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Contractions and unilateral shifts

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A contraction T on a separable Hilbert space is said to be a weak contraction if $I-T^*T\in(\tau, C)$ which denotes the trace class, and $\sigma(T)\neq\bar{D}$, where D is the open unit disk. It is well known that there is a C_0-C_{11} decomposition for a weak contraction ([3]). Therefore we can easily show that if T is of class C_{10} (about $C_{10}, C_{\cdot 0}$, etc., see p. 72 of [3]) and if $I-T^*T\in(\tau, C)$, then

$$\sigma_p(T^*)=D \quad \text{and} \quad \sigma_p(T)\cap D=\emptyset.$$

In this note, we shall investigate a contraction T such that $I-T^*T\in(\tau, C)$ and $\sigma(T)=\bar{D}$.

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1. Operator valued functions

For $T\in I+(\tau, C)$, Bercovici and Voiculescu defined the algebraic adjoint T^{a}, which satisfies

$$T^{\mathrm{a}}T=TT^{\mathrm{a}}=(\det T)I.$$

They showed that if $\Theta(\lambda)$ is a contractive holomorphic function and if $\Theta(\lambda)\in I+$ $+(\tau, C)$ for every $\lambda\in D$, then $\Theta(\lambda)^{\mathrm{a}}$ is a contractive holomorphic function. In this case, if $\det\Theta(e^{it})\neq 0$ a.e., then $\Theta(e^{it})$ is invertible and its inverse is $\Theta(e^{it})^{\mathrm{a}}/$ $\det\Theta(e^{it})$ a.e.

Theorem 1. *Let $\Theta(\lambda)$ be an inner function (that is, $\Theta(\lambda)$ is a contractive holomorphic function defined on D and $\Theta(e^{it})$ is isometric a.e.) with values in $\mathscr{L}(E, E')$, where E, E' are separable Hilbert spaces. If there is an isometry V in $\mathscr{L}(E, E')$ such that for every $\lambda\in D$*

(1.1) $$I_E-V^*\Theta(\lambda)\in(\tau, C),$$

(1.2) $$\det V^*\Theta(\lambda)\not\equiv 0,$$

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then there is a bounded holomorphic function $\Delta(\lambda)$ *with values in* $\mathscr{L}(E', F)$ *for a suitable Hilbert space* F *such that*

$$\Theta(e^{it})E \oplus \Delta(e^{it})^* F = E' \ a.e. \tag{1.3}$$

Proof. If V is unitary, then $\Theta(e^{it})$ is invertible a.e. Hence we may assume that V is not unitary. Set $F=E'\ominus VE$. Let $E_0=E\oplus F$ be the direct sum of E and F. For $\lambda\in D$, define $\Theta'(\lambda)\in\mathscr{L}(E_0, E')$ by

$$\Theta'(\lambda)|_E = \Theta(\lambda) \quad \text{and} \quad \Theta'(\lambda)|_F = I_F.$$

For simplicity, set $d(\lambda)=\det V^*\Theta(\lambda)$ and $A(\lambda)=(V^*\Theta(\lambda))^a$. Determine $\Delta(\lambda)\in \in\mathscr{L}(E', F)$ by

$$\Delta(\lambda) = -P_F\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F \tag{1.4}$$

and $\Delta'(\lambda)\in\mathscr{L}(E', E_0)$ by

$$\Delta'(\lambda) = A(\lambda)V^* + \Delta(\lambda).$$

Then we have

$$\Delta'(\lambda)\Theta'(\lambda)|_E = \Delta'(\lambda)\Theta(\lambda) = A(\lambda)V^*\Theta(\lambda) + \Delta(\lambda)\Theta(\lambda) =$$
$$= d(\lambda)I_E - P_F\Theta(\lambda)\, d(\lambda)I_E + d(\lambda)P_F\Theta(\lambda) = d(\lambda)I_E,$$
$$\Delta'(\lambda)\Theta'(\lambda)|_F = A(\lambda)V^*I_F + \Delta(\lambda)I_F = d(\lambda)I_F,$$

and

$$\Theta'(\lambda)\Delta'(\lambda) = \Theta(\lambda)A(\lambda)V^* + \Delta(\lambda) = (I-P_F)\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F =$$
$$= VV^*\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F = V\,d(\lambda)V^* + d(\lambda)P_F = d(\lambda)I_{E'}.$$

Thus we have

$$\Delta'(\lambda)\Theta'(\lambda) = d(\lambda)I_{E_0}, \quad \Theta'(\lambda)\Delta'(\lambda) = d(\lambda)I_{E'}.$$

Since the inverse of $\Theta'(e^{it})$ is $\Delta'(e^{it})/d(e^{it})$ a.e., the orthogonal complement of $\Theta(e^{it})E=\Theta'(e^{it})E$ is

$$\frac{\Delta'(e^{it})^*}{\overline{d(e^{it})}}(E_0\ominus E) = \Delta(e^{it})^*F.$$

It is clear that $\Delta(\lambda)$ is a bounded holomorphic function.

Cambern showed that the orthogonal complement of a finite dimensional holomorphic range function is conjugate holomorphic (c.f. p. 94 of [2]). Now, we can show this result as a corollary.

Corollary 1. *Let* $\Theta(\lambda)$ *be an inner function with values in* $\mathscr{L}(E, E')$. *Suppose* $\dim E=m<\infty$. *Then there is a bounded holomorphic function* $\Delta(\lambda)$ *satisfying* (1.3).

Proof. We may assume that $E\subset E'$ and $\Theta(e^{it})$ is a matrix. Since

$$1=\det\left(\Theta(e^{it})^*\Theta(e^{it})\right)=\sum_{\sigma}|\det \Theta_\sigma(e^{it})|^2,$$

a.e., where $\sum_{\sigma}$ is taken over all $m\times m$ submatrices of $\Theta(e^{it})$, there is at least one σ such that $\det\Theta_{\sigma}(e^{it})\neq 0$ a.e. Thus there is an isometry V such that

$$\det V^{*}\Theta(e^{it})=\det\Theta_{\sigma}(e^{it})\neq 0 \quad \text{a.e.}$$

(see [4]). Hence V and $\Theta(\lambda)$ satisfy (1.1), (1.2).

2. Quasi unilateral shifts

We begin with a short review about the canonical model theory of B. Sz.-Nagy and C. Foiaş. Let T be a contraction of class $C_{\cdot 0}$ on a separable Hilbert space H. Set $D_T=(I-T^{*}T)^{1/2}$, and let E and E' be the closures of $D_T H$ and $D_{T^*}H$, respectively. Then the characteristic function $\Theta(\lambda)$ of T determined by

$$\Theta(\lambda)=\{-T+\lambda D_{T^*}(I-\lambda T^{*})^{-1}D_T\}|_E \quad \text{for} \quad \lambda\in D \tag{2.1}$$

is an inner function with values in $\mathscr{L}(E,E')$. Therefore

$$\dim E\leqq\dim E'.$$

Moreover T is unitary equivalent to $S(\Theta)$ on $H(\Theta)$ defined by

$$H(\Theta)=H^{2}(E')\ominus\Theta H^{2}(E),\quad S(\Theta)^{*}h=\frac{1}{\lambda}(h(\lambda)-h(0)) \quad \text{for } h \text{ in } H(\Theta). \tag{2.2}$$

T is of class $C_{1\cdot}$ if and only if $\Theta(\bar\lambda)^{*}H^{2}(E')$ is dense in $H^{2}(E)$ (that is, Θ is $*$-outer).

In this note, for simplicity, we call T a *quasi unilateral shift* if T is a contraction of class $C_{\cdot 0}$ such that

$$I-T^{*}T\in(\tau,C),\quad \mathscr{K}(T)=\{0\} \quad\text{and}\quad \mathscr{K}(T^{*})\neq\{0\},$$

where $\mathscr{K}(T)$ denotes the kernel of T.

Theorem 2. *If T is a quasi unilateral shift on H, then there is a bounded operator X with dense range satisfying*

$$XT=SX, \tag{2.3}$$

where S is a unilateral shift satisfying

$$0>\operatorname{index}S=\operatorname{index}T\geqq-\infty,$$

where $\operatorname{index}T=\dim\mathscr{K}(T)-\dim\mathscr{K}(T^{*})$.

Proof. We may assume $I-T^{*}T\neq 0$. From $T(I-T^{*}T)=(I-TT^{*})T$, it follows that $TE\subset E'$, $T(H\ominus E)=H\ominus E'$, where E and E' are the spaces defined above. Thus we have

$$H\ominus TH = E'\ominus TE \neq \{0\}. \tag{2.4}$$

Let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be the C.O.N.B. of E such that $(I-T^*T)e_n=\mu_n e_n$, $\mu_n>0$. Then $f_n=(1-\mu_n)^{-1/2}Te_n$ $(n=1, 2, \ldots)$ is a C.O.N.B. of TE and $T^*f_n=(1-\mu_n)^{1/2}e_n$ (see p. 324 of [3]). Setting $Ve_n=-f_n$ $(n=1, 2, \ldots)$, V is an isometry from E to E', and

$$V+T|_E\in(\tau, C) \quad \text{(see [1])}. \tag{2.5}$$

Setting $F=E'\ominus VE$, from (2.4) it follows that

$$\dim F = -\ \text{index}\ T. \tag{2.6}$$

$I-T^*T\in(\tau, C)$ implies $D_T\in(\sigma, C)$ which denotes the Hilbert—Schmidt class. Since $(I-TT^*)|_{TE}$ is unitarily equivalent to $I-T^*T$, we have $D_{T^*}|_{TE}\in(\sigma, C)$. Thus

$$\lambda V^*D_{T^*}(I-\lambda T^*)^{-1}D_T = \lambda V^*(D_{T^*}|_{TE})(I-\lambda T^*)^{-1}D_T \quad (\lambda\in D)$$

belongs to (τ, C). Thus, from (2.1), (2.5), we have

$$I-V^*\Theta(\lambda)\in(\tau, C) \quad \text{for each } \lambda.$$

Since

$$|\det(V^*\Theta(0))|^2 = \det(\Theta(0)^*VV^*\Theta(0)) = \det(T^*VV^*T|_E) = \det(T^*T|_E) \neq 0,$$

we have $\det V^*\Theta(\lambda)\not\equiv 0$. Thus V and $\Theta(\lambda)$ satisfy the conditions of Theorem 1. Hence $\Delta(\lambda)$ defined by (1.4) satisfies (1.3). Since $\Delta(\lambda)\Theta(\lambda)=0$, setting

$$X_0 h = \Delta h \quad \text{for } h \text{ in } H(\Theta), \tag{2.7}$$

we have $X_0\in\mathscr{L}(H(\Theta), H^2(F))$ and $X_0S(\Theta)=S_0X_0$, where S_0 is the unilateral shift on $H^2(F)$. Since

$$H^2(F) \supset X_0H(\Theta) = \Delta H^2(E') \supset \Delta H^2(F) = (\det V^*\Theta(\lambda))\, H^2(F),$$

it follows that $S=S_0|_{\overline{X_0H(\Theta)}}$ is unitarily equivalent to S_0. Thus, from (2.6), we have

$$\text{index}\ S = \text{index}\ S_0 = -\dim F = \text{index}\ T.$$

Consequently an operator X from $H(\Theta)$ to $\overline{X_0H(\Theta)}$ defined by

$$Xh = X_0 h \quad \text{for } h \text{ in } H(\Theta) \tag{2.8}$$

satisfies (2.3).

Corollary 1. *Let T be a contraction of class C_{00} such that $I-T^*T$ and $I-TT^*$ belong to (τ, C). Then, for $a\in D$, $\mathscr{K}(T-aI)=\{0\}$ if and only if $\mathscr{K}(T^*-\bar{a}I)=\{0\}$.*

Proof. Set $T_a=(T-aI)(I-\bar{a}T)^{-1}$ and $A=(1-|a|^2)^{1/2}(I-\bar{a}T)^{-1}$. Then we have $I-T_a^*T_a=A^*(I-T^*T)A$, $I-T_aT_a^*=A(I-TT^*)A^*$, and T_a is of class C_{00} (see p. 240 and p. 257 of [3]). Suppose $\mathscr{K}(T-aI)=\{0\}$ and $\mathscr{K}(T^*-\bar{a}I)\neq\{0\}$. Then T_a is a quasi unilateral shift. Therefore, there is an X satisfying $XT_a=SX$, which implies that T is not of class C_{00}. This is a contradiction. Thus $\mathscr{K}(T-aI)=\{0\}$ implies $\mathscr{K}(T^*-\bar{a}I)=\{0\}$. Similarly we can prove the converse assertion.

For a contraction T on H, we have

$$\|I-T^*T\|_p^p+\dim\mathscr{K}(T^*) = \|I-TT^*\|_p^p+\dim\mathscr{K}(T), \tag{2.9}$$

where $\| \ \|_p$ denotes the p-Schatten norm. Indeed, from $T(I-T^*T)=(I-TT^*)T$, $(I-T^*T)|_{\overline{T^*H}}$ and $(I-TT^*)|_{\overline{TH}}$ are unitarily equivalent. $(I-T^*T)\,|_{\mathscr{K}(T)}=I_{\mathscr{K}(T)}$ and $(I-TT^*)\,|_{\mathscr{K}(T^*)}=I_{\mathscr{K}(T^*)}$ imply that

$$\|I-T^*T\|_p^p = \|(I-T^*T)|_{\overline{T^*H}}\|_p^p+\dim\mathscr{K}(T),$$

$$\|I-TT^*\|_p^p = \|(I-TT^*)|_{\overline{TH}}\|_p^p+\dim\mathscr{K}(T^*).$$

Thus we have (2.9). Similarly we have

$$\operatorname{rank}(I-T^*T)+\dim\mathscr{K}(T^*) = \operatorname{rank}(I-TT^*)+\dim\mathscr{K}(T). \tag{2.9$'$}$$

Proposition 1. *Let T be a Fredholm quasi unilateral shift. Suppose X with dense range satisfies $XT=SX$, where S is a unilateral shift with* index $S=$index T. *Then $T\,|_{\mathscr{K}(X)}$ is of class C_0.*

Proof. Let $T=\begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$ be a decomposition of T corresponding to $H=\mathscr{K}(X)\oplus\mathscr{K}(X)^\perp$. Then T_1 is injective and, from (2.3), also T_2 is injective. From the assumption and (2.9), it follows that $I-T^*T\in(\tau, C)$ and $I-TT^*\in(\tau, C)$, which implies

$$I-T_1^*T_1\in(\tau, C), \tag{2.10}$$

$$I-(T_1T_1^*+T_{12}T_{12}^*)\in(\tau, C), \tag{2.11}$$

$$I-(T_{12}^*T_{12}+T_2^*T_2)\in(\tau, C), \tag{2.12}$$

$$I-T_2T_2^*\in(\tau, C). \tag{2.13}$$

From $\mathscr{K}(T_2^*)\subset\mathscr{K}(T^*)$, it follows that

$$\operatorname{index} T = -\dim\mathscr{K}(T^*) \leqq -\dim\mathscr{K}(T_2^*) \leqq -\dim\mathscr{K}(S^*)=\operatorname{index} T,$$

which implies index $T=$index T_2. From (2.9) and (2.13), we have $I-T_2^*T_2\in(\tau, C)$, which, by (2.12), implies $T_{12}\in(\sigma, C)$. Therefore, from (2.10) and (2.11), T_1 is a Fredholm operator. Since

$$\operatorname{index} T = \operatorname{index}\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \operatorname{index} T_1+\operatorname{index} T_2,$$

we have index $T_1=0$. Thus T_1 is invertible. Hence T_1 is a weak contraction of class $C_{\cdot 0}$. Consequently T_1 is of class C_0.

Corollary 2. *Let T be a Fredholm quasi unilateral shift of class C_{10}. Then $\mathscr{K}(A)=\{0\}$ provided $AT=TA$ and $\mathscr{K}(A^*)=\{0\}$* (cf. [6]).

Proof. For X defined in Theorem 2, we have $(XA)T=S(XA)$. From Proposition 1, we have $\mathscr{K}(XA)=\{0\}$.

Proposition 2. *Let T be of class $C_{\cdot 0}$. Then T is of class C_{10} if and only if*

$$\Theta L^2(E)\cap H^2(E')=\Theta H^2(E). \tag{2.14}$$

Proof. Since, for h in $H^2(E')$ and f in $H^2(E)$, we have

$$(\Theta(\bar\lambda)^*h(\lambda),f(\lambda))_{H^2(E)}=\frac{1}{2\pi}\int_0^{2\pi}(\Theta(e^{-it})^*h(e^{it}),f(e^{it}))_E\,dt=$$

$$=-\frac{1}{2\pi}\int_0^{-2\pi}(\Theta(e^{it})^*h(e^{-it}),f(e^{-it}))_E\,dt=\frac{1}{2\pi}\int_0^{2\pi}(\Theta(e^{it})^*h(e^{-it}),f(e^{-it}))_E\,dt=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}(\Theta(e^{it})^*e^{-it}h(e^{-it}),e^{-it}f(e^{-it}))_E\,dt=(\Theta(\lambda)^*\bar\lambda h(\bar\lambda),\bar\lambda f(\bar\lambda))_{L^2(E)},$$

$\Theta(\bar\lambda)^*H^2(E')$ is dense in $H^2(E)$ if and only if $\Theta(\lambda)^*(H^2(E'))^\perp$ is dense in $(H^2(E))^\perp$, where $\perp$ denotes the orthogonal complement. We have always

$$\Theta L^2(E)\cap H^2(E')\supset\Theta H^2(E).$$

At first, assume that T is of class C_{10}. Suppose

$$\Theta g\in\{\Theta L^2(E)\cap H^2(E')\}\ominus\Theta H^2(E).$$

Then $\Theta g\in H^2(E')$ and $g\perp H^2(E)$, because Θ is an isometry from $L^2(E)$ to $L^2(E')$. Thus $g\perp\Theta^*(H^2(E'))^\perp$ and $g\in(H^2(E))^\perp$. Since $\Theta(\lambda)$ is $*$-outer, we have $g=0$. Consequently (2.14) follows.

Conversely assume (2.14). Suppose $f\perp\Theta(\lambda)^*(H^2(E'))^\perp$ and $f\in(H^2(E))^\perp$. Then $\Theta f\in H^2(E')$ and $\Theta f\perp\Theta H^2(E)$. Thus from (2.14), we have $\Theta f=0$ and hence $f=0$. Consequently $\Theta(\lambda)$ is $*$-outer.

Theorem 3. *Let T be a quasi unilateral shift. Then $T\prec S$ (that is, there is an X such that $\mathscr{K}(X)=\mathscr{K}(X^*)=\{0\}$, $XT=SX$), where S is a unilateral shift with* index $S=$index T, *if and only if T is of class C_{10}.*

Proof. Assume that T is of class C_{10}. From Theorem 2, there is an X with dense range satisfying (2.3). If $Xh=0$ for h in $H(\Theta)$, then, from (2.7) and (2.8), $\Delta(e^{it})h(e^{it})=0$ a.e. Thus, from (1.3), $h\in\Theta L^2(E)$, so that, from (2.14), $h\in\Theta H^2(E)$. Consequently $h=0$. Thus we have $T\prec S$.

Conversely, assume $XT=SX$ and $\mathcal{K}(X)=\mathcal{K}(X^*)=\{0\}$. From $XT^n=S^nX$ $(n=1, 2, \dots)$ it follows that T is of class C_{10}.

Remark 1. If T is a Fredholm operator, then, from Theorem 2 and Proposition 1, it is clear that $T\prec S$ if T is of class C_{10}.

Remark 2. Theorem 3 implies that the Jordan model of a quasi unilateral shift of class C_{10} is a unilateral shift.

Corollary 3. *Let T be a quasi unilateral shift of class C_{10}. Then T^* has a cyclic vector.*

Proof. $T\prec S$ implies that $S^*\prec T^*$. Since S^* has a cyclic vector, also T^* does.

Proposition 3. *Let T be a quasi unilateral shift. Then there is an injection Y such that*

$$YS = TY, \tag{2.15}$$

where S is a unilateral shift with index $S=$index T.

Proof. Consider $S(\Theta)$ defined by (2.2) instead of T. Let V be an isometry defined in the proof of Theorem 2. Then

$$E' = VE \oplus F \quad \text{and} \quad \det V^*\Theta(e^{it}) \neq 0 \quad \text{a.e.}.$$

Define an operator Y from $H^2(F)$ to $H(\Theta)$ by

$$Yh = P_{H(\Theta)}h \quad \text{for } h \text{ in } \quad H^2(F).$$

Then we have

$$YSh = P_{H(\Theta)}Sh = P_{H(\Theta)}SP_{H(\Theta)}h = S(\theta)Yh,$$

which implies (2.15). Suppose $Yh=0$. Then $h=\Theta f$ for some $f\in H^2(E)$. Thus $0=V^*h(e^{it})=V^*\Theta(e^{it})f(e^{it})$ a.e. Since $V^*\Theta(e^{it})$ is invertible a.e., $f(e^{it})=0$ a.e. Consequently Y is injective.

Proposition 4. *Let T be a quasi unilateral shift of class C_{10}. Then, if $T\prec S'$, where S' is a unilateral shift, then* index $S'=$index T.

Proof. From $S'^*\prec T^*$, $\dim \mathcal{K}(S'^*)\leqq \dim \mathcal{K}(T^*)$. The proposition above implies that there is an injection Y' such that

$$Y'S = S'Y', \quad \text{index } S = \text{index } T,$$

which implies that $0>$index $S\geqq$index S' (cf. [4]). Thus we have

$$\text{index } T = \text{index } S \geqq \text{index } S' \geqq \text{index } T,$$

from which index $T=$index S' follows.

Remark 3. P. Y. Wu [6] showed that if $I-T^*T$ is a finite rank operator, and if $T\prec S'$, then

$$\operatorname{rank}(I-TT^*)-\operatorname{rank}(I-T^*T) = -\operatorname{index} S'.$$

From (2.9)′, our proposition is an extension of this result.

3. Cyclic vector

In this section, we consider a quasi unilateral shift of class C_{10} which has a cyclic vector. The next proposition is a partial extension of Proposition 2 of [4] and Theorem 3.1 of [5].

Proposition 5. *Let T be a quasi unilateral shift of class C_{10}. Then next conditions are equivalent:*

(a) *T has a cyclic vector;*

(b) *there is a bounded operator Y satisfying*

$$YS_1 = TY, \quad \mathscr{K}(Y^*) = \{0\}, \tag{3.1}$$

where S_1 is a unilateral shift with $\operatorname{index} S_1 = -1$;

(c) $S_1\prec T$;

(d) $S_1\prec T$ *and* $T\prec S_1$;

(e) $\|I-TT^*\|_1-\|I-T^*T\|_1=1$, *and there is a bounded holomorphic function Γ with values in $\mathscr{L}(\mathrm{C}, E')$ satisfying*

$$\|\Gamma(e^{it})\| \leqq 1 \quad a.e., \tag{3.2}$$

$$\Gamma H^2(\mathrm{C})\vee \Theta H^2(E) = H^2(E'), \tag{3.3}$$

where Θ is the characteristic function of T defined by (2.1).

Proof. (a)→(e). From Theorem 3, for a unilateral shift S with $\operatorname{index} S = \operatorname{index} T$, we have $T\prec S$. That T has a cyclic vector implies that also S does. Thus $\operatorname{index} S = -1$. Consequently, from (2.9), we have

$$\|I-TT^*\|_1-\|I-T^*T\|_1 = 1.$$

We can construct a function Γ in the same way as in [4].

(e)→(b). The contraction Y defined by $Yh=P_{H(\Theta)}\Gamma h$ for h in $H^2(\mathrm{C})$ satisfies (3.1).

(b)→(c). Suppose $\mathscr{K}(Y)\neq\{0\}$. Since $S_1\mathscr{K}(Y)\subset\mathscr{K}(Y)$, there is a scalar inner function ψ such that $\mathscr{K}(Y)=\psi H^2(\mathrm{C})$. Thus

$$\mathscr{K}(Y)^{\perp} = H(\psi) \quad (= H^2(\mathrm{C})\ominus\psi H^2(\mathrm{C})), \quad Y|_{H(\psi)}S(\psi) = TY|_{H(\psi)},$$

where $S(\psi)=P_{H(\psi)}S\,|_{H(\psi)}$. Since $S(\psi)$ is of class C_0, T must be of class C_0. This is a contradiction. Consequently $\mathscr{K}(Y)=\{0\}$.

(c)→(d). $S_1\prec T$ implies $T^*\prec S_1^*$, from which it follows that $\dim\mathscr{K}(T^*)\leqq\dim\mathscr{K}(S_1^*)=1$. That T is a quasi unilateral shift, implies index $T<0$. Thus index $T=-1$. By Theorem 3, we have $T\prec S_1$.

(d)→(a). This is obvious.

(3.3) implies that $[\Gamma,\Theta]$ is an outer function from $H^2(\mathbf{C})\oplus H^2(E)$ to $H^2(E')$. Generally $[\Gamma,\Theta]$ is not contractive. Therefore $d(\lambda)=\det[\Gamma(\lambda),\Theta(\lambda)]\in H^\infty$ and $d(\lambda)\leqq 1$ are not obvious. We shall show these results.

Let $A\in\mathscr{L}(E,E')$ be a contraction and $V\in\mathscr{L}(E,E')$ an isometry with index $V=-1$. Let $\{e_1,e_2,\dots,e_n,\dots\}$ be a C.O.N.B. in E. Then, setting $d_n=Ve_n$ $(n=1,2,\dots)$, $\{d_0,d_1,\dots,d_n,\dots\}$ is a C.O.N.B. in E', where d_0 is a unit vector in $\mathscr{K}(V^*)$. For $i=1,2,\dots$, define an isometry $V_i\in\mathscr{L}(E,E')$ by

$$V_ie_1=d_0,\dots,V_ie_i=d_{i-1},\ V_ie_{i+1}=d_{i+1},\ V_ie_{i+2}=d_{i+2},\dots.$$

Let $a_{ij}=(Ae_j,d_i)$ $(i\geqq 0,j\geqq 1)$. Then, in the base $\{e_1,e_2,\dots\}$, we have

$$V_i^*A=\begin{bmatrix} a_{01}, & \dots, & a_{0j}, & \dots\\ \vdots & & \vdots & \\ a_{i-1\,1}, & \dots, & a_{i-1\,j}, & \dots\\ a_{i+1\,1}, & \dots, & a_{i+1\,j}, & \dots\\ \vdots & & \vdots & \end{bmatrix}\quad (i=1,2,\dots).$$

Let $E_0=\mathbf{C}\oplus E$ be a direct sum of $\mathbf{C}$ and E, and e_0 a unit vector in $\mathbf{C}$. Let x_n $(n=0,1,2,\dots)$ be a scalar number such that $\sum_{n=0}^{\infty}|x_n|^2\leqq 1$. Let $B\in\mathscr{L}(E_0,E')$ be an operator defined by

$$(Be_0,d_i)=x_i,\ (Be_j,d_i)=a_{ij}\quad (i\geqq 0,j\geqq 1).$$

Determine a unitary $U\in\mathscr{L}(E_0,E')$ by $Ue_i=d_i\,(i\geqq 0)$. Then in the base $\{e_0,e_1,\dots,\dots,e_i,\dots\}$ of E_0 we have

$$U^*B=\begin{bmatrix} x_0, & a_{01}, & \dots, & a_{0j}, & \dots\\ x_1, & a_{11}, & \dots, & a_{1j}, & \dots\\ \vdots & & & \vdots & \\ x_i, & a_{i1}, & \dots, & a_{ij}, & \dots\\ \vdots & & & \vdots & \end{bmatrix}.$$

Let $I_E-V^*A\in(\tau,C)$. Then, since $(V_i^*Ae_j,e_k)=(V^*Ae_j,e_k)$ for $j\geqq 1$ and $k\geqq i+1$, $I_E-V_i^*A\in(\tau,C)$ for every i.

$$P_E(I_{E_0}-U^*B)|_E=I_E-V^*A$$

implies $I_{E_0}-U^*B\in(\tau,C)$.

Lemma. *Let $I_E - V^*A \in (\tau, C)$. Set $V_0 = V$. Then*

$$\det U^*B = \sum_{i=0}^{\infty} x_i(-1)^i \det(V_i^*A),$$

and

$$\sum_{i=1}^{\infty} |x_i(-1)^i \det(V_i^*A)| \leqq 1.$$

Proof. For simplicity, let $[A]_n$ denote the first $n \times n$ submatrix of A, and write A_n for $A|_{E_n}$, where $E_n = \langle e_1, \ldots, e_n \rangle$. For any k and n as $n \geqq k$, we have

$$\sum_{i=0}^{k} |\det[V_i^*A]_n|^2 \leqq \det(A_n^* A_n) = \det[A^*A]_n \leqq 1, \tag{3.4}$$

because A is a contraction. Since for each i

$$\det[V_i^*A]_n \to \det(V_i^*A) \quad (n \to \infty),$$

we have $\sum_{i=0}^{k} |\det(V_i^*A)|^2 \leqq 1$, which implies

$$\sum_{i=0}^{\infty} |\det(V_i^*A)|^2 \leqq 1. \tag{3.5}$$

Consequently $\sum_{i=0}^{\infty} |x_i(-1)^i \det(V_i^*A)| \leqq 1$. For any $\varepsilon > 0$, take an m such that

$$\sum_{i=m+1}^{\infty} |x_i|^2 < \varepsilon^2. \tag{3.6}$$

Since $\det[U^*B]_n \to \det(U^*B)$, and $\det[V_i^*A]_n \to \det(V_i^*A)$ as $n \to \infty$, we can take an N such that

$$n \geqq N \to |\det[U^*B]_n - \det(U^*B)| < \varepsilon, \tag{3.7}$$

and

$$n \geqq N \to \sum_{i=0}^{m} |\det[V_i^*A]_n - \det(V_i^*A)|^2 < \varepsilon^2. \tag{3.8}$$

Fix a k as $k \geqq N+1$ and $k \geqq m+1$. Then it follows that

$$\left|\det(U^*B) - \sum_{i=0}^{\infty} x_i(-1)^i \det(V_i^*A)\right| \leqq$$

$$\leqq |\det(U^*B) - \det[U^*B]_k| + \left|\det[U^*B]_k - \sum_{i=0}^{m} x_i(-1)^i \det[V_i^*A]_{k-1}\right| +$$

$$+ \left|\sum_{i=0}^{m} x_i(-1)^i \{\det[V_i^*A]_{k-1} - \det(V_i^*A)\}\right| + \left|\sum_{i=m+1}^{\infty} x_i(-1)^i \det(V_i^*A)\right|.$$

From (3.7) $|\det(U^*B)-\det[U^*B]_k|<\varepsilon$, and from (3.8)

$$\left|\sum_{i=0}^{m} x_i(-1)^i\{\det[V_i^*A]_{k-1}-\det(V_i^*A)\}\right| \leqq$$

$$\leqq \left(\sum_{i=0}^{m}|x_i|^2\right)^{1/2}\left(\sum_{i=0}^{m}|\det[V_i^*A]_{k-1}-\det(V_i^*A)|^2\right)^{1/2} < \varepsilon.$$

(3.5) and (3.6) implies that

$$\left|\sum_{i=m+1}^{\infty} x_i(-1)^i\det(V_i^*A)\right| < \varepsilon.$$

By finite matrix theory

$$\left|\det[U^*B]_k-\sum_{i=0}^{m} x_i(-1)^i\det[V_i^*A]_{k-1}\right| = \left|\sum_{i=m+1}^{k-1} x_i(-1)^i\det[V_i^*A]_{k-1}\right| < \varepsilon,$$

because the last inequality follows from (3.4), (3.6). Consequently, for any $\varepsilon>0$ we have

$$\left|\det(U^*B)-\sum_{i=0}^{\infty} x_i(-1)^i\det(V_i^*A)\right| < 4\varepsilon.$$

In (e) of Proposition 5, set $(\Gamma(\lambda)e_0, d_i)=h_i(\lambda)$ for $i\geqq 0$. Then we have:

Proposition 6. $|\det(U^*[\Gamma(\lambda), \Theta(\lambda)])|\leqq 1$, *and*

(3.9) $$\det(U^*[\Gamma(\lambda), \Theta(\lambda)]) = \sum_{i=0}^{\infty} h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))$$

is holomorphic on D.

Proof. From (3.2), we have $\sum_{i=0}^{\infty}|h_i(\lambda)|^2\leqq 1$. Since $V_i^*\Theta(\lambda)$ is a contractive holomorphic function, $\det(V_i^*\Theta(\lambda))\in H^\infty$. Since $\Theta(\lambda)$ is a contraction for every $\lambda\in D$, it follows that

$$\sum_{i=1}^{\infty}\left|h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))\right| \leqq 1,$$

which implies that $\sum_{i=0}^{\infty} h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))$ is holomorphic. Equality (3.9) follows from Lemma.

Problem. Is $\det(U^*[\Gamma(\lambda), \Theta(\lambda)])$ outer?

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