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On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator

I. JOÓ and V. KOMORNIK

The equiconvergence theorems play an important role in the theory of expansions. One of the first results of this type was proved by A. HAAR [1] in 1910—11. Later on general equiconvergence theorems were proved for the self-adjoint Schrödinger operator. However, many problems of practical interest require the investigation of the non-selfadjoint case. Under general conditions there do not exist complete orthonormal systems of eigenfunctions. However, introducing the eigenfunctions of higher order (a notion similar to what is known from the Jordan theorem in linear algebra), the existence of a Riesz basis consisting of eigenfunctions of higher order was proved for such cases, too [4], [5], [6]. During the investigation of a non-classical heat transfer problem a concrete Riesz basis consisting of eigenfunctions of higher order with infinitely many eigenfunctions of order $\geqq 1$ was found by A. A. SAMARSKIĬ and N. I. IONKIN [12].

The aim of the present paper is to prove a general equiconvergence theorem with respect to Riesz bases, which extends the previous results for the case of discrete spectrum in several directions. Namely we consider a complex potential function from the class $L^1_{\mathrm{loc}}(G)$ where G is an arbitrary interval and the eigenvalues may be arbitrary complex numbers. This theorem was first obtained by the authors independently. The present proof is a synthesis which is based on a fruitful method of V. A. Il'in [11] and uses also some new ideas of the papers [7]—[10].

1. Bessel-systems of eigenfunctions

Let G be an arbitrary open interval on the real line, $q \in L^1_{\mathrm{loc}}(G)$ an arbitrary complex function and consider the formal Schrödinger operator $Lu = -u'' + qu$. Given a complex number λ, the function $u: G \to \mathbf{C}$, $u \equiv 0$ is called an eigenfunction

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of order -1 of the operator L with the eigenvalue λ. Furthermore, a function $u: G\to \mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order m ($m=0, 1, \dots$) of the operator L with the eigenvalue λ if u and its derivative u' are locally absolutely continuous on G and $Lu=\lambda u-u^*$ almost everywhere on G, where u^* is an eigenfunction of order $m-1$ of the operator L with the same eigenvalue λ.

Let us introduce for any $\mu\in\mathbf{C}$, $t>0$ the functions

$$f_1(\mu, t) := t\frac{\sin\mu t}{\mu},$$

$$f_i(\mu, t) := \int_0^t \frac{\sin\mu(t-t_{i-1})}{\mu}\int_0^{t_{i-1}}\dots\int_0^{t_2}\frac{\sin\mu(t_2-t_1)}{\mu}\, t_1\frac{\sin\mu t_1}{\mu}\,dt_1\dots dt_{i-1} \quad (i=2, 3, \dots)$$

and for any $u\in L^\infty_{\mathrm{loc}}(G)$, $\mu\in\mathbf{C}$, $x\pm t\in G$, $t>0$ the functions

$$g_0(u, \mu, x, t) := \int_{x-t}^{x+t}\frac{\sin\mu(t-|x-\xi|)}{\mu}\,q(\xi)\,u(\xi)\,d\xi,$$

$$g_i(u, \mu, x, t) := \int_0^t\frac{\sin\mu(t-t_i)}{\mu}\int_0^{t_i}\dots\int_0^{t_2}\frac{\sin\mu(t_2-t_1)}{\mu}\times$$

$$\times\int_{x-t_1}^{x+t_1}\frac{\sin\mu(t_1-|x-\xi|)}{\mu}\,q(\xi)u(\xi)\,d\xi\,dt_1\dots dt_i \quad (i=1, 2, \dots).$$

Lemma 1. *Let u_m be an eigenfunction of order $\leqq m$ of the operator L with the eigenvalue $\lambda=\mu^2$ and put $u_{j-1}:=\lambda u_j-Lu_j$, $j=0, 1, \dots, m$. Then*

$$(1)\quad u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t = \sum_{i=1}^m f_i(\mu, t)\,u_{m-i}(x)+\sum_{i=0}^m g_i(u_{m-i}, \mu, x, t)$$

whenever $x\pm t\in G$, $t>0$. Moreover, putting $\nu:=\operatorname{Im}\mu$, the following estimates are valid:

$$(2)\quad \begin{aligned} |f_i(\mu, t)\,u_{m-i}(x)| &\leqq \left|\frac{t}{\mu}\right|^i |u_{m-i}(x)\operatorname{ch}\nu t|,\\ |g_i(u_{m-i}, \mu, x, t)| &\leqq \frac{\|q\|_{L^1(x-t,x+t)}}{|\mu|}\left|\frac{t}{\mu}\right|^i \sup_{|x-\xi|\leqq t}\big|u_{m-i}(\xi)\operatorname{ch}\nu(t-|x-\xi|)\big|.\end{aligned}$$

Proof. We recall the generalized Titchmarsh formula of Joó [7]:

$$(3)\quad \begin{aligned} &u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t =\\ &= \int_{x-t}^{x+t}\frac{\sin\mu(t-|x-\xi|)}{\mu}\,[q(\xi)\,u_m(\xi)+u_{m-1}(\xi)]\,d\xi.\end{aligned}$$

One can easily see that

$$\int_{x-t}^{x+t} \frac{\sin\mu(t-|x-\xi|)}{\mu} u_{m-1}(\xi)\,d\xi = t\frac{\sin\mu t}{\mu} u_{m-1}(x)+ \tag{4}$$
$$+\int_0^t \frac{\sin\mu(t-\tau)}{\mu}[u_{m-1}(x+\tau)+u_{m-1}(x-\tau)-2u_{m-1}(x)\cos\mu\tau]\,d\tau.$$

Combining (3) and (4) we obtain

$$u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t =$$
$$= \int_{x-t}^{x+t} \frac{\sin\mu(t-|x-\xi|)}{\mu} q(\xi)u_m(\xi)\,d\xi + t\frac{\sin\mu t}{\mu} u_{m-1}(x)+ \tag{5}$$
$$+\int_0^t \frac{\sin\mu(t-\tau)}{\mu}[u_{m-1}(x+\tau)+u_{m-1}(x-\tau)-2u_{m-1}(x)\cos\mu\tau]\,d\tau.$$

Now the formula (1) can be proved by an easy induction on m. Indeed, for $m=0$ both formulas (3) and (5) coincide with (1). Assume (1) is true for $m-1$ instead of m. Then applying this formula in the last integral of (5), we obtain (1) for m.

The estimates (2) follow immediately from the definition of f_i and g_i. The lemma is proved.

Let us now given a system $(u_k)\subset L^2(G)$ of eigenfunctions of the operator L. Let λ_k (resp. o_k) denote the eigenvalue (resp., the order) of u_k and assume that the following conditions are satisfied:

$$\sup o_k < \infty, \tag{6}$$

$$\text{in case } o_k > 0,\ \lambda_k u_k - Lu_k = u_{k-1}, \tag{7}$$

(8) (u_k) is a Bessel system, i.e., for any $w\in L^2(G)$, $\sum_k |\langle u_k, w\rangle|^2 \le C_0\|w\|^2_{L^2(G)}$ where C_0 is a constant independent of w.

The purpose of this section is to prove the following

Proposition 1. *Given any compact interval $K\subset G$, there exists an $R>0$ with*

$$\sup_{\mu>0} \sum_{|\mu-|\mathrm{Re}\sqrt{\lambda_k}||\le 1} (\|u_k\|_{L^\infty(K)} \operatorname{ch}(R\,\mathrm{Im}\sqrt{\lambda_k}))^2 < \infty.$$

We need some preliminary lemmas. For brevity, let us denote by μ_k a square root of λ_k for which $\mathrm{Re}\,\mu_k\ge 0$, and put $\varrho_k := \mathrm{Re}\,\mu_k$, $\nu_k := \mathrm{Im}\,\mu_k$. We shall repeatedly deal with compact intervals $K=[a,b]$ having the property

$$K_R := [a-R, b+R] \subset G \quad \textit{for} \quad R := (b-a)/4. \tag{9}$$

We introduce in this case the functions $d, v_k\colon K_R \to \mathbf{C}$ defined by

$$d(\xi) := \min\{\xi-(a-R),\ (b+R)-\xi\},\quad v_k(\xi) := u_k(\xi)\operatorname{ch}(\nu_k d(\xi)).$$

Lemma 2. *Given any compact interval* $K=[a, b]$ *having property* (9),

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]\,R^{-0.5}\|v_k\|_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)} \quad (|\nu_k|\to\infty) \tag{10}$$

uniformly in k.

Proof. Using the inequalities $|\operatorname{sh}(\operatorname{Im} z)| \leqq |\cos z|$ $(z\in\mathbf{C})$, $R\leqq d(x)\leqq 3R$ $(x\in K)$, and applying Lemma 1 for any $x\in K$ with $t:=d(x)$, we get

$$|v_k(x)| \leqq |0.5\operatorname{cth}\nu_k t|\,|2u_k(x)\cos\mu_k t| \leqq$$
$$\leqq [0.5+o(1)]\Big\{|u_k(x-t)|+|u_k(x+t)|+$$
$$+\sum_{i=0}^{\sigma_k}|\mu_k|^{-i-1}\|q\|_{L^1(K_R)}t^i\|v_{k-i}\|_{L^\infty(K_R)}+\sum_{i=1}^{\sigma_k}|\mu_k|^{-i}t^i|v_{k-i}(x)|\Big\} \leqq$$
$$\leqq [0.5+o(1)]\Big\{2\|v_k\|_{L^\infty(K_R\setminus K)}+(\operatorname{ch}\nu_k R)^{-1}\|v_k\|_{L^\infty(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}\Big\} \leqq$$
$$\leqq [1+o(1)]\,\|v_k\|_{L^\infty(K_R\setminus K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

This is obviously true for all $x\in K_R\setminus K$, too, therefore

$$\|v_k\|_{L^\infty(K_R)} \leqq [1+o(1)]\,\|v_k\|_{L^\infty(K_R\setminus K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}. \tag{11}$$

Now for any $a-R\leqq x\leqq a$ and $R\leqq t\leqq 2R$, we apply Lemma 1 with $x+t$ instead of x, and multiply by $\operatorname{ch}\nu_k d(x)$, to derive

$$|v_k(x)| \leqq |v_k(x+2t)|+2\,|v_k(x+t)|+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Applying the transformation $R^{-1}\int_R^{2R}\cdot\,dt$ and using in the first two integrals of the right side the Hölder inequality, we have

$$|v_k(x)| \leqq 3R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

This is true analogously for all $b\leqq x\leqq b+R$, too, therefore

$$\|v_k\|_{L^\infty(K_R\setminus K)} \leqq 3R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Substituting this into the right side of (11), we obtain

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)},$$

whence

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}. \tag{12}$$

Now we prove (10) by induction on σ_k. For $\sigma_k=-1$ (10) is trivial because $v_k\equiv 0$. Suppose (10) is true for $\sigma_k<m$ ($m\geqq 0$). Then it is true also for $\sigma_k=m$. Indeed, using (12) and the induction hypothesis,

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+$$

$$+o(1)\sum_{i=1}^{\sigma_k}\Big\{[3+o(1)]R^{-0.5}\|v_{k-i}\|_{L^2(K)}+o(1)\sum_{j=1}^{\sigma_{k-i}}\|v_{k-i-j}\|_{L^2(K)}\Big\}=$$

$$=[3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)}$$

(in the last step we used that $\sigma_{k-i}=\sigma_k-i$). The lemma is proved.

Lemma 3. *Each point of G has a neighbourhood K having property* (9) *such that*

$$\sup_{\mu>0}\sum_{\substack{|\mu-\varrho_k|\leqq 1\\ \varrho_k\geqq B|v_k|\\ |v_k|\geqq B}}(\|u_k\|_{L^\infty(K)}\operatorname{ch}0.5v_kR)^2=O(1)\quad(B\to\infty). \tag{13}$$

Proof. Putting for brevity

$$I_\mu=I_\mu(B):=\{k\colon |\mu-\varrho_k|\leqq 1,\ \varrho_k\geqq B|v_k|,\ |v_k|\geqq B\}, \tag{14}$$

it suffices to show in view of (6) that to any $m\in\{-1,0,1,\dots\}$, each point $y\in G$ has a neighbourhood $K_{y,m}$ with property (9) such that

$$\sup_{\mu>0}\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}(\|u_k\|_{L^\infty(K_{y,m})}\operatorname{ch}0.5v_kR_{y,m})^2=O(1)\quad(B\to\infty). \tag{15}$$

This is obvious for $m=-1$: each point of G has a neighbourhood having property (9). Let now $m\geqq 0$ and assume (15) is true for $m-1$. Let now $K=K_{y,m}$ be an arbitrary compact subinterval of $K_{y,m-1}$ which is 6 times shorter than $K_{y,m-1}$ and contains y. It follows then from the inductive hypothesis that

$$\sup_{\mu>0}\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)}=O(1)\quad(B\to\infty). \tag{16}$$

Indeed, for any k,

$$\|v_k\|_{L^2(K)} \leqq (\|u_k\|_{L^\infty(K)} \operatorname{ch} 3\nu_k R)(4R)^{0.5} =$$
$$= (4R)^{0.5} \|u_k\|_{L^\infty(K)} \operatorname{ch}(0.5\nu_k R_{y,m-1}) \leqq (4R)^{0.5} \|u_k\|_{L^\infty(K_{y,m-1})} \operatorname{ch}(0.5\nu_k R_{y,m-1}).$$

Let us now fix $\mu>0$ and $x\in K$ arbitrarily and introduce the function $w: G\to\mathbf{R}$:

$$w(x+t) := \begin{cases} \cos\mu t & \text{if } \ |t| \leqq d(x), \\ 0 & \text{otherwise.} \end{cases}$$

In the sequel we shall consider always $k\in I_\mu$ and the estimates o, O $(B\to\infty)$ will be uniform in $\mu>0, x\in K$ and $k\in I_\mu$. Obviously,

(17) $$\|w\|^2_{L^2(G)} = O(1).$$

For any $k\in I_\mu$, multiplying (1) by $\cos\mu t$ and integrating by t from 0 to $d(x)$, we obtain

(18) $$\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt\cdot u_k(x) = \langle u_k, w\rangle -$$
$$- \sum_{i=1}^{\sigma_k}\int_0^{d(x)} \cos\mu t f_i(\mu_k, t)\,dt\cdot u_{k-i}(x) - \sum_{i=0}^{\sigma_k}\cos\mu t\, g_i(u_{k-i}, \mu_k, x, t)\,dt.$$

Here

$$\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt = \frac{\sin(\mu-\mu_k)\,d(x)}{\mu-\mu_k} + \frac{\sin(\mu+\mu_k)\,d(x)}{\mu+\mu_k};$$

using $\varrho_k\geqq 0$, $d(x)\geqq R$, the definition (14) of I_μ and the inequalities $|\operatorname{sh}\operatorname{Im} z|\leqq \leqq|\sin z|\leqq\operatorname{ch}\operatorname{Im} z$ $(z\in\mathbf{C})$, we get that

$$\left|\frac{\sin(\mu-\mu_k)\,d(x)}{\mu-\mu_k}\right| \geqq \frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}|\operatorname{th}\nu_k d(x)|\sqrt{\frac{\nu_k^2}{1+\nu_k^2}} = [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|},$$
$$\left|\frac{\sin(\mu+\mu_k)\,d(x)}{\mu+\mu_k}\right| \leqq \frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}\sqrt{\frac{\nu_k^2}{B^2\nu_k^2+\nu_k^2}} = o(1)\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|},$$

whence

(19) $$\left|\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt\right| \geqq [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}.$$

(18), (19), (2) and (14) imply

$$[1-o(1)]|\nu_k|^{-1}|v_k(x)| \leqq |\langle u_k, w\rangle| +$$
$$+\|q\|_{L^1(K_R)}d(x)\left|\frac{\nu_k}{\mu_k}\right|\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)},$$

$$\left|\frac{v_k(x)}{\nu_k}\right| \leqq O(1)|\langle u_k, w\rangle| + o(1)\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Applying (10), we obtain

$$\left|\frac{v_k(x)}{\nu_k}\right| \leqq O(1)\,|\langle u_k, w\rangle| + o(1)\left\|\frac{v_k}{\nu_k}\right\|_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)}.$$

Summing up the square of this inequality for an arbitrary finite index set $I\subset$ $\subset\{k\in I_\mu\colon o_k\leqq m\}$, then applying (17) and the Bessel inequality (8) to the first sum on the right side, we obtain

$$\sum_{k\in I}\left|\frac{v_k(x)}{\nu_k}\right|^2 \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{k\in I}\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)} \leqq$$

$$\leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)}$$

(we used (16) in the last step), whence, integrating by x on K, we get

$$\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)},$$

$$\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} \leqq O(1).$$

Since I was chosen arbitrarily,

$$\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} = O(1).$$

Using (10) and (16) again, we see that

$$\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^\infty(K_R)} \leqq O(1)\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} =$$

$$= O(1)+o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} = O(1),$$

and hence (15) follows with $K=K_{y,m}$ because for any k,

$$\|u_k\|_{L^\infty(K)}\operatorname{ch}(0.5\,\nu_k R) \leqq \|u_k\|_{L^\infty(K)}\frac{\operatorname{ch}\nu_k R}{|\nu_k R|} \leqq R^{-1}\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)}.$$

The lemma is proved.

Lemma 4. *Each point of G has a neighbourhood K having property* (9) *such that for any fixed $B>0$,*

$$\sum_{\substack{\varrho_k\leqq B|\nu_k|\\ |\nu_k|\geqq C}}(\|u_k\|_{L^\infty(K)}\operatorname{ch}0.5\nu_k R)^2 = O(1)\quad (C\to\infty). \tag{20}$$

Proof. Setting

$$J = J(C) := \{k:\ \varrho_k \leqq B\,|\nu_k|,\ |\nu_k| \geqq C\}, \tag{21}$$

it suffices to show in view of (6) that for any $m \in \{-1, 0, \dots\}$ and $y \in G$, there exists a neighbourhood $K_{y,m}$ of y having property (9) such that

$$\sum_{\substack{k\in J\\ \sigma_k \leqq m}} (\|u_k\|_{L^\infty(K_{y,m})} \operatorname{ch} 0.5\nu_k R_{y,m})^2 = O(1) \quad (C \to \infty). \tag{22}$$

This is obvious for $m = -1$. Assume $m \geqq 0$ and (22) is true for $m-1$. Let $K = K_{y,m}$ be an arbitrary compact subinterval of $K_{y,m-1}$ containing y which is at least 6 times shorter than $K_{y,m-1}$ and which satisfies the following condition:

$$81(m+2)R^2\|q\|^2_{L^1(K_R)} < 8^{-1}. \tag{23}$$

As in the preceding lemma, we have

$$\sum_{\substack{k\in J\\ \sigma_k < m}} \|v_k\|^2_{L^2(K)} = O(1) \quad (C \to \infty). \tag{24}$$

Let us fix $x \in K$ arbitrarily and define $w: G \to \mathbf{R}$ by

$$w(x+t) := \begin{cases} 1 & \text{if } \ |t| \leqq d(x), \\ 0 & \text{otherwise.} \end{cases}$$

In the following considerations the estimates o, O will be uniform in $x \in K$ and $k \in J$ $(C \to \infty)$. Obviously,

$$\|w\|^2_{L^2(G)} = O(1). \tag{25}$$

For any $k \in J$, $o_k \leqq m$, integrating (1) by t from 0 to $d(x)$, we get

$$\begin{aligned} \int_0^{d(x)} 2\cos\mu_k t\,dt\,u_k(x) &= \langle u_k, w\rangle - \\ - \sum_{i=1}^{\sigma_k} \int_0^{d(x)} f_i(\mu_k, t)\,dt\,u_{k-i}(x) &- \sum_{i=0}^{\sigma_k} g_i(u_{k-i}, \mu_k, x, t)\,dt. \end{aligned} \tag{26}$$

Here, by the inequality $d(x) \geqq R$,

$$\left|\int_0^{d(x)} 2\cos\mu_k t\,dt\right| = \left|\frac{\sin\mu_k d(x)}{\mu_k}\right| \geqq \left|\frac{\operatorname{sh}\nu_k d(x)}{\mu_k}\right| = [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\mu_k|},$$

and therefore (26), (2), and (21) imply

$$[1-o(1)]\left|\frac{v_k(x)}{\mu_k}\right| \leqq |\langle u_k, w\rangle| + 3R\|q\|_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|_{L^\infty(K_R)} + \\ + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}$$

(we used that $d(x)\leqq 3R$), whence in view of (10),

$$\left|\frac{v_k(x)}{\mu_k}\right| \leqq O(1)|\langle u_k, w\rangle| + [9+o(1)]\, R^{0.5}\|q\|_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|_{L^2(K)} + \\ + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)},$$

and

$$\left|\frac{v_k(x)}{\mu_k}\right|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 + \\ +(2+o_k)[81+o(1)]\,R\,\|q\|^2_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)}.$$

$$\leqq O(1)\,|\langle u_k, w\rangle|^2 + (8R)^{-1}[1+o(1)]\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)}$$

(we used (23) and $o_k\leqq m$). Summing up this inequality for an arbitrary finite index set $I\subset\{k\in J;\ o_k\leqq m\}$, then applying (8) and (25) on the right side, and finally integrating by x on K (the length of which is $4R$), we obtain

$$\sum_{k\in I}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} \leqq O(1)+[0.5+o(1)]\sum_{k\in I}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in J\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)}.$$

Hence, using (24) and the choice of I, we have

$$\sum_{\substack{k\in J\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} = O(1),$$

and taking into account the estimate $|\mu_k|\leqq(1+B)|v_k|$, we get

$$\sum_{\substack{k\in J\\ \sigma_k\leqq m}}\left\|\frac{v_k}{v_k}\right\|^2_{L^2(K)} = O(1).$$

Now the proof can be finished exactly as in the preceeding lemma, using (24) instead of (16). The lemma is proved.

Lemma 5. *Given any compact interval $K=[a,b]\subset G$ and any number $D>0$, we have*

(27) $$\sup_{\mu>0}\sum_{\substack{|\mu-\varrho_k|\leqq 1\\ |v_k|\leqq D}}\|u_k\|^2_{L^\infty(K)}<\infty.$$

Proof. Putting $I_\mu := \{k: |\mu-\varrho_k| \leqq 1, |\nu_k| \leqq D\}$, we will show by induction on m that

$$\sup_{\mu>0} \sum_{\substack{k\in I_\mu \\ \sigma_k \leqq m}} \|u_k\|^2_{L^2(K)} < \infty \quad (m=-1, 0, \ldots). \tag{28}$$

Hence (27) will follow in view of (6) because by a result of Joó [7] there exists a constant $C_{m,D}$ such that

$$\|u_k\|_{L^\infty(K)} \leqq C_{m,D} \|u_k\|_{L^2(K)} \quad \text{whenewer} \quad o_k \leqq m \text{ and } |\nu_k| \leqq D. \tag{29}$$

(28) is true for $m=-1$. Let now $m \geqq 0$ and assume (28) is true for $m-1$, i.e.,

$$\sup_{\mu>0} \sum_{\substack{k\in I_\mu \\ \sigma_k < m}} \|u_k\|^2_{L^2(K)} < \infty. \tag{30}$$

In the following arguments the estimates O, o $(R\to 0)$ will be uniform in $\mu>0$, $k\in I_\mu$, and $x\in K$.

For any $\mu>0$ and $a \leqq x \leqq 2^{-1}(a+b)$ define $w: G\to \mathbf{R}$ by

$$w(y) := \begin{cases} 2\cos\mu(y-x)-0.5 & \text{if } x \leqq y \leqq x+R, \\ -0.5 & \text{if } x+R < y \leqq x+2R, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$\|w\|^2_{L^2(G)} = O(1) \quad (R\to 0). \tag{31}$$

Applying (1) for any $k\in I_\mu$, $o_k \leqq m$ with $0 \leqq t \leqq R \leqq 4^{-1}(b-a)$ and with $x+t$ instead of x, we obtain that

$$u_k(x) = 2u_k(x+t)\cos\mu t - u_k(x+2t) + 2u_k(x+t)[\cos\mu_k t - \cos\mu t] +$$
$$+ \sum_{i=1}^{\sigma_k} f_i(\mu_k, t) u_{k-i}(x+t) + \sum_{i=0}^{\sigma_k} g_i(u_{k-i}, \mu_k, x+t, t);$$

integrating by t from 0 to R, we see that

$$Ru_k(x) = \langle u_k, w\rangle + \int_0^R 2u_k(x+t)[\cos\mu_k t - \cos\mu t]\,dt +$$
$$+ \sum_{i=1}^{\sigma_k} \int_0^R f_i(\mu_k, t)\, u_{k-i}(x+t)\,dt + \sum_{i=0}^{\sigma_k} \int_0^R g_i(u_{k-i}, \mu_k, x+t, t)\,dt.$$

Taking into account that $\mu\in\mathbf{R}$ and $|\mu-\mu_k| \leqq D+1$, we get

$$|\cos\mu_k t - \cos\mu t| = O(t) \quad (t\to 0).$$

Furthermore using instead of (2) the estimates

$$(32)\qquad \begin{aligned} &|f_i(\mu_k, t)\,u_{k-i}(x)| = O(t^{2i})\,|u_{k-i}(x)| \quad (t\to 0),\\ &|g_i(u_{k-i}, \mu_k, x, t)| = O(t^{2i+1})\,\|u_{k-i}\|_{L^\infty(x-t,x+t)} \quad (t\to 0),\end{aligned}$$

which follow from $|v_k|\leqq D$ and from the definition of f_i, g_i, we obtain that

$$R\,|u_k(x)| \leqq |\langle u_k, w\rangle| + O(R^2)\,\|u_k\|_{L^\infty(K)} +$$

$$+\sum_{i=1}^{\sigma_k} O(R^{2i+1})\,\|u_{k-i}\|_{L^\infty(K)} + \sum_{i=1}^{\sigma_k} O(R^{2i+2})\,\|u_{k-i}\|_{L^\infty(K)} \leqq$$

$$\leqq |\langle u_k, w\rangle| + O(R^2)\,\|u_k\|_{L^\infty(K)} + O(1)\sum_{i=1}^{\sigma_k}\|u_{k-i}\|_{L^\infty(K)} \quad (R\to 0),$$

yielding

$$R^2\,|u_k(x)|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 + o(R^2)\,\|u_k\|^2_{L^\infty(K)} + O(1)\sum_{i=1}^{\sigma_k}\|u_{k-i}\|^2_{L^\infty(K)} \quad (R\to 0)$$

and in view of (29),

$$R^2\,|u_k(x)|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 + o(R^2)\,\|u_j\|^2_{L^2(K)} + O(1)\sum_{i=1}^{\sigma_k}\|u_{k-i}\|^2_{L^2(K)} \quad (R\to 0).$$

Summing up this inequality for any finite set $I\subset\{k\in I_\mu : o_k\leqq m\}$, and applying (8), (31) and (30), we have

$$\sum_{k\in I} R^2\,|u_k(x)|^2 \leqq O(1) + o(R^2)\sum_{k\in I}\|u_k\|^2_{L^2(K)}.$$

By a similar argument, this inequality is true for all $2^{-1}(a+b)\leqq x\leqq b$, too. Thus, integrating by x on K we get that

$$\sum_{k\in I} R^2\,\|u_k\|^2_{L^2(K)} \leqq O(1) + o(R^2)\sum_{k\in I}\|u_k\|^2_{L^2(K)},$$

$$\sum_{k\in I}\|u_k\|^2_{L^2(K)} = O(R^{-2})$$

and $I\subset\{k\in I_\mu : o_k\leqq m\}$ being arbitrary,

$$\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\|u_k\|^2_{L^2(K)} = O(R^{-2}) \quad (R\to 0).$$

Hence (28) follows and the lemma is proved.

Now we can prove the proposition formulated after Lemma 1. Given a point $y\in G$ arbitrarily, there exists by Lemma 3 a neighbourhood K_1 of y and two numbers $R_1, B>0$ with

$$(33)\qquad \sup_{\mu>0}\sum_{\substack{|\mu-\varrho_k|\leqq 1\\ \varrho_k\geqq B|v_k|\\ |v_k|\geqq B}} (\|u_k\|_{L^\infty(K_1)}\,\mathrm{ch}\, v_k R_1)^2 < \infty.$$

Fixing B, there exists by Lemma 4 another neighbourhood K_2 of y and two numbers $R_2, C>0$ such that

(34) $$\sum_{\substack{\varrho_k \leq B|\nu_k| \\ |\nu_k| \geq C}} (\|u_k\|_{L^\infty(K_2)} \operatorname{ch} \nu_k R_2)^2 < \infty.$$

Finally, for

(35) $$K := K_1 \cap K_2, \quad R := \min\{R_1, R_2\} \quad \text{and} \quad D := \max\{B, C\},$$

it follows from Lemma 5 that

(36) $$\sup_{\mu>0} \sum_{\substack{|\mu-\varrho_k|\leq 1 \\ |\nu_k| \leq D}} (\|u_k\|_{L^\infty(K)} \operatorname{ch} \nu_k R)^2 < \infty.$$

(33)—(36) imply

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k|\leq 1} (\|u_k\|_{L^\infty(K)} \operatorname{ch} \nu_k R)^2 < \infty,$$

i.e., each point $y \in G$ has a neighbourhood K_y such that for some $R_y > 0$,

(37) $$\sup_{\mu>0} \sum_{|\mu-\varrho_k|\leq 1} (\|u_k\|_{L^\infty(K_y)} \operatorname{ch} \nu_k R_y)^2 < \infty.$$

Hence the proposition follows by an elementary compactness argument. Indeed, given any compact interval $K \subset G$, it can be covered by a finite system $\{K_{y_i}: i=1,2,\dots,N\}$ of intervals having property (37) with some $R_i > 0$, $i=1,\dots,N$. Setting $R := \min\{R_1,\dots,R_N\}$, we have obviously

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k|\leq 1} (\|u_k\|_{L^2(K)} \operatorname{ch} \nu_k R)^2 < \infty,$$

completing the proof of Proposition 1.

2. An equiconvergence theorem

Let G be an arbitrary open interval on the real line, $q, \hat{q} \in L^1_{\text{loc}}(G)$ arbitrary complex functions. Let (u_k) (resp. $(\hat{u}_k)$) be a Riesz basis in $L^2(G)$ consisting of eigenfunctions of the operator $Lu = -u'' + qu$ (resp. $\hat{L}u = -u'' + \hat{q}u$) and having the following properties:

(38) $$\sup o_k < \infty, \ \sup \hat{o}_k < \infty,$$

(39) $$\text{in case } o_k > 0 \quad (\text{resp. } \hat{o}_k > 0)$$

$$\lambda_k u_k - L u_k = u_{k-1} \quad (\text{resp. } \hat{\lambda}_k \hat{u}_k - \hat{L}\hat{u}_k = \hat{u}_{k-1}).$$

where λ_k and o_k (resp. $\hat{\lambda}_k$ and $\hat{o}_k$) are the eigenvalue and the order of u_k (resp. $\hat{u}_k$).

Now let us introduce some notations:

$$\sigma_\mu(f,x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, v_k\rangle u_k(x),$$

(40)

$$\hat{\sigma}_\mu(f,x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, \hat{v}_k\rangle \hat{u}_k(x) \quad (f\in L^2(G),\ x\in G,\ \mu>0)$$

where (v_k) (resp. $(\hat{v}_k)$) is the dual system of (u_k) (resp. $(\hat{u}_k)$), i.e., (v_k), $(\hat{v}_k)\subset L^2(G)$ and $\langle v_k, u_j\rangle = \langle \hat{v}_k, \hat{u}_j\rangle = \delta_{kj}$. The following result holds:

Theorem. *Given any compact interval* $K\subset G$, *for all* $f\in L^2(G)$

$$\lim_{\mu\to\infty}\sup_{x\in K}|\sigma_\mu(f,x)-\hat{\sigma}_\mu(f,x)| = 0. \tag{41}$$

For $f\in L^2(G)$, $\mu>0$ and $x\pm R\in G$, define

$$S_\mu(f,x) = S_\mu(f,x,R) := \int_{x-R}^{x+R} \frac{\sin\mu(y-x)}{\pi(y-x)} f(y)\,dy. \tag{42}$$

The theorem will follow obviously from the following assertion:

Proposition 2. *Given any compact interval* $K\subset G$, *for any sufficiently small* $R>0$, *and for all* $f\in L^2(G)$, *we have*

$$\lim_{\mu\to\infty}\sup_{x\in K}|S_\mu(f,x)-\sigma_\mu(f,x)| = 0. \tag{43}$$

Indeed, an analogous result holds for $\hat{\sigma}_\mu(f,x)$, too, and it remains only to apply the triangle inequality.

For the sake of brevity, from now on we shall denote by μ_k a square root of λ_k with $\mathrm{Re}\,\mu_k\geqq 0$ and we set $\varrho_k := \mathrm{Re}\,\mu_k$, $\nu_k := \mathrm{Im}\,\mu_k$. For the proof of Proposition 2 we shall need two preliminary lemmas.

Lemma 6. *Given any* $R>0$, *there exists a constant* $C=C(R)$ *such that with the notation*

$$\delta(\mu,\varrho_k) := \begin{cases} 1 & \text{if } \mu>\varrho_k, \\ 1/2 & \text{if } \mu=\varrho_k, \\ 0 & \text{if } \mu<\varrho_k, \end{cases} \tag{44}$$

for any $\mu>0$ *and* k, *we have*

$$\left|\frac{2}{\pi}\int_0^R \frac{\sin\mu t\cos\mu_k t}{t}\,dt - \delta(\mu,\varrho_k)\right| \leqq C(R)\frac{\mathrm{ch}\,\nu_k R}{2+|\mu-\varrho_j|}. \tag{45}$$

Proof. We recall that

$$\int_0^\infty \frac{\sin t}{t}\,dt = 2^{-1}\pi \tag{46}$$

and

$$\left|\int_x^\infty \frac{\sin t}{t}\,dt\right| < 6(1+x)^{-1} \quad \text{for all} \quad x \geqq 0 \tag{47}$$

((47) can be seen by integrating by parts). Setting for brevity $\varrho_k=\varrho$ and $\nu_k=\nu$, we can write

$$(2/\pi)\int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\,dt =$$

$$= (2/\pi)\int_0^R \frac{\sin \mu t}{t}(\cos \varrho t \operatorname{ch} \nu t - i \sin \varrho t \operatorname{sh} \nu t)\,dt =$$

$$= (2/\pi)\int_0^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt - (2/\pi)\int_R^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt + \tag{48}$$

$$+(2/\pi)\int_0^R \sin \mu t \cos \varrho t \frac{\operatorname{ch}(\nu t)-1}{t}\,dt - (2i/\pi)\int_0^R \sin \mu t \sin \varrho t \frac{\operatorname{sh}(\nu t)}{t}\,dt \equiv$$

$$\equiv I_1+I_2+I_3+I_4.$$

Here, by (46),

$$I_1 = \pi^{-1}\int_0^\infty \frac{\sin(\mu+\varrho)t+\sin(\mu-\varrho)t}{t}\,dt =$$

$$= \pi^{-1}(\operatorname{sgn}(\mu+\varrho)+\operatorname{sgn}(\mu-\varrho))\int_0^\infty \frac{\sin t}{t}\,dt = \delta(\mu, |\varrho|) \tag{49}$$

and

$$I_2 = -\pi^{-1}\int_R^\infty \frac{\sin(\mu+\varrho)t+\sin(\mu-\varrho)t}{t}\,dt =$$

$$= -\pi^{-1}\left((\operatorname{sgn}\mu+\varrho)\int_{|\mu+\varrho|R}^\infty \frac{\sin t}{t}\,dt + \operatorname{sgn}(\mu-\varrho)\int_{|\mu-\varrho|R}^\infty \frac{\sin t}{t}\,dt\right),$$

whence, in view of (47),

$$|I_2| \leqq \pi^{-1}\left(\frac{6}{1+|\mu+\varrho|R}+\frac{6}{1+|\mu-\varrho|R}\right) \leqq \frac{4}{1+|\mu-\varrho|R}. \tag{50}$$

Considering now the quantities I_3, I_4, we obviously have

$$|I_3| \leqq (2/\pi)\int_0^R \frac{\operatorname{ch}(\nu t)-1}{t}\,dt \leqq (2/\pi)(\operatorname{ch}\nu R - 1) \tag{51}$$

and

$$|I_4| \leqq (2/\pi)\int_0^R \frac{\operatorname{sh}(|\nu|t)}{t}\,dt \leqq (2/\pi)\operatorname{sh}\nu R \tag{52}$$

On the other hand, in case $\mu\neq\varrho$ we can write

$$I_3=\pi^{-1}\int_0^R(\sin(\mu+\varrho)t+\sin(\mu-\varrho)t)\frac{\operatorname{ch}\nu t-1}{t}\,dt=$$

$$=\pi^{-1}\left[\left(\frac{-\cos(\mu+\varrho)t}{\mu+\varrho}+\frac{-\cos(\mu-\varrho)t}{\mu-\varrho}\right)\frac{\operatorname{ch}\nu t-1}{t}\right]_0^R+$$

$$+\pi^{-1}\int_0^R\left(\frac{\cos(\mu+\varrho)t}{\mu+\varrho}+\frac{\cos(\mu-\varrho)t}{\mu-\varrho}\right)\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'dt$$

and

$$I_4=(i/\pi)\int_0^R(\cos(\mu+\varrho)t-\cos(\mu-\varrho)t)\frac{\operatorname{sh}\nu t}{t}\,dt=$$

$$=(i/\pi)\left[\left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho}-\frac{\sin(\mu-\varrho)t}{\mu-\varrho}\right)\frac{\operatorname{sh}\nu t}{t}\right]_0^R-$$

$$-(i/\pi)\int_0^R\left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho}-\frac{\sin(\mu-\varrho)t}{\mu-\varrho}\right)\left(\frac{\operatorname{sh}\nu t}{t}\right)'dt,$$

Hence, taking into account that the functions $\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'$, $\left(\frac{\operatorname{sh}\nu t}{t}\right)'$ do not change sign, we obtain that

$$|I_3|\leqq\frac{2}{\pi|\mu-\varrho|}\left(\frac{\operatorname{ch}\nu R-1}{R}+\left|\int_0^R\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'dt\right|\right)=\frac{4\operatorname{ch}\nu R}{\pi R|\mu-\varrho|}. \tag{53}$$

and

$$|I_4|\leqq\frac{2}{\pi|\mu-\varrho|}\left(\left|\frac{\operatorname{sh}\nu R}{R}-\nu\right|+\left|\int_0^R\left(\frac{\operatorname{sh}\nu t}{t}\right)'dt\right|\right)\leqq\frac{4\,|\operatorname{sh}\nu R|}{\pi R\,|\mu-\varrho|}. \tag{54}$$

Now the lemma follows from the relations (48)—(54).

Lemma 7. *Given any $R>0$, there exist constants $C_i=C_i(R)$ such that for the functions f_i, g_i of Lemma* 1,

$$\int_0^R\left|\frac{f_i(\mu,t)}{t}\right|dt\leqq C_i\operatorname{ch}(R\operatorname{Im}\mu)(1+|\mu|)^{-i}\quad(\mu\in\mathbf{C},\ i=1,2,\ldots); \tag{55}$$

furthermore, for any $u\in L^\infty_{\mathrm{loc}}(G)$, $\mu\in\mathbf{C}$ and $x\pm R\in G$,

$$\int_0^R\left|\frac{g_i(u,\mu,x,t)}{t}\right|dt\leqq C_i\|q\|_{L^1(x-R,x+R)}\|u\|_{L^\infty(x-R,x+R)}\times$$
$$\times\operatorname{ch}(R\operatorname{Im}\mu)(1+|\mu|)^{-i-1}\quad(i=1,2,\ldots), \tag{56}$$

$$\int_0^R\left|\frac{g_0(u,\mu,x,t)}{t}\right|dt\leqq C_0\|q\|_{L^1(x-R,x+R)}\|u\|_{L^\infty(x-R,x+R)}\times$$
$$\times\operatorname{ch}(R\operatorname{Im}\mu)\,(1+\ln(1+|\mu|))(1+|\mu|)^{-1}. \tag{57}$$

Proof. Using the inequalities

$$\left|\frac{\sin z}{z}\right| \leqq 2\,\mathrm{ch}\,(\mathrm{Im}\, z), \quad |\sin z| \leqq \mathrm{ch}\,(\mathrm{Im}\, z) \quad (z\in \mathbf{C}),$$

$$\mathrm{ch}\,\alpha\,\mathrm{ch}\,\beta \leqq \mathrm{ch}\,(\alpha+\beta), \quad \min\{\alpha, \beta\} \leqq \frac{1+\alpha}{1+1/\beta} \quad (\alpha, \beta > 0),$$

and the notations

$$\nu := \mathrm{Im}\,\mu, \quad M := \|q\|_{L^1(x-R,\, x+R)} \|u\|_{L^\infty(x-R,\, x+R)}$$

we see from the definition of f_i, g_i that

$$|f_i(\mu, t)| \leqq t^i\,\mathrm{ch}\,\nu t \min\{2t, |1/\mu|\}^i \leqq t^i(1+2t)^i\,\mathrm{ch}\,\nu t(1+|\mu|)^{-i},$$

$$|g_i(u, \mu, x, t)| \leqq t^i\,\mathrm{ch}\,\nu t \min\{2t, |1/\mu|\}^{i+1} M \leqq$$

$$\leqq t^i(1+2t)^{i+1}\,\mathrm{ch}\,\nu t\,(1+|\mu|)^{-i-1} M.$$

Hence (55), (56) and the case $R \leqq |1/\mu|$ of (57) follows at once:

$$\int_0^R \left|\frac{f_i(\mu, t)}{t}\right| dt \leqq RR^{i-1}(1+2R)^i\,\mathrm{ch}\,\nu R\,(1+|\mu|^{-i},$$

$$\int_0^R \left|\frac{g_i(u, \mu, x, t)}{t}\right| dt \leqq RR^{i-1}(1+2R)^{i+1}\,\mathrm{ch}\,\nu R(1+|\mu|)^{-i-1} M \quad (i = 1, 2, \ldots),$$

$$\int_0^R \left|\frac{g_0(u, \mu, x, t)}{t}\right| dt \leqq \int_0^R t^{-1}\,\mathrm{ch}\,\nu t\, 2t\, M\, dt \leqq$$

$$\leqq 2RM\,\mathrm{ch}\,\nu R \leqq 2M(R+1)\,\mathrm{ch}\,\nu R\,(1+|\mu|)^{-1}.$$

In view of this last estimate, for the case $R > |1/\mu|$ it remains to remark that

$$\int_{|1/\mu|}^R \left|\frac{g_0(u, \mu, x, t)}{t}\right| dt \leqq \int_{|1/\mu|}^R t^{-1}(1+2t)\,\mathrm{ch}\,\nu t\,(1+|\mu|)^{-1} M\, dt \leqq$$

$$\leqq (\ln R - \ln |1/\mu|)(1+2R)\,\mathrm{ch}\,\nu R\,(1+|\mu|)^{-1} M =$$

$$= (1+2R)\, M\,\mathrm{ch}\,\nu R\,(\ln R + \ln |\mu|)(1+|\mu|)^{-1},$$

and the lemma is proved.

Let us now turn to the proof of Proposition 2. Since $(u_k) \subset L^2(G)$ is a Riesz basis and $(v_k) \subset L^2(G)$ is the dual system of (u_k), there exists a constant C_0 such that for all $f, w \in L^2(G)$,

(58) $$\sum_k |\langle u_k, w\rangle|^2 \leqq C_0 \|w\|^2_{L^2(G)},$$

(59) $$\sum_k |\langle f, v_k\rangle|^2 \leqq C_0 \|f\|^2_{L^2(G)},$$

(60) $$\langle f, w\rangle = \sum_k \langle f, v_k\rangle \langle u_k, w\rangle.$$

Given a compact interval $K=[a, b]\subset G$ arbitrarily, we can fix by Proposition 1 an $R>0$ such that $K_R:=[a-R, b+R]\subset G$ and

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k|\leqq 1} (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} v_k R)^2 < A < \infty. \tag{61}$$

Applying Lemmas 6, 7 and using $\|q\|_{L^1(K_R)}<\infty$ and (38), we can fix a constant $C=C(R)$ such that

$$\left|\frac{2}{\pi}\int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\,dt - \delta(\mu, \varrho_k)\right| \leqq C \operatorname{ch} v_k R(2+|\mu-\varrho_k|)^{-1}, \tag{62}$$

$$\left|\int_0^R \frac{\sin \mu t}{\pi t} f_i(\mu_k, t)\,dt\right| \leqq C \operatorname{ch} v_k R(1+|\mu_k|)^{-1}, \tag{63}$$

$$\left|\int_0^R \frac{\sin \mu t}{\pi t} g_i(u_{k-i}, \mu_k, x, t)\,dt\right| \leqq C \|u_{k-i}\|_{L^\infty(K_R)} \operatorname{ch} v_k R(1+|\mu_k|)^{-3/4} \tag{64}$$

for all $k, \mu>0$ and $i=1, 2, \ldots, o_k$ in (63), $i=0, 1, \ldots, o_k$ in (64).

Fixing $x\in K$ and $\mu>0$ arbitrarily, define $w\colon G\to\mathbf{R}$ by

$$w(x+t) := \begin{cases} \dfrac{\sin \mu t}{\pi t} & \text{if} \quad |t|\leqq R, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any $f\in L^2(G)$, (42) and (60) imply that

$$S_\mu(f, x) = \sum_k \langle f, v_k\rangle\langle u_k, w\rangle. \tag{65}$$

Applying Lemma 1 with $m:=\sup o_k$ $(<\infty)$ and using (40), (62)—(65), we obtain the inequality

$$\begin{aligned} |S_\mu(f, x)-\sigma_\mu(f, x)| \leqq 2^{-1} &\sum_{\varrho_k=\mu} |\langle f, v_k\rangle u_k(x)| + \\ &+\sum_k |\langle f, v_k\rangle u_k(x)|\, C \operatorname{ch} v_k R\,(2+|\mu-\varrho_k|)^{-1} + \\ &+\sum_k \sum_{i=1}^{\sigma_k} |\langle f, v_k\rangle u_{k-i}(x)|\, C \operatorname{ch} v_k R\,(1+|\mu_k|)^{-1} + \\ &+\sum_k \sum_{i=0}^{\sigma_k} |\langle f, v_k\rangle|\, \|u_{k-i}\|_{L^\infty(K_R)}\, C \operatorname{ch} v_k R(1+|\mu_k|)^{-3/4}. \end{aligned}$$

Using for each sum the Cauchy—Schwarz inequality, (50) and $o_k \leqq m$, we get that

$$|S_\mu(f, x)-\sigma_\mu(f, x)| \leqq \sqrt{C_0}\,\|f\|_{L^2(G)}\{2^{-1}(\sum_{\varrho_k=\mu}|u_k(x)|^2)^{1/2}+$$
$$+C(\sum_k(u_k(x)\operatorname{ch}\nu_k R)^2(2+|\mu-\varrho_k|)^{-2})^{1/2}+$$
$$+mC(\sum_k(u_k(x)\operatorname{ch}\nu_k R)^2(1+|\mu_k|)^{-2})^{1/2}+$$
$$+mC(\sum_k\|u_k\|_{L^\infty(K_R)}\operatorname{ch}\nu_k R)^2(1+|\mu_k|)^{-3/2})^{1/2}\}.$$

Applying to these expressions the estimate (52), we have

$$\sum_{\varrho_k=\mu}|u_k(x)|^2<A,$$
$$\sum_k(u_k(x)\operatorname{ch}\nu_k R)^2(2+|\mu-\varrho_k|)^{-2}\leqq$$
$$\leqq\sum_{i=0}^{\infty}(1+|\mu-i|)^{-2}\sum_{i\leqq\varrho_k<i+1}(u_k(x)\operatorname{ch}\nu_k R)^2\leqq 2A\sum_{j=1}^{\infty}j^{-2},$$
$$\sum_k(u_k(x)\operatorname{ch}\nu_k R)^2(1+|\mu_k|)^{-2}\leqq$$
$$\leqq\sum_{i=0}^{\infty}(1+i)^{-2}\sum_{i\leqq\varrho_k<i+1}(u_k(x)\operatorname{ch}\nu_k R)^2\leqq A\sum_{j=1}^{\infty}j^{-2},$$

and similarly

$$\sum_k(\|u_k\|_{L^\infty(K_R)}\operatorname{ch}\nu_k R)^2(1+|\mu_k|)^{-3/2}\leqq A\sum_{j=1}^{\infty}j^{-3/2}.$$

Therefore there exists a constant $D>0$ such that

(66) $$\sup_{\mu>0}\sup_{x\in K}|S_\mu(f, x)-\sigma_\mu(f, x)|\leqq D\|f\|_{L^2(G)}\quad\text{for all }f\in L^2(G).$$

Given now $f\in L^2(G)$ and $\varepsilon>0$ arbitrarily, let us choose a finite linear combination $P:=\sum_{k=1}^{n}c_k u_k$ with

(67) $$\|f-P\|_{L^2(G)}<\varepsilon/2D.$$

P being continuously differentiable, it is well known [3] that

$$\lim_{\mu\to\infty}\sup_{x\in K}|S_\mu(P, x)-P(x)|=0.$$

Thus we can fix $N>0$ so that

(68) $$\sup_{x\in K}|S_\mu(P, x)-P(x)|<\varepsilon/2\quad\text{whenever}\quad\mu>N.$$

Let now $\mu > \max\{N, \varrho_1, \ldots, \varrho_n\}$ be arbitrary; then $\sigma_\mu(P, x) \equiv P(x)$ and therefore (66)—(68) imply that

$$\sup_{x \in K} |S_\mu(f, x) - \sigma_\mu(f, x)| \leqq \sup_{x \in K} |S_\mu(f-P, x) - \sigma_\mu(f-P, x)| +$$

$$+ \sup_{x \in K} |S_\mu(P, x) - P(x)| \leqq D \frac{\varepsilon}{2D} + \varepsilon/2 = \varepsilon;$$

this finishes the proof of Proposition 2 and also that of the Theorem.

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DEPARTMENT OF MATH. ANALYSIS II
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY