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(0, A)-semigroups on $L_p(G)$ commuting with translations are (C_0)

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1. Introduction. Let X be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X with the operator norm. Suppose that $\{T(\xi);\ \xi\geqq 0\}$ is a family of operators in $B(X)$ satisfying the following conditions:

(i) $T(\xi_1+\xi_2)=T(\xi_1)T(\xi_2)$ for $\xi_1, \xi_2\geqq 0,\ T(0)=I$;

(ii) $T(\xi)$ is strongly measurable on $\xi>0$.

It is well known that (i) and (ii) imply that $T(\xi)$ is strongly continuous for $\xi>0$ [2, p. 305] and we shall call the family $\{T(\xi)\}$ a *strongly continuous* semigroup of operators on X. In studying semigroups of operators, it is usual to assume that $T(\xi)$ converges to an operator J in one sense or another as $\xi\to 0^+$. In particular, semigroups have been classified in terms of the sense in which $T(\xi)$ converges to the identity operator. Thus a strongly continuous semigroup of operators satisfying

(iii) $\lim\limits_{\xi\to 0^+} T(\xi)x=x$ for all $x\in X$

is called a semigroup of class (C_0) [2, 10.6].

A semigroup $\{T(\xi)\}$ satisfying $\lim\limits_{\xi\to 0^+} T(\xi)x=Jx$ for all $x\in X$, where J is a bounded linear operator on X is said to *converge strongly in the sense of Cauchy* with J as its Cauchy limit. If $\lim\limits_{\xi\to 0^+} T(\xi)=J$ in the uniform operator topology then $\{T(\xi)\}$ is said to *converge uniformly in the sense of Cauchy* with J as its Cauchy limit.

To define the second class of semigroups that we shall be concerned with, we need the notion of the type of a semigroup. For any strongly continuous semigroup $\{T(\xi)\}$, the real number

$$\omega_0=\inf_{\xi>0}\frac{1}{\xi}\log\|T(\xi)\|=\lim_{\xi\to\infty}\frac{1}{\xi}\log\|T(\xi)\|$$

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is called the *type* of $\{T(\xi)\}$. (See [2, 10.2].) A strongly measurable semigroup of operators $\{T(\xi)\}$ on X of type ω_0 is said to be of *class* $(0, A)$ if it satisfies the following conditions:

(iv) $\int_0^1 \|T(\xi)x\|\, d\xi < \infty$ for each $x \in X$;

(v) for all λ with $\mathrm{re}(\lambda) > \omega_0$, the linear operator

$$R(\lambda)x \equiv \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi$$

is defined and bounded for all $x \in X$;

(vi) $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for each $x \in X$.

A semigroup of class (C_0) is of class $(0, A)$ [2, Theorem 10.6.1]. There are a number of classes between (C_0) and $(0, A)$ which we shall not define here. For a full discussion of the basic classes of semigroups, the reader is referred to [2, 10.6].

A semigroup $\{T(\xi)\}$ satisfying $\lim_{\lambda\to\infty} \lambda R(\lambda)x = Jx$ for all $x \in X$, where J is a bounded linear operator on X is said to be *strongly Abel-ergodic at zero* with the operator J as its *Abel limit.* The condition is then written

$$\text{(A)-}\lim_{\xi\to 0^+} T(\xi)x \equiv \lim_{\lambda\to\infty} \lambda R(\lambda)x = Jx \quad \text{for all} \quad x \in X.$$

If

$$\text{(A)-}\lim_{\xi\to 0^+} T(\xi) \equiv \lim_{\lambda\to\infty} \lambda R(\lambda) = J$$

in the uniform operator topology, then $\{T(\xi)\}$ is said to be *uniformly Abel-ergodic* at zero with J as its *Abel limit* [2, 18.4.3].

In this paper, we shall be concerned with semigroups $\{T(\xi)\}$ defined on $L_p(G)$ where G is an infinite compact group and $1 \leqq p < \infty$. Two of the results proved in [3] may be stated as follows:

1.1. Theorem. *Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Then $\{E_\xi\}$ converges uniformly in the sense of Cauchy to the identity operator if and only if $\{T(\xi)\}$ converges strongly in the sense of Cauchy to the identity operator.*

Our first result in the present paper is in the same spirit: Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Then $\{E_\xi\}$ is uniformly measurable if and only if $\{T(\xi)\}$ is strongly measurable.

In our next theorem we show that if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then $\{E_\xi\}$ is uniformly Abel-ergodic at zero with the identity operator as its Abel limit. These results and the result quoted from [3] suggest that the strong version of a property of $\{T(\xi)\}$ implies the uniform version of the corresponding property of $\{E_\xi\}$.

Our main result is Theorem 2.5 in which the above results are used to prove that if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then $T(\xi)$ actually converges strongly to the identity operator in the sense of Cauchy.

The work in this paper shows again the usefulness of studying semigroups of multipliers for a function space in order to obtain results about operators on the function space itself. In this connection, see [3], [4] and [5].

2. Semigroups of operators on $L_p(G)$. For G an infinite compact group with dual object Σ, we denote by $\mathfrak{G}(\Sigma)$ the set $\underset{\sigma\in\Sigma}{PB}(H_\sigma)$ where H_σ is the representation space of the representation U^σ [1, 28.24]. If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of $\mathfrak{G}(\Sigma)$, then an element $E\in\mathfrak{G}(\Sigma)$ is said to be an $(\mathfrak{A}, \mathfrak{B})$-*multiplier* if $EA\in\mathfrak{B}$ for all $A\in\mathfrak{A}$ [1, 35.1]. An $(\mathfrak{A}, \mathfrak{A})$-multiplier will be described simply as an $\mathfrak{A}$-multiplier and an $L_p(G)^\wedge$-multiplier will be called an $L_p(G)$-multiplier. Here $L_p(G)^\wedge$ denotes the set of Fourier transforms $\hat{f}$ of $f\in L_p(G)$.

A family $\{E_\xi;\ \xi\geqq 0\}$ of functions $E_\xi\in\mathfrak{G}(\Sigma)$ is called a *semigroup of $L_p(G)$-multipliers* [3] if

(i) for each $\xi\geqq 0$, E_ξ is an $L_p(G)$-multiplier;

(ii) $E_{\xi_1+\xi_2}=E_{\xi_1}\cdot E_{\xi_2}$ for all $\xi_1, \xi_2\geqq 0$.

Condition (ii) means that for each $\sigma\in\Sigma$, $\{E_\xi(\sigma);\ \xi\geqq 0\}$ is a semigroup of operators on the space H_σ and $\{E_\xi\}$ is called a strongly (uniformly) continuous semigroup of $L_p(G)$-multipliers if each semigroup $\{E_\xi(\sigma)\}$ is strongly (uniformly) continuous.

Throughout the rest of this paper, $\{T(\xi)\}$ will denote a semigroup of operators on $L_p(G)$ each of which commutes with right translations. Such a semigroup defines a semigroup $\{E_\xi\}$ of $L_p(G)$-multipliers, the functions E_ξ being defined by

$$(T(\xi)f)^\wedge(\sigma) = E_\xi(\sigma)\hat{f}(\sigma), \quad f\in L_p(G),\ \sigma\in\Sigma$$

(see [3]). The following lemma is contained in Theorem 28.39 of [1].

2.1. Lemma. *Let $\sigma\in\Sigma$ and for $U^{(\sigma)}$ in σ with representation space H_σ, let $\mathfrak{T}_\sigma(G)$ denote the set of all finite complex linear combinations of functions of the form $x\to\langle U_x^{(\sigma)}\xi, \eta\rangle$ as ξ, η vary over H_σ. Then $\{\hat{f}(\sigma): f\in\mathfrak{T}_\sigma(G)\}=B(H_\sigma)$.*

Following [2, 3.5.1], we shall say that $T(\xi)$ is *strongly measurable in* $(0, \infty)$ if for each $f\in L_p(G)$, there exists a sequence $\{u_n(\xi)\}$ of countably-valued functions (depending on f) from $(0, \infty)$ into $L_p(G)$ converging almost everywhere to $T(\xi)f$ in the topology of $L_p(G)$. For $\sigma\in\Sigma$, the semigroup $E_\xi(\sigma)$ is said to be *uniformly measurable in* $(0, \infty)$ if there exists a sequence of countably-valued functions $\{U_n(\xi)\}$ from $(0, \infty)$ into $B(H_\sigma)$ converging almost everywhere to $E_\xi(\sigma)$ in the uniform operator topology of $B(H_\sigma)$.

We can now state our first result.

2.2. Theorem. *Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of multipliers. Then $\{E_\xi\}$ is uniformly measurable if and only if $\{T(\xi)\}$ is strongly measurable.*

Proof. Suppose that $\{T(\xi)\}$ is strongly measurable and let σ be an arbitrary but fixed element of Σ. By Lemma 2.1, there exists $t\in\mathfrak{T}_\sigma(G)$ such that $\hat{t}(\sigma)=I_\sigma$, the identity operator on H_σ. The strong measurability of $\{T(\xi)\}$ implies that there exist a sequence $\{u_n\}$ of countably-valued functions on $(0, \infty)$ into $L_p(G)$ and a null set $E_0\subset(0, \infty)$ such that $\lim_{n\to\infty} \|T(\xi)t-u_n(\xi)\|_p=0$ for all $\xi\in(0, \infty)\sim E_0$. Then clearly $\{\hat{u}_n(\xi)(\sigma)\}$ is a sequence of countably-valued functions on $(0, \infty)$ into $B(H_\sigma)$. Moreover we have

$$\|E_\xi(\sigma)-\hat{u}_n(\xi)(\sigma)\|_{B(H_\sigma)} = \|E_\xi(\sigma)\hat{t}(\sigma)-\hat{u}_n(\xi)(\sigma)\|_{B(H_\sigma)} =$$
$$= \|[T(\xi)t-u_n(\xi)]\hat{\ }(\sigma)\|_{B(H_\sigma)} \leqq \|T(\xi)t-u_n(\xi)\|_p \to 0$$

as $n\to\infty$, for all $\xi\in(0, \infty)\sim E_0$. Hence $\{\hat{u}_n(\xi)(\sigma)\}$ converges almost everywhere on $(0, \infty)$ to $E_\xi(\sigma)$ in the uniform norm and so $E_\xi(\sigma)$ is uniformly measurable on $(0, \infty)$. Since σ was arbitrary, $\{E_\xi\}$ is uniformly measurable.

Conversely, let $\{E_\xi\}$ be uniformly measurable for $\sigma\in\Sigma$; there exist a sequence $\{U_n^\sigma\}$ of countably-valued functions on $(0, \infty)$ into $B(H_\sigma)$ and a null set $E_0^\sigma\subset(0, \infty)$ such that

$$\lim_{n\to\infty} \|E_\xi(\sigma)-U_n^\sigma(\xi)\|_{B(H_\sigma)} = 0 \quad \text{for all} \quad \xi\in(0, \infty) \sim E_0^\sigma.$$

By Lemma 2.1, this means there exists a sequence $\{t_n\}$ of countably valued functions on $(0, \infty)$ to $\mathfrak{T}_\sigma(G)$ such that $\hat{t}_n(\xi)(\sigma)=U_n^\sigma(\xi)$ and

$$\lim_{n\to\infty} \|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{B(H_\sigma)} = 0 \quad \text{for all} \quad \xi\in(0, \infty) \sim E_0^\sigma.$$

Then for any coordinate function $u_{jk}^{(\sigma)}$, using the notation in the proof Theorem 3.3 of [3], we have

$$\|T(\xi)u_{jk}^{(\sigma)}-t_n(\xi)*u_{jk}^{(\sigma)}\|_p \leqq d_\sigma\left\|(T(\xi)u_{jk}^{(\sigma)})\hat{\ }(\sigma)-(t_n(\xi)*u_{jk}^{(\sigma)})\hat{\ }(\sigma)\right\|_{\Phi_1} =$$
$$= d_\sigma\|E_\xi(\sigma)\hat{u}_{jk}^{(\sigma)}(\sigma)-\hat{t}_n(\xi)(\sigma)\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} \leqq d_\sigma\|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{\Phi_\infty}\|\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} =$$
$$= d_\sigma\|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{B(H_\sigma)}\|\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} \to 0 \quad \text{as } n\to\infty \text{ and for all } \quad \xi\in(0, \infty) \sim E_0.$$

Hence for every coordinate function u, the sequence $\{t_n(\xi)*u\}$ of countably-valued functions on $(0, \infty)$ converges almost everywhere to $T(\xi)u$ in the $L_p(G)$-norm. That $\{t_n(\xi)*f\}$ converges almost everywhere to $T(\xi)f$ for each $f\in L_p(G)$

in the $L_p(G)$-norm now follows from the fact that the operators $T(\xi)$ are linear and continuous and the trigonometric polynomials are dense in $L_p(G)$. This concludes the proof.

2.3. Theorem. *Let $\{T(\xi)\}$ be a strongly measurable semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Suppose that $\{T(\xi)\}$ is of type ω_0 and that for each $f\in L_p(G)$ the integral $R(\lambda)f=\int_0^\infty e^{-\lambda\xi}T(\xi)f d\xi$ exists for all λ with* re $(\lambda)>\omega_0$. *Then for each $\sigma\in\Sigma$, the integral $P(\lambda)(\sigma)=\int_0^\infty e^{-\lambda\xi}E_\xi(\sigma)d\xi$ exists as an element of $B(H_\sigma)$ for all λ with* re $(\lambda)>\omega_0$. *Moreover, if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then for each $\sigma\in\Sigma$, $\{E_\xi(\sigma)\}$ is uniformly Abel-ergodic at zero with the identity operator as its Abel limit.*

Note. Here and throughout this paper, the integrals are in the sense of Bochner [2, 3.7].

Proof. Since $\{T(\xi)\}$ is strongly measurable, $\{E_\xi(\sigma)\}$ is uniformly measurable for each $\sigma\in\Sigma$, by Theorem 2.2. If t is chosen as in the proof of Theorem 2.2, we have for all λ with re $(\lambda)>\omega_0$,

$$\int_0^\infty \|e^{-\lambda\xi}E_\xi(\sigma)\|_{B(H_\sigma)}\,d\xi = \int_0^\infty \left\|e^{-\lambda\xi}(T(\xi)t)\hat{\ }(\sigma)\right\|_{B(H_\sigma)}d\xi \leqq \int_0^\infty \|e^{-\lambda\xi}T(\xi)t\|_p\,d\xi <\infty.$$

Hence by [2, Theorem 3.7.4], the Bochner integral $\int_0^\infty e^{-\lambda\xi}E_\xi(\sigma)\,d\xi$ exists as an element of $B(H_\sigma)$ for each λ with re $(\lambda)>\omega_0$. Moreover, for all such λ, we have

$$\left\|\lambda\int_0^\infty e^{-\lambda\xi}E_\xi(\sigma)\,d\xi-E_0(\sigma)\right\|_{B(H_\sigma)}=\left\|\lambda\int_0^\infty e^{-\lambda\xi}(T(\xi)t)\hat{\ }(\sigma)\,d\xi-(T(0)t)\hat{\ }(\sigma)\right\|_{B(H_\sigma)}=$$

$$=\left\|\left[\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)t\,d\xi-T(0)t\right]\hat{\ }(\sigma)\right\|_{B(H_\sigma)}\leqq\left\|\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)t\,d\xi-t\right\|_p\to 0$$

as $\lambda\to\infty$, which completes the proof of the theorem.

The proof of our main result depends on the following very striking ergodic theorem which holds for a much wider class of semigroups than needed here [2, 18.8.3].

2.4. Theorem. *Let $\{S(\xi)\}$ be a semigroup of class* $(0, A)$ *on a Banach space X and suppose that $\{S(\xi)\}$ is uniformly Abel-ergodic at zero with J as its Abel limit. Then $S(\xi)=J\exp(\xi A)$ where $J^2=J$, $A\in B(X)$; $AJ=JA=A$ and* uniform $\lim_{\xi\to 0^+} S(\xi)=J$, i.e., *$S(\xi)$ converges uniformly to J in the sense of Cauchy.*

2.5. Theorem. *Let* $\{T(\xi)\}$ *be a semigroup of class* $(0, A)$ *on* $L_p(G)$ *each of which commutes with right translations. Then* $\{T(\xi)\}$ *is a semigroup of class* (C_0).

Proof. Let $\{E_\xi\}$, as before, denote the associated semigroup of $L_p(G)$-multipliers. Then for each $\sigma \in \Sigma$, $\{E_\xi(\sigma)\}$ is, by Theorem 2.3, uniformly Abel-ergodic with the identity operator as its Abel limit.

Since $\{E_\xi(\sigma)\}$ is clearly of class $(0, A)$, it follows from Theorem 2.4 that $\lim_{\xi \to 0^+} \|E_\xi(\sigma) - E_0(\sigma)\|_{B(H_\sigma)} = 0$, $E_0(\sigma) = I_\sigma$, the identity operator on H_σ. Thus $E_\xi(\sigma)$ is uniformly continuous for all $\xi \geqq 0$ and the same is true for each $\sigma \in \Sigma$. Now $\{T(\xi);\ \xi \geqq 0\}$ is, in the terminology of [3], the semigroup of operators on $L_p(G)$ defined by the semigroup of $L_p(G)$-multipliers $\{E_\xi(\sigma);\ \xi \geqq 0, \sigma \in \Sigma\}$. Hence by Theorem 1.1, $\{T(\xi);\ \xi \geqq 0\}$ is strongly continuous for all $\xi \geqq 0$ and is therefore of class (C_0). This concludes the proof.

As stated in the Introduction, there are a number of classes between (C_0) and $(0, A)$. Theorem 2.5 shows that if $T(\xi)$ commutes with right translations, then all these classes collapse into (C_0).

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