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A characterization of quasi-varieties in equality-free languages

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1. The result

1.1. By a type t, we mean an ordered quintuple $t=\langle\mathcal{R}, \mathcal{F}, \mathcal{C}, t_{\mathcal{R}}, t_{\mathcal{F}}\rangle$ where $\mathcal{R}, \mathcal{F}, \mathcal{C}$ are pairwise disjoint sets, $\mathcal{C}$ is not empty, $t_{\mathcal{R}}: \mathcal{R}\to\omega$, $t_{\mathcal{F}}: \mathcal{F}\to\omega$. By a structure of type t, an ordered quadruplet $\langle A, \langle R_r\rangle_{r\in\mathcal{R}}, \langle F_f\rangle_{f\in\mathcal{F}}, \langle C_c\rangle_{c\in\mathcal{C}}\rangle$ is meant where A is a nonvoid set and $R_r\in\mathbf{P}({}^{(t_{\mathcal{R}}(r))}A)$, $F_f: {}^{(t_{\mathcal{F}}(f))}A\to A$, $C_c\in A$, for every $r\in\mathcal{R}$, $f\in\mathcal{F}$, $c\in\mathcal{C}$. (For any set B, $\mathbf{P}(B)$ stands for the power set of B; and if $n\in\omega$, then ${}^{n}B$ denotes the n-th Cartesian power of B.) We shall use German capitals for denoting structures. If $\mathfrak{A}$ is a structure of type t, then the universe A, the relations R_r, the functions F_f and the constants C_c of $\mathfrak{A}$ will also be denoted by $|\mathfrak{A}|$, $R_r^{(\mathfrak{A})}$, $F_f^{(\mathfrak{A})}$, $C_c^{(\mathfrak{A})}$, respectively.

1.2. For $i\in\{0, 1\}$, let $\mathfrak{A}_i=\langle A^i, \langle R_r^i\rangle_{r\in\mathcal{R}}, \langle F_f^i\rangle_{f\in\mathcal{F}}, \langle C_c^i\rangle_{c\in\mathcal{C}}\rangle$ be two structures for a fixed type $t=\langle\mathcal{R}, \mathcal{F}, \mathcal{C}, t_{\mathcal{R}}, t_{\mathcal{F}}\rangle$. We define

$$\mathfrak{A}_0\cap\mathfrak{A}_1=\langle A^0\cap A^1, \langle R_r^0\cap R_r^1\rangle_{r\in\mathcal{R}}, \langle F_f^0\cap F_f^1\rangle_{f\in\mathcal{F}}, \langle C_c^0\rangle_{c\in\mathcal{C}}\cap\langle C_c^1\rangle_{c\in\mathcal{C}}\rangle.$$

The *meet* of $\mathfrak{A}_0$ and $\mathfrak{A}_1$, in notation, $\mathfrak{A}_0\sqcap\mathfrak{A}_1$ is then defined as being identical to $\mathfrak{A}_0\cap\mathfrak{A}_1$ iff $\mathfrak{A}_0\cap\mathfrak{A}_1$ is itself a structure of type t.

Obviously, the meet of structures is a partial operation.

A class $\mathbf{K}$ of structures with the same type t is *closed under finite meets* iff for every $\mathfrak{A}_0, \mathfrak{A}_1\in\mathbf{K}$, if $\mathfrak{A}_0\sqcap\mathfrak{A}_1$ exists, then it is in $\mathbf{K}$.

1.3. Our result is the following preservation theorem: *If Σ is a set of equality-free sentences and $\mathbf{K}$ is the class of all models of Σ, then $\mathbf{K}$ is closed under finite meets iff Σ is equivalent to a set of universal equality-free Horn sentences* (Theorem 2.14).

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We note, that as a simple example will show, this theorem fails to hold for Σ with equality.

From this result we shall derive a characterization of quasi-varieties in equality-free languages. To be more specific, we shall prove, that *if* **K** *is an arbitrary class of structures for an equality-free language, then* **K** *is a quasi-variety* (i.e. **K** is axiomatizable by universal equality-free Horn sentences) *iff* **K** *is closed under finite meets, ultraproducts and equality-free elementary equivalence.*

1.4. Model theorists are generally uninterested in equality-free languages, since their expressive power is restricted in comparison with languages containing equality, and, on the other hand, several results and methods developed in the model theory for general first order languages apply directly to the equality-free case.

Recently, however, equality-free languages play a role in theoretical, as well as in practical computer science. For example, computability can be formalized in equality-free languages as was noticed by R. Hill and proved in general by H. ANDRÉKA and I. NÉMETI [1]. R. KOWALSKI [7] used this observation to show how equality-free languages can be considered as programming languages. On this basis, a practical programming language PROLOG was implemented by A. COLMERAUER et al [3]. A special case of our Lemma 2.13 was used by R. HILL [6] to prove a completeness theorem for a particularly efficient deduction system. Also, M. H. VAN EMDEN and R. KOWALSKI [4] investigated logically based programming languages by means of a special instance (for Herbrandian models) of the "easy direction" of the preservation theorem above, and questioned whether the converse was true. Our result shows that, generally, the answer is in the negative if equality is present in the language, but is affirmative if the equality is excluded. These applications make the belief plausible, that studying equality-free languages has some theoretical and practical value. This paper takes a step in this direction. Although motivations come from computer science, the preservation theorem mentioned above and its proof are purely model theoretic in character.

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2. The proof

2.1. From now on, we shall fix an arbitrary type t. When we are speaking of an arbitrary structure without a closer indication of its type, then we shall always mean that it is a structure of the fixed type t.

2.2. Let $\mathfrak{A}=\langle|\mathfrak{A}|, \langle R_r^{(\mathfrak{A})}\rangle_{r\in\mathcal{R}}, \langle F_f^{(\mathfrak{A})}\rangle_{f\in\mathcal{F}}, \langle C_c^{(\mathfrak{A})}\rangle_{c\in\mathcal{C}}\rangle$ be an arbitrary structure. Let us suppose, that Δ is a congruence relation on $\mathfrak{A}$ (in the sense of [5, Definition

0.2.20, p. 73.]). We say, that Δ is a *universal congruence* relation on $\mathfrak{A}$, if

(1) $(\forall r\in\mathscr{R})(\forall a, b\in{}^{(t_{\mathscr{R}}(r))}|\mathfrak{A}|)\left[(\forall i < t_{\mathscr{R}}(r))\left(\langle a_i, b_i\rangle\in\Delta \to (a\in R_r^{(\mathfrak{A})} \leftrightarrow b\in R_r^{(\mathfrak{A})})\right)\right]$

where a_i, b_i denote the i-th component of a, b, respectively, for all $i<t_{\mathscr{R}}(r)$.

If Δ is a universal congruence on $\mathfrak{A}$, then the *quotient structure* $\mathfrak{A}/\Delta = \langle|\mathfrak{A}/\Delta|, \langle R_r^{(\mathfrak{A}/\Delta)}\rangle_{r\in\mathscr{R}}, \langle F_f^{(\mathfrak{A}/\Delta)}\rangle_{f\in\mathscr{F}}, \langle C_c^{(\mathfrak{A}/\Delta)}\rangle_{c\in\mathscr{C}}\rangle$ of $\mathfrak{A}$ *over* Δ can be defined in the traditional manner: for all $a\in|\mathfrak{A}|$, let $a/\Delta=\{b\mid b\in|\mathfrak{A}|\wedge\langle a, b\rangle\in\Delta\}$ and set

(2) $$|\mathfrak{A}/\Delta| = \{a/\Delta \mid a\in|\mathfrak{A}|\};$$

for all $f\in\mathscr{F}$, such that $t_{\mathscr{F}}(f)=n+1$ and for arbitrary $a_0, \dots, a_n\in|\mathfrak{A}|$, we define

(3) $$F_f^{(\mathfrak{A}/\Delta)}(a_0/\Delta, \dots, a_n/\Delta) = \left(F_f^{(\mathfrak{A})}(a_0, \dots, a_n)\right)/\Delta;$$

for all $c\in\mathscr{C}$, let

(4) $$C_c^{(\mathfrak{A}/\Delta)} = C_c^{(\mathfrak{A})}/\Delta$$

and finally, for all $r\in\mathscr{R}$ such that $t_{\mathscr{R}}(r)=n+1$ and for every $a_0, \dots, a_n\in|\mathfrak{A}|$, we define

(5) $$\langle a_0/\Delta, \dots, a_n/\Delta\rangle\in R_r^{(\mathfrak{A}/\Delta)} \quad\text{iff}\quad \langle a_0, \dots, a_n\rangle\in R_r^{(\mathfrak{A})}.$$

Δ being an universal congruence on $\mathfrak{A}$, the definition of $\mathfrak{A}/\Delta$ is correct and obviously, it is a structure of type t.

Proposition 2.3. *Let* $\mathfrak{A}$ *be an arbitrary structure and suppose, that* Δ *is a universal congruence on* $\mathfrak{A}$. *Then* $\mathfrak{A}\equiv\mathfrak{A}/\Delta$, *where elementary equivalence is meant in the equality-free sense.*

Proof. Let V be the set of variables and $k\colon V\to|\mathfrak{A}|$ be an arbitrary assignment relative to $\mathfrak{A}$. We define $\tilde{k}\colon V\to|\mathfrak{A}/\Delta|$ for all $v\in V$, by

(6) $$\tilde{k}(v) = k(v)/\Delta.$$

We shall prove by a straight-forward induction, that for arbitrary (equality-free) formula φ and $k\colon V\to|\mathfrak{A}|$,

(7) $$\mathfrak{A}\models\varphi[k] \quad\text{iff}\quad \mathfrak{A}/\Delta\models\varphi[\tilde{k}].$$

Since Δ is a congruence on $\mathfrak{A}$, it is easily seen, that for any term τ and $k\colon V\to|\mathfrak{A}|$,

(8) $$\tau^{(\mathfrak{A})}[k]/\Delta = \tau^{(\mathfrak{A}/\Delta)}[\tilde{k}].$$

(The notations used here are standard and can be found e.g. in [2, 1.3.1, 1.3.2, 1.3.3, pp. 22—23 and 1.3.13, 1.3.14, 1.3.15, pp. 27—28].)

Indeed, if $\tau\in V$ or $\tau\in\mathscr{C}$, then (6) or (4) is the same as (8). Let τ be of the form $f(\tau_0, \dots, \tau_n)$ for some $f\in\mathscr{F}$, such that $t_{\mathscr{F}}(f)=n+1$ where τ_i is a term for which

(8) holds, for all $i<t_{\mathscr{F}}(f)$. Then,

$$\tau^{(\mathfrak{A})}[k]/\Delta = (F_f^{(\mathfrak{A})}(\tau_0^{(\mathfrak{A})}[k], \ldots, \tau_n^{(\mathfrak{A})}[k]))/\Delta \overset{(3)}{=} F_f^{(\mathfrak{A}/\Delta)}(\tau_0^{(\mathfrak{A})}[k]/\Delta, \ldots, \tau_n^{(\mathfrak{A})}[k]/\Delta) \overset{\text{(i.h.)}}{=}$$
$$\overset{\text{(i.h.)}}{=} F_f^{(\mathfrak{A}/\Delta)}(\tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}], \ldots, \tau_n^{(\mathfrak{A}/\Delta)}[\tilde{k}]) = \tau^{(\mathfrak{A}/\Delta)}[\tilde{k}].$$

(Here, and everywhere below (i.h.) stands for "by the induction hypothesis".)

Turning to the proof of (7), we proceed similarly. Let φ be a prime formula of the form $r(\tau_0, \ldots, \tau_n)$, where $r\in\mathscr{R}, t_{\mathscr{R}}(r)=n+1$ and for all $i<t_{\mathscr{R}}(r)$, τ_i is a term. Then

$$\mathfrak{A} \models \varphi[k] \Leftrightarrow \mathfrak{A} \models r(\tau_0, \ldots, \tau_n)[k] \Leftrightarrow \langle \tau_0^{(\mathfrak{A})}[k], \ldots, \tau_n^{(\mathfrak{A})}[k]\rangle \in R_r^{(\mathfrak{A})} \overset{(5)}{\Leftrightarrow}$$
$$\overset{(5)}{\Leftrightarrow} \langle \tau_0^{(\mathfrak{A})}[k]/\Delta, \ldots, \tau_n^{(\mathfrak{A})}[k]/\Delta\rangle \in R_r^{(\mathfrak{A}/\Delta)} \overset{(8)}{\Leftrightarrow} \langle \tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}], \ldots, \tau_0^{(\mathfrak{A}/\Delta)}[\tilde{k}]\rangle \in R_r^{(\mathfrak{A}/\Delta)} \Leftrightarrow$$
$$\Leftrightarrow \mathfrak{A}/\Delta \models r(\tau_0, \ldots, \tau_n)[\tilde{k}] \Leftrightarrow \mathfrak{A}/\Delta \models \varphi[\tilde{k}].$$

If φ is of the form $\neg\psi$ and (7) is true for ψ, then

$$\mathfrak{A} \models \varphi[k] \Leftrightarrow \mathfrak{A} \models \neg\psi[k] \Leftrightarrow \mathfrak{A} \not\models \psi[k] \overset{\text{(i.h.)}}{\Leftrightarrow} \mathfrak{A}/\Delta \not\models \psi[\tilde{k}] \Leftrightarrow$$
$$\Leftrightarrow \mathfrak{A}/\Delta \models \neg\psi[\tilde{k}] \Leftrightarrow \mathfrak{A}/\Delta \models \varphi[\tilde{k}].$$

Obviously, the induction goes through for $\varphi=\psi_1\wedge\psi_2$.

Finally, let us suppose, that φ is of the form $\exists v\psi$, where $v\in V$ and (7) is true for ψ. Then,

(9) $\mathfrak{A} \models \varphi[k] \Leftrightarrow \mathfrak{A} \models \exists v\psi[k] \Leftrightarrow$ (there exists an assignment $k': V\to|\mathfrak{A}|$ such that for all $w\in V, k'(w) = k(w)$, provided $v \neq w$ and $\mathfrak{A} \models \psi[k']$).

By the induction hypothesis, $\mathfrak{A}\models\psi[k'']$ iff $\mathfrak{A}/\Delta\models\psi[\tilde{k}'']$ for arbitrary $k'': V\to|\mathfrak{A}|$; moreover, for k and k' in (9), we have $k(w)=k'(w)$, for all $w\in V$ such that $v\neq w$. Thus (9) is equivalent to the assertion

(there exists an assignment $\tilde{k}': V\to|\mathfrak{A}/\Delta|$ such that for all $w\in V$ if $v\neq w$ then $k(w)=k'(w)$ and $\mathfrak{A}/\Delta\models\psi[\tilde{k}']$),

which, in turn, is equivalent to $\mathfrak{A}/\Delta\models\exists v\psi[\tilde{k}]$.

This lemma has been proposed to me by H. Andréka and I. Németi to replace my original stronger but much less true assertion.

It is easy to construct a simple counterexample, using the obvious fact that an equation can hold in the quotient structure even if it is false in the initial one, which shows that Proposition 2.3 does not generalize for languages with equality.

2.4. Let X be an arbitrary set and consider the absolutely free algebra $\mathfrak{Fr}_{X\cup\mathscr{C}}$ of type t generated by the set $X\cup\mathscr{C}$ (cf. [5, Definition 0.4.19 (i), Remarks 0.4.20, pp. 130—131]).

Let $\mathfrak{A}$ be any structure of type t. It is well-known that, for arbitrary $h: X\cup\mathscr{C}\to|\mathfrak{A}|$ such that for all $c\in\mathscr{C}$, $h(c)=C_c^{(\mathfrak{A})}$ holds, there exists a unique homomorphism $\bar{h}$ from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$ for which $h\subseteq\bar{h}$ (cf. Definition 0.4.23., Theorem 0.4.24, Theorem 0.4.27 (i), pp. 131—132, in [5]).

We define the *free structure* $\mathfrak{Fr}_h\mathfrak{A}$ induced by h and $\mathfrak{A}$ as follows. Let

(10) (i) $$|\mathfrak{Fr}_h\mathfrak{A}| = |\mathfrak{Fr}_{X\cup\mathscr{C}}|.$$

(ii) For every $r\in\mathscr{R}$, such that $t_{\mathscr{R}}(r)=n+1$ and for arbitrary $a_0, \ldots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

(11) $$\langle a_0, \ldots, a_n\rangle\in R_r^{(\mathfrak{Fr}_h\mathfrak{A})} \quad \text{iff} \quad \langle\bar{h}(a_0), \ldots, \bar{h}(a_n)\rangle\in R_r^{(\mathfrak{A})},$$

where $\bar{h}$ is the unique extension of h to a homomorphism from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$.

(iii) For every $f\in\mathscr{F}$ such that $t_{\mathscr{F}}(f)=n+1$ and for arbitrary $a_0, \ldots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

(12) $$F_f^{(\mathfrak{Fr}_h\mathfrak{A})}(a_0, \ldots, a_n) = F_f^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}(a_0, \ldots, a_n).$$

(iv) Finally, for all $c\in\mathscr{C}$, let

(13) $$C_c^{(\mathfrak{Fr}_h\mathfrak{A})} = C_c^{(\mathfrak{Fr}_{X\cup\mathscr{C}})}.$$

It is obvious, that the homomorphism $\bar{h}$ from $\mathfrak{Fr}_{X\cup\mathscr{C}}$ into $\mathfrak{A}$ is a homomorphism as well from $\mathfrak{Fr}_h\mathfrak{A}$ into $\mathfrak{A}$. Moreover, the relation

(14) $$\Delta_{\bar{h}} = \{\langle a, b\rangle | a, b\in|\mathfrak{Fr}_h\mathfrak{A}|\wedge\bar{h}(a) = \bar{h}(b)\}$$

is a universal congruence on $\mathfrak{Fr}_h\mathfrak{A}$. Thus, $\mathfrak{Fr}_h\mathfrak{A}$ is correctly defined and is a structure of type t. Additionally, $\mathfrak{Fr}_h\mathfrak{A}$ has the following useful property.

Lemma 2.5. *Let $\mathfrak{A}$ be an arbitrary structure, X be a set and $h: X\cup\mathscr{C}\to|\mathfrak{A}|$ be such that $h(c)=C_c^{(\mathfrak{A})}$ for all $c\in\mathscr{C}$. If h is onto, then $\mathfrak{Fr}_h\mathfrak{A}\equiv\mathfrak{A}$ where elementary equivalence is meant in the equality-free sense.*

Proof. Since $\Delta_{\bar{h}}$, defined by (14) is a universal congruence on $\mathfrak{Fr}_h\mathfrak{A}$, we have from Proposition 2.3 that $\mathfrak{Fr}_h\mathfrak{A}\equiv\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}$. On the other hand, $\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}\cong\mathfrak{A}$ by the isomorphism g defined as $g(a/\Delta_{\bar{h}})=\bar{h}(a)$. Hence, $\mathfrak{Fr}_h\mathfrak{A}/\Delta_{\bar{h}}\equiv\mathfrak{A}$, which yields to the assertion.

An important consequence of this lemma is formulated as follows.

Corollary 2.6. *Let $\mathbf{K}\neq\emptyset$ be any class of structures which is closed under equality-free elementary equivalence. Let $\mathfrak{A}, \mathfrak{B}\in\mathbf{K}$. Then, there exist $\mathfrak{A}', \mathfrak{B}'\in\mathbf{K}$ such that the following conditions are satisfied: $\mathfrak{A}\equiv\mathfrak{A}'$, $\mathfrak{B}\equiv\mathfrak{B}'$ and the meet $\mathfrak{A}'\sqcap\mathfrak{B}'$ exists. (Elementary equivalence is meant in the equality-free sense.)*

Proof. Let X be a set with cardinality large enough such that $\operatorname{card}(X \cup \mathscr{C}) \geqq \operatorname{card}(|\mathfrak{A}| \cup |\mathfrak{B}|)$. Define $h_1 : X \cup \mathscr{C} \to |\mathfrak{A}|$ and $h_2 : X \cup \mathscr{C} \to |\mathfrak{B}|$ in such a way that both h_1 and h_2 are onto and for all $c \in \mathscr{C}$, $h_1(c) = C_c^{(\mathfrak{A})}$ and $h_2(c) = C_c^{(\mathfrak{B})}$. Let $\mathfrak{A}' = \mathfrak{Fr}_{h_1}\mathfrak{A}$ and $\mathfrak{B}' = \mathfrak{Fr}_{h_2}\mathfrak{B}$. By Lemma 2.5., $\mathfrak{A} \equiv \mathfrak{A}'$ and $\mathfrak{B} \equiv \mathfrak{B}'$. Since $\mathbf{K}$ is closed under equality-free elementary equivalence, we have $\mathfrak{A}' \in \mathbf{K}$ and $\mathfrak{B}' \in \mathbf{K}$. It remains to establish that $\mathfrak{A}' \sqcap \mathfrak{B}'$ exists.

Obviously,

$$|\mathfrak{A}' \cap \mathfrak{B}'| = |\mathfrak{Fr}_{h_1}\mathfrak{A}| \cap |\mathfrak{Fr}_{h_2}\mathfrak{B}| \overset{(10)}{=} |\mathfrak{Fr}_{X \cup \mathscr{C}}| \cap |\mathfrak{Fr}_{X \cup \mathscr{C}}| = |\mathfrak{Fr}_{X \cup \mathscr{C}}| \supseteqq X \cup \mathscr{C},$$

hence, the universe of $\mathfrak{A}' \sqcap \mathfrak{B}'$ is not empty. Similarly, for all $f \in \mathscr{F}$,

$$F_f^{(\mathfrak{A}')} \cap F_f^{(\mathfrak{B}')} = F_f^{(\mathfrak{Fr}_{h_1}\mathfrak{A})} \cap F_f^{(\mathfrak{Fr}_{h_2}\mathfrak{B})} \overset{(12)}{=} F_f^{(\mathfrak{Fr}_{X \cup \mathscr{C}})} \cap F_f^{(\mathfrak{Fr}_{X \cup \mathscr{C}})} = F_f^{(\mathfrak{Fr}_{X \cup \mathscr{C}})},$$

hence the set theoretic meet of the functions rendered to f in $\mathfrak{A}'$ and $\mathfrak{B}'$, respectively, is again a function.

For all $c \in \mathscr{C}$,

$$C_c^{(\mathfrak{A}')} = C_c^{(\mathfrak{Fr}_{h_1}\mathfrak{A})} \overset{(13)}{=} C_c^{(\mathfrak{Fr}_{X \cup \mathscr{C}})} \overset{(13)}{=} C_c^{(\mathfrak{Fr}_{h_2}\mathfrak{B})} = C_c^{(\mathfrak{B}')},$$

thus the meet of the two sequences $\langle C_c^{(\mathfrak{A}')}\rangle_{c \in \mathscr{C}}$ and $\langle C_c^{(\mathfrak{B}')}\rangle_{c \in \mathscr{C}}$ is the sequence $\langle C_c^{(\mathfrak{Fr}_{X \cup \mathscr{C}})}\rangle_{c \in \mathscr{C}}$.

Finally, for any $r \in \mathscr{R}$, the meet of the two relations $R_r^{(\mathfrak{A}')}$ and $R_r^{(\mathfrak{B}')}$ is again a relation with arity $t_{\mathscr{R}}(r)$.

Hence $\mathfrak{A}' \cap \mathfrak{B}'$ is a structure of type t, whence $\mathfrak{A}' \sqcap \mathfrak{B}'$ exists.

This corollary shows that the property "a class $\mathbf{K}$ of structures is closed under finite meets" is not trivial in the case when $\mathbf{K}$ is the class of all models of a set of sentences.

2.7. We shall need the well-known preservation theorem under submodels (cf. [2], Theorem 3.2.2, p. 124, or Theorem 5.2.4, p. 228) in a somewhat stronger form: i.e. for equality-free languages. (Insufficiency of the original form for our purposes in the proof of Lemma 2.11, below, was pointed out to me by H. Andréka and I. Németi.) Before formulating the stronger version some preparation is required.

Let $\mathfrak{A}$ be an arbitrary structure and let Γ be the diagram of $\mathfrak{A}$ (cf. [2], p. 68). Let

$$\Gamma_{\mathfrak{A}} = \{\varphi \mid \varphi \in \Gamma \text{ and } \varphi \text{ does not contain the equality symbol}\}.$$

The set $\Gamma_{\mathfrak{A}}$ will be called the *equality-free diagram* of $\mathfrak{A}$.

Lemma 2.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures. If the equality-free diagram $\Gamma_{\mathfrak{A}}$ holds in $\mathfrak{B}$, then there exist two structures $\mathfrak{A}'$ and $\mathfrak{B}'$ of the diagram types* *of* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively, such that the following conditions hold:* $\mathfrak{A}\subseteq\mathfrak{A}'$, $\mathfrak{B}'\subseteq\mathfrak{B}$ *and* $\mathfrak{A}'\equiv\mathfrak{B}'$ *in the equality-free sense.*

Proof. For all $a\in|\mathfrak{A}|$, let c_a be a completely new constant symbol. Let $\mathfrak{A}'=(\mathfrak{A}, a)_{a\in|\mathfrak{A}|}$ and let $\mathfrak{B}'$ be that submodel of $\mathfrak{B}$ which is generated by the set $\{C_{c_a}^{(\mathfrak{B})}\mid a\in|\mathfrak{A}|\}$ where c_a is the new constant symbol for $a\in|\mathfrak{A}|$. Clearly, $\mathfrak{A}\subseteq\mathfrak{A}'$ and $\mathfrak{B}'\subseteq\mathfrak{B}$; moreover it follows that $\mathfrak{B}'\models\Gamma_{\mathfrak{A}}$.

Let φ be an equality-free sentence. We prove by induction on the construction of φ, that

(15) $$\mathfrak{A}'\models\varphi \quad\text{iff}\quad \mathfrak{B}'\models\varphi.$$

Let φ be a prime sentence of the form $r(\tau_0, \ldots, \tau_n)$, where $\tau_0, \ldots, \tau_n$ are terms in which no variables occur, $r\in\mathscr{R}$, $t_{\mathscr{R}}(r)=n+1$. If $\mathfrak{A}'\models r(\tau_0, \ldots, \tau_n)$, then $r(\tau_0, \ldots, \tau_n)\in\Gamma_{\mathfrak{A}}$, thus $\mathfrak{B}'\models r(\tau_0, \ldots, \tau_n)$. Similarly, if $\mathfrak{A}'\not\models r(\tau_0, \ldots, \tau_n)$, then $\neg r(\tau_0, \ldots, \tau_n)\in\Gamma_{\mathfrak{A}}$ and so $\mathfrak{B}'\models\neg r(\tau_0, \ldots, \tau_n)$, i.e. $\mathfrak{B}'\not\models r(\tau_0, \ldots, \tau_n)$.

It is obvious, that the induction goes through for the cases $\neg\psi$ and $\psi_1\wedge\psi_2$.

Let φ be a sentence of the form $\exists v\psi$. If $\mathfrak{A}'\models\varphi$, then there exists $k\colon V\to|\mathfrak{A}'|$ such that $\mathfrak{A}'\models\psi[k]$. It follows, that the sentence ψ^*, obtained from ψ by substituting every occurrence of v by $c_{k(v)}$, holds in $\mathfrak{A}'$. By the induction hypothesis, (15) is true for ψ^*, hence $\mathfrak{B}'\models\psi^*$. Let $k'\colon V\to|\mathfrak{B}'|$ be such that $k'(v)=C_{c_{k(v)}}^{(\mathfrak{B}')}$ (and arbitrary otherwise). Then, $\mathfrak{B}'\models\psi[k']$. Hence, there is a $k'\colon V\to|\mathfrak{B}'|$ for which $\mathfrak{B}'\models\psi[k']$, whence $\mathfrak{B}'\models\exists v\psi$.

Conversely, let us suppose, that $\mathfrak{B}'\models\exists v\psi$. Then there exists an assignment $k'\colon V\to|\mathfrak{B}'|$, such that $\mathfrak{B}'\models\psi[k']$. By the definition of $\mathfrak{B}'$, there exists a term τ, in which no variables occur, such that $k'(v)=\tau^{(\mathfrak{B}')}$ and so, the sentence ψ^*, obtained again from ψ by substituting v everywhere by the term τ, holds in $\mathfrak{B}'$. By the induction hypothesis, we have $\mathfrak{A}'\models\psi^*$. But then, there exists a $k\colon V\to|\mathfrak{A}'|$ such that $k(v)=\tau^{(\mathfrak{A}')}$ and thus $\mathfrak{A}'\models\psi[k]$. So, there is an assignment $k\colon V\to|\mathfrak{A}'|$ for which $\mathfrak{A}'\models\psi[k]$, whence $\mathfrak{A}'\models\exists v\psi$.

We quote Lemma 3.2.1 from [2], p. 124.

Lemma 2.9. *Let* Σ *be a consistent set of sentences in an arbitrary first order language* L *and let* Γ *be a set of sentences in* L *which is closed under finite disjunctions. Then, the following two assertions are equivalent:*

(i) Σ *has a set of axioms* Σ_1 *such that* $\Sigma_1\subseteq\Gamma$.

(ii) *If* $\mathfrak{A}$ *is a model of* Σ *and every sentence* $\varphi\in\Gamma$ *holding in* $\mathfrak{A}$ *holds in* $\mathfrak{B}$, *then* $\mathfrak{B}$ *is a model of* Σ.

It is easy to check that the proof of this lemma in [2], p. 124, does not depend on the presence or lack of equality, hence we can use it for equality-free languages, as well.

The next assertion is the stronger form of the preservation theorem concerning submodels.

Lemma 2.10. *Let Σ be a set of sentences in an arbitrary equality-free language L. Then, the following two conditions are equivalent:*

(i) *Σ is preserved under submodels,*

(ii) *Σ has a set of equality-free universal axioms.*

Proof. It is immediate that (ii) entails (i.) To prove the converse, let us suppose that Σ is preserved under submodels, and let $\mathfrak{A}$ be a model of Σ. Let $\mathfrak{B}$ be such that every equality-free universal sentence holding in $\mathfrak{A}$ holds in $\mathfrak{B}$. Then every existential equality-free sentence true in $\mathfrak{B}$ is true in $\mathfrak{A}$. For if this is not the case, i.e. there exists an existential equality-free sentence, say φ, such that $\mathfrak{B}\models\varphi$ and $\mathfrak{A}\not\models\varphi$, then $\mathfrak{A}\models\neg\varphi$. But $\neg\varphi$ is a universal equality-free sentence, thus, by assumption $\mathfrak{B}\models\neg\varphi$.

Consider the theory $\Sigma'=\Sigma\cup\Gamma_{\mathfrak{B}}$, where $\Gamma_{\mathfrak{B}}$ is the equality-free diagram of $\mathfrak{B}$. Σ' is consistent (provided Σ is such), since for any finite set

$$\{\theta_0(b_0, \dots, b_n), \dots, \theta_m(b_0, \dots, b_n)\}\subseteq\Gamma_{\mathfrak{B}},$$

the existential equality-free sentence

$$\psi = (\exists x_0 \dots \exists x_n)\big(\theta_0(x_0, \dots, x_n)\wedge \dots \wedge\theta_m(x_0, \dots, x_n)\big)$$

is true in $\mathfrak{B}$, hence in $\mathfrak{A}$, too. Thus, ψ is consistent with Σ. By compactness; Σ' is consistent, and has a model $\mathfrak{C}$. So we have $\mathfrak{C}\models\Sigma$ and $\mathfrak{C}\models\Gamma_{\mathfrak{B}}$. By Lemma 2.8 there exist structures $\mathfrak{B}', \mathfrak{C}'$ such that $\mathfrak{C}\supseteq\mathfrak{C}'$, $\mathfrak{B}\subseteq\mathfrak{B}'$ and $\mathfrak{B}'\equiv\mathfrak{C}'$ in the equality-free sense. Σ is preserved under submodels, thus $\mathfrak{C}'\models\Sigma$ and so $\mathfrak{B}'\models\Sigma$. Again, by the preservation property of Σ, $\mathfrak{B}\models\Sigma$.

Let Γ be the set of all sentences, which are equivalent to universal equality-free sentences. Obviously, Γ is closed under finite disjunctions. Thus, the conditions of Lemma 2.9 (ii) are satisfied, and we obtain from Lemma 2.9 (i), that Σ has a set of universal equality-free axioms.

This proof follows closely the proof of Theorem 3.2.2 in [2], p. 128. The only difference, that we use the equality-free diagram of $\mathfrak{B}$ in place of the diagram of $\mathfrak{B}$ in the original proof.

Lemma 2.11. *Let L be an equality-free language and Σ be a set of sentences in L. If Σ is preserved under finite meets, then Σ has a set of universal axioms in L.*

Proof. Let us suppose, that Σ is preserved under finite meets, i.e. if $\mathfrak{A}\models\Sigma$ and $\mathfrak{B}\models\Sigma$, then $\mathfrak{A}\sqcap\mathfrak{B}\models\Sigma$, provided the meet exists. In contrary to the assertion, let us assume, that no set of universal equality-free axioms exists for Σ. Then Σ is consistent and by Lemma 2.10, there exist $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{B}\subseteq\mathfrak{A}$, $\mathfrak{A}\models\Sigma$, $\mathfrak{B}\not\models\Sigma$ hold. Let us define $\mathfrak{A}'$ as follows. We set first

$$|\mathfrak{A}'| = |\mathfrak{B}|\cup((|\mathfrak{A}|-|\mathfrak{B}|)\times\{|\mathfrak{A}|\}).$$

Let $h\colon |\mathfrak{A}|\to|\mathfrak{A}'|$ be a mapping such that

(16) $$h(b)=b \text{ for all } b\in|\mathfrak{B}|,$$

(17) $$h(a)=\langle a, |\mathfrak{A}|\rangle \text{ for all } a\in|\mathfrak{A}|-|\mathfrak{B}|.$$

Clearly, h is one-one and onto, hence we can define:

(18) $$R_r^{(\mathfrak{A}')} = \{\langle a_0, \dots, a_n\rangle \mid a_0, \dots, a_n\in|\mathfrak{A}'|\wedge\langle h^{-1}(a_0), \dots, h^{-1}(a_n)\rangle\in R_r^{(\mathfrak{A})}\}$$

for all $r\in\mathscr{R}$, such that $t_{\mathscr{R}}(r)=n+1$;

(19) $$F_f^{(\mathfrak{A}')}(a_0, \dots, a_n) = h\big(F_f^{(\mathfrak{A})}(h^{-1}(a_0), \dots, h^{-1}(a_n))\big)$$

for all $f\in\mathscr{F}$, $t_{\mathscr{F}}(f)=n+1$ and $a_0, \dots, a_n\in|\mathfrak{A}'|$;

(20) $$C_c^{(\mathfrak{A}')} = h(C_c^{(\mathfrak{A})}).$$

It follows from (16), (17), (18), (19), (20), that $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ (by h) and so $\mathfrak{A}'\equiv\mathfrak{A}$. Thus, we have $\mathfrak{A}'\models\Sigma$. The meet of $\mathfrak{A}$ and $\mathfrak{A}'$ exists by the construction and $\mathfrak{A}\cap\mathfrak{A}'=\mathfrak{B}$.

We obtain that Σ is not preserved under finite meets, a contradiction.

Lemma 2.12. *Let L be an equality-free language and Σ be a set of sentences in L. If Σ is preserved under finite meets, then Σ has a set of axioms consisting of universal equality-free Horn sentences.*

Proof. Let us suppose that Σ is preserved under finite meets, but, in contrary to the assertion, Σ has no set of axioms consisting of universal equality-free Horn sentences. By Lemma 2.11, however, Σ has a set Γ of universal equality-free axioms. It follows from the indirect assumption that Σ, hence Γ is consistent. Again, by the absurd hypothesis, there exists (at least one) sentence $\varphi\in\Gamma$ such that φ is equivalent to a sentence of the form

(21) $$(\forall x_0 \dots \forall x_n) \bigwedge_{u=1}^{z} \left(\bigwedge_{i=1}^{s_u} p_{iu} \to \bigvee_{j=1}^{m_u} q_{ju}\right)$$

where $z, n\in\omega$ and for all u $(1\leqq u\leqq z)$, $s_u, m_u\in\omega$; moreover each p_{iu}, q_{ju} $(1\leqq i\leqq s_u$, $1\leqq j\leqq m_u)$ is a prime formula in which at most $x_0, \dots, x_n$ can occur; and φ is not equivalent to any sentence of the form

$$(22)\qquad (\forall x_0 \dots \forall x_n) \bigwedge_{u=1}^{z} \left(\bigwedge_{i=1}^{s_u} p_{iu} \rightarrow q_{ju} \right)$$

where $n, z, s_u, p_{iu}, q_{ju}$ are just as in (21) and $j=1, \dots, m_u$. For the sake of simplicity, let us suppose that $z=1$ and $m_1=2$. Let $\mathfrak{A}$ be a structure such that $\mathfrak{A} \models \varphi$, but

$$\mathfrak{A} \not\models (\forall x_0 \dots \forall x_n) \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_1 \right) \quad \text{and} \quad \mathfrak{A} \not\models (\forall x_0 \dots \forall x_n) \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_2 \right).$$

(Such a structure $\mathfrak{A}$ exists by (21) and (22), and by the fact that Γ is consistent.) Then, for some $k_1: V \rightarrow |\mathfrak{A}|$ and $k_2: V \rightarrow |\mathfrak{A}|$, we have

$$\mathfrak{A} \not\models \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_1 \right) [k_1] \quad \text{and} \quad \mathfrak{A} \not\models \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_2 \right) [k_2].$$

Let B be a completely new set of constant symbols such that $\operatorname{card}(B \cup \mathscr{C}) \geqq \operatorname{card}(|\mathfrak{A}|)$. Let $b_0, \dots, b_n \in B$ be $n+1$ distinct elements. We can define the mappings $h_1: B \cup \mathscr{C} \rightarrow |\mathfrak{A}|$ and $h_2: B \cup \mathscr{C} \rightarrow |\mathfrak{A}|$ in such a way that both h_1 and h_2 are onto and the following conditions hold: $h_1(c)=h_2(c)=C_c^{(\mathfrak{A})}$ for all $c \in \mathscr{C}$, and $h_1(b_l)=k_1(x_l)$, $h_2(b_l)=k_2(x_l)$ for all $l \in \{0, 1, \dots, n\}$. Let $\mathfrak{A}_1 = \mathfrak{Fr}_{h_1}\mathfrak{A}$ and $\mathfrak{A}_2 = \mathfrak{Fr}_{h_2}\mathfrak{A}$. By Lemma 2.5, $\mathfrak{A}_1 \equiv \mathfrak{A} \equiv \mathfrak{A}_2$ in the equality-free sense and thus

$$\mathfrak{A}_1 \models \varphi, \quad \mathfrak{A}_2 \models \varphi,$$

$$(23)\qquad \mathfrak{A}_1 \not\models \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_1 \right) [k_1'], \quad \mathfrak{A}_2 \not\models \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_2 \right) [k_2'[$$

where $k_1': V \rightarrow |\mathfrak{A}_1|$, $k_2': V \rightarrow |\mathfrak{A}_2|$ such that $k_1'(x_l) = b_l = k_2'(x_l)$ for all $l \in \{0, 1, \dots, n\}$ (and arbitrary otherwise). Notice, that $|\mathfrak{A}_1| = |\mathfrak{A}_2|$ and $\{b_0, \dots, b_n\} \subseteq |\mathfrak{A}_1|$, hence the definitions of k_1' and k_2' are correct. We may assume that $k_1' = k_2'$. Moreover, using an analogous argument to the proof of Corollary 2.6, it is easily seen that $\mathfrak{A}_1 \sqcap \mathfrak{A}_2$ exists.

It follows from (23) and from $k_1' = k_2'$ that

$$\mathfrak{A}_1 \sqcap \mathfrak{A}_2 \not\models \left(\bigwedge_{i=1}^{s} p_i \rightarrow q_1 \vee q_2 \right) [k_1'],$$

hence $\mathfrak{A}_1 \sqcap \mathfrak{A}_2 \not\models \varphi$, contradicting the assumption that Σ is preserved under finite meets.

Using a simple induction, one obtains contradictions in a similar way for all $z \geqq 1$ and $m_u \geqq 2$.

We remark that this lemma is false if equality is present in the language. For example, let

$$\Sigma = \{(\forall x \forall y \forall z)(x \equiv y \vee x \equiv z)\}.$$

Clearly, Σ is consistent and is preserved under finite meets, for any structure $\mathfrak{A}$ is a model of Σ iff card $|\mathfrak{A}| \leqq 2$.

The following assertion is the converse of the lemma above, and establishes the easy direction of our preservation theorem. It is true, however, for arbitrary first order languages with equality.

Lemma 2.13. *Let L be an arbitrary first order language and Σ be a consistent set of sentences. If Σ has a set of axioms consisting of universal Horn sentences, then Σ is preserved under finite meets.*

Proof. It suffices to prove that every universal Horn sentence φ is preserved under finite meets. We proceed by induction on the construction of φ.

Let φ be a quantifier-free basic Horn formula and assume that the free variables of φ are among $\{x_0, \dots, x_m\}$. By definition, φ is equivalent to one of the following two forms:

$$(24) \qquad (\forall x_0 \dots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right),$$

or

$$(25) \qquad (\forall x_0 \dots \forall x_m)\left(\neg \bigwedge_{i=1}^{s} p_i\right),$$

where $p_i\,(1 \leqq i \leqq s)$ and q are prime formulae and $s \in \omega$. (As usual, we allow $s=0$, in which case φ is equivalent either to q or is inconsistent; the latter possibility is, however, ruled out by the conditions.)

For illustration, we consider the case when φ is equivalent to a sentence of the form (24); the other one can be treated similarly.

Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be two structures such that $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ exists and both structures are models of φ, i.e.

$$(26) \qquad \mathfrak{A}_j \models (\forall x_0 \dots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right) \quad (j = 0, 1).$$

Let $k\colon V \to |\mathfrak{A}_0 \sqcap \mathfrak{A}_1|$ be arbitrary and distinguish the following two subcases:

$$(27) \qquad \mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models q[k],$$

$$(28) \qquad \mathfrak{A}_0 \sqcap \mathfrak{A}_1 \not\models q[k].$$

If (27) holds, then by propositional logic, we have immediately that

$$(29) \qquad \mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k].$$

If (28) is the case, then by the definition of the meet, it follows that for some $j\in\{0, 1\}$, $\mathfrak{A}_j\not\models q[k']$ where $k'\colon V\to|\mathfrak{A}_j|$ is such that $k'(x_l)=k(x_l)$ for all $l\in\{0, 1, \dots, m\}$. From (26), we have that

$$\mathfrak{A}_j \not\models \bigwedge_{i=1}^{s} p_i[k'].$$

Again, by the definition of the meet, it follows, that

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \not\models \bigwedge_{i=1}^{s} p_i[k],$$

thus, by propositional logic:

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k]. \tag{30}$$

Putting (29) and (30) together, we obtain that for arbitrary k, (26) entails that

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \left(\bigwedge_{i=1}^{s} p_i \to q\right)[k],$$

which is equivalent to

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models (\forall x_0 \dots \forall x_m)\left(\bigwedge_{i=1}^{s} p_i \to q\right),$$

hence $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ is a model of φ.

It is clear, that the induction goes through for conjunctions of quantifier-free basic Horn formulae.

Let us suppose now, that φ is equivalent to a sentence of the form $(\forall x)\psi$, where ψ is a conjunction of quantifier-free basic Horn formulae. Assume that $\mathfrak{A}_0\models(\forall x)\psi$, $\mathfrak{A}_1\models(\forall x)\psi$ and that $\mathfrak{A}_0 \sqcap \mathfrak{A}_1$ exists.

Then, for arbitrary $k_0\colon V\to|\mathfrak{A}_0|$ and $k_1\colon V\to|\mathfrak{A}_1|$,

$$\mathfrak{A}_0 \models \psi[k_0] \quad \text{and} \quad \mathfrak{A}_1 \models \psi[k_1],$$

respectively. It follows that for arbitrary $k\colon V\to|\mathfrak{A}_0 \sqcap \mathfrak{A}_1|$, we have

$$\mathfrak{A}_0 \sqcap \mathfrak{A}_1 \models \psi[k]$$

and thus $\mathfrak{A}_0 \sqcap \mathfrak{A}_1\models(\forall x)\psi$.

Applying a trivial induction on the number of universal quantifiers (in the prenex form of φ), the lemma is established.

We note that the proof does not depend on the form of prime formulae occurring in φ, hence the assertion is true for languages with equality, too.

Theorem 2.14. *Let L be an equality-free first order language; let Σ be a set of sentences in L and assume that $\mathbf{K}$ is the class of models for Σ. Then, the following two assertions are equivalent:*

(i) **K** *is closed under finite meets.*

(ii) Σ *is equivalent to a set of universal equality-free Horn sentences (in L).*

Proof. (i) entails (ii), by Lemma 2.12. The converse follows from Lemma 2.13, provided Σ is consistent. If Σ is inconsistent, then (i) trivially holds by definition.

We note that this theorem does not generalize to arbitrary first order languages. More precisely, (i) does not entail (ii) if equality is present in the language, as was shown by the counterexample after Lemma 2.12. The converse implication (ii)$\Rightarrow$(i) is, however, true in general.

2.16. Let L be an arbitrary equality-free language and **K** be a class of structures for L. **K** is an *elementary class* in L iff there exists a set Σ of sentences in L such that

$$\mathbf{K} = \{\mathfrak{A} \mid \mathfrak{A} \models \Sigma\}.$$

If Σ consists of universal equality-free Horn sentences, then **K** is said to be a *quasi-variety in* L.

The following assertion is a version of the well-known theorem of Łoś (cf. [2], Theorem 4.1.12, p. 173) for equality-free languages.

Lemma 2.17. *Let* L *be an arbitrary equality-free first order language and* **K** *be a class of structures for* L. *Then,* **K** *is an elementary class in* L *iff* **K** *is closed under equality-free elementary equivalence and ultraproducts.*

Proof. Completely the same as the proof of Theorem 4.1.12 in [2], p. 173.

Corollary 2.18. *Let* L *be an arbitrary equality-free first order language and* **K** *be a class of structures for* L. *Then the following two assertions are equivalent.*

(i) **K** *is closed under finite meets, ultraproducts and equality-free elementary equivalence.*

(ii) **K** *is a quasi-variety in* L.

Proof. Immediate by Theorem 2.15 and Lemma 2.17.

We note that this corollary does not generalize for languages with equality; more precisely (i) does not imply (ii) if the equality is present (cf. the counterexample after Lemma 2.12). The converse, however, holds for arbitrary first order languages, by Lemma 2.13 and Lemma 2.17.

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