 2. [6, Ch. 3] *Let* $\{\mathscr{E}^m, \mathscr{E}^n, \theta\}$ *be a bounded analytic function. Then* $S(C(\theta))$ *is a* C_0 *contraction if and only if* θ *admits a factorization of the form*

$\theta=\bar{z}G^*\psi$, where $\{\mathscr{E}^m, \mathscr{E}^m, \psi\}$ is inner from both sides, $\{\mathscr{E}^n, \mathscr{E}^m, G\}$ is a bounded analytic function, and the only common, inner from both sides, left factor to both ψ and G_i is a unitary constant. (The inner part of G is denoted by G_i.) Furthermore, when $S(C(\theta))$ is a C_0 contraction then $S(C(\theta))$ and $S(\psi)$ are quasi-similar. In particular, $S(C(\theta))$ and $S(\psi)$ admit the same Jordan model.

Theorem 1. *Let $\{\mathscr{E}^m, \mathscr{E}^n, \theta\}$ be the outer factor for N. Assume there exists a scalar inner function c such that $cN=zH$ is a bounded analytic function. Then*

(i) *$S(C(\theta))$ is a C_0 contraction.*

(ii) *The Jordan model for $S(C(\theta))$ is $S(m_1)\oplus S(m_2)\oplus\ldots\oplus S(m_k)$ where k is the number of nonzero invariant factors for $\{\mathscr{E}^m, \mathscr{E}^m, H\}$ and $m_i=c/(\mathscr{E}_i(H)\wedge c)$ for $i=1, \ldots, k$.*

Proof. Part (i) is an obvious consequence of Lemma 1. The proof of part (ii) is similar to Theorem 1 in [3]. Let

$$D' = \operatorname{diag}[\mathscr{E}_1(H), \ldots, \mathscr{E}_k(H), 0, 0, \ldots, 0] \tag{3}$$

be the normal form for H, where $\mathscr{E}_k(H)\neq 0$. Choose any two bounded analytic functions $\{\mathscr{E}^m, \mathscr{E}^m, A\}$ and $\{\mathscr{E}^m, \mathscr{E}^m, B\}$ with $\det(A)\cdot\det(B)=a$ such that a is prime to $c\mathscr{E}_k(H)$ and $HB=AD'$. Lemma 2 and $N=\theta^*\theta$ gives $\psi^*G\theta=H\bar{c}$ where ψ and G satisfy the conclusion of Lemma 2. Applying B yields

$$\psi^*G\theta B = AD'\bar{c}. \tag{4}$$

Let

$$M = \operatorname{diag}[m_1, m_2, \ldots, m_k\ 1, 1, \ldots, 1], \tag{5}$$

$$D = \operatorname{diag}[d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0]$$

where the m_i's are defined in statement (ii) above and $d_i:=\mathscr{E}_i(H)/(\mathscr{E}_i(H)\wedge c)$ for $i=1, \ldots, k$. By [12, Lemma 2b] we have d_i divides d_{i+1}. Using $D'\bar{c}=DM^*$ in (4):

$$G\theta BM = \psi AD. \tag{6}$$

Equation (6) and [11, Theorem 3.6, p. 258] or [8], [14] implies $S(\psi)X=XS(M)$ where

$$X = P_{\mathscr{H}(\psi)}G\theta B|\mathscr{H}(M). \tag{7}$$

To complete the proof it is sufficient to show that X is a quasiaffinity. By the results in [1], [12], [13], [14] this implies $S(M)$ is the Jordan model for $S(\psi)$. Then by Lemma 2, $S(M)$ is also the Jordan model for $S(C(\theta))$.

First it is shown that X is densely onto. By equation (6):

$$P_{\mathscr{H}(\psi)}G\theta BMH^2(\mathscr{E}^m) = \{0\}.$$

Using this in the following calculation with the fact that θ is outer gives:

$$\overline{X\mathscr{H}(M)} = \overline{P_{\mathscr{H}(\psi)}G\theta B(\mathscr{H}(M)\vee MH^2(\mathscr{E}^m))} = \overline{P_{\mathscr{H}(\psi)}G\theta BH^2(\mathscr{E}^m)} \supseteqq \tag{8}$$
$$\supseteqq \overline{P_{\mathscr{H}(\psi)}G\theta aH^2(\mathscr{E}^m)} = \overline{P_{\mathscr{H}(\psi)}GaH^2(\mathscr{E}^n)} = \overline{P_{\mathscr{H}(\psi)}GaH^2(\mathscr{E}^n)\vee\psi H^2(\mathscr{E}^m)} = \mathscr{H}(\psi).$$

The last equality follows from Lemma 3 in [3] which shows that

$$GaH^2(\mathscr{E}^n)\vee\psi H^2(\mathscr{E}^m) = H^2(\mathscr{E}^m). \tag{9}$$

Hence X is densely onto.

Finally we verify that X is one-to-one. Our technique is similar to some of the arguments in [14]. Assume $h\in\mathscr{H}(M)$ and $Xh=0$. Let $g\in L^2(\mathscr{E}^m)$ be such that $h=Mg$. To show that X is one-to-one we simply show that $g\in H^2(\mathscr{E}^m)$. Then $h\in MH^2(\mathscr{E}^m)\cap\mathscr{H}(M)=\{0\}$.

By using (6):

$$0 = P_{\mathscr{H}(\psi)}G\theta BMg = P_{\mathscr{H}(\psi)}\psi ADg. \tag{10}$$

Since Mg is analytic, ψADg is analytic. Equation (10) implies ψADg is in $\psi H^2(\mathscr{E}^m)$. Thus ADg is in $H^2(\mathscr{E}^m)$. Using $A'A=aI$ for the appropriate bounded analytic $\{\mathscr{E}^m, \mathscr{E}^m, A'\}$ yields $aDg\in H^2(\mathscr{E}^m)$. This with the definition of D places $ad_k g$ in $H^2(\mathscr{E}^m)$. (This follows because $m_j=1$ if $j>k$ where k is defined in (3) or (5). Notice that $h=Mg$ is in $\mathscr{H}(M)$. Thus $g_j=0$ for all $j>k$. Here g_j is the jth component of the m-vector g.) Clearly $h=Mg$ is in $H^2(\mathscr{E}^m)$. Therefore cg is in $H^2(\mathscr{E}^m)$. By [11, Proposition 1.5, p. 108] we have $(c\wedge(ad_k))g\in H^2(\mathscr{E}^m)$. By construction c and ad_k are prime. Hence g is in $H^2(\mathscr{E}^m)$, X is one-to-one and the proof is complete.

Lemma 3. ([5], [6]) *Let $\{\mathscr{E}^p, \mathscr{E}^m, \Omega\}$ be a bounded analytic function.*

(i) *$S(C(\Omega))$ is a C_0 contraction if and only if $S(C(\tilde{\Omega}))$ is a C_0 contraction.*

(ii) *If $S(C(\Omega))$ is a C_0 contraction then $S(C(\Omega))$ and $S^*(C(\tilde{\Omega}))$ are quasi-similar. In particular, they have the same Jordan model.*

Proof. This lemma follows from Theorem 2.1 in [5]. One can also obtain this result by using either Theorem 14.11, p. 206 and Theorem 3.5, p. 254 in [6] or Theorem 1 in [3].

Finally we are ready for

Corollary 1. *Assume there exists a scalar valued inner function c such that $cN=zH$ is a bounded analytic function. Then*

i) *N admits a *-outer factorization $N(z)=\Omega(z)\Omega^*(z)$ a.e. where $\{\mathscr{E}^p, \mathscr{E}^m, \Omega\}$ is *-outer.*

(ii) *$S(C(\Omega))$ is a C_0 contraction. Furthermore, $S(C(\Omega))$ and $S(C(\theta))$ have the same Jordan model. (θ is the outer factor for N.) In particular, the Jordan model for $S(C(\Omega))$ can be obtained directly from Theorem 1.*

Proof. (i) $\tilde{c}\tilde{N}=z\tilde{H}$ is a bounded analytic function. By Remark 1 or [2] N admits a *-outer factorization.

Now for part (ii). Clearly $\tilde{N}=\tilde{\Omega}^*\tilde{\Omega}$ is an outer factorization of $\tilde{N}$ and $\tilde{c}\tilde{N}=z\tilde{H}$. Lemmas 1 and 3 imply that $S(C(\Omega))$ and $S(C(\tilde{\Omega}))$ are C_0 contractions. By Theorem 1 the Jordan model for $S(C(\tilde{\Omega}))$ is $S(\tilde{m}_1)\oplus\ldots\oplus S(\tilde{m}_k)$ where k is the number of nonzero invariant factors for H and

$$\tilde{m}_j = [c/(c\wedge\mathscr{E}_j(H))]^{\sim} = [\tilde{c}/(\tilde{c}\wedge\mathscr{E}_j(\tilde{H}))]. \tag{11}$$

Recall [11] that $S(\tilde{m})$ is unitarily equivalent to $S^*(m)$ for an inner function m. Equation (11), Theorem 1 and Lemma 3 imply that $S(C(\Omega))$ and $S(C(\theta))$ have the same Jordan model.

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