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Characterizations of some classes of semigroups

GYÖRGY POLLÁK and OTTÓ STEINFELD

Dedicated to E. S. Lyapin on his 70th birthday

In his paper [1], L. RÉDEI determined the structure of all rings A such that A contains divisors of zero but its proper subrings do not. In [2], O. STEINFELD proved that these rings coincide with those having the property that 0 is not a prime ideal in A but it is prime in every proper subring of A. R. WIEGANDT [4] gave some new characterizations of the same class, and in [5] he determined the structure of a larger class of rings. In [3], F. SZÁSZ proved the mentioned result of Steinfeld by a quite elementary method.

The purpose of the present note is to prove some theorems which can be considered as semigroup theoretical analogues of the structure theorems mentioned above. It turns out that Rédei's and Steinfeld's results as well as part of Wiegandt's equivalent conditions do have their exact counterparts for semigroups; however, as soon as one considers conditions involving left ideals, they fail to be equivalent to the former ones, and determine larger classes. Note that, to a certain extent, this was the case already for rings: the condition "R is non-cancellative but every proper left ideal of R is" was equivalent to the rest under the assumption of the descending chain condition [4] (for semigroups, the equivalence fails even in this case).

In Theorem 1, the equivalence $(i_1)\Leftrightarrow(iii_1)$ is the analogue of Rédei's result, $(i_1)\Leftrightarrow(ii_1)$ is that of Wiegandt's ([4], Theorem 1), and $(i_1)\Leftrightarrow(iv_1)$ corresponds to Steinfeld's theorem.

Theorem 1. *The following conditions are equivalent for a semigroup S with* 0:

(i_1) *S is either the 0-direct union of two groups of prime order with 0 or a two-element zero semigroup;*

(ii_1) *S is not a group with 0, but every proper subsemigroup of S is either a subgroup or a subgroup with 0 of S;*

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(iii_1) *S contains divisors of* 0, *but every proper subsemigroup of S is free from divisors of* 0;

(iv_1) (0) *is not a prime ideal of S but it is prime in every proper subsemigroup with* 0 *of S.*

Remark 1. Note that by a subgroup with 0 in (ii_1) we mean a subsemigroup having 0 for zero element; else the assertion were false (cf. the example in Remark 2).

Proof. (i_1)⇒(ii_1) is obvious.

(ii_1)⇒(iii_1). If S did not contain zero divisors, then $S\setminus\{0\}$ were a proper subsemigroup of S and therefore a group, which contradicts the assumption. The second assertion is obvious.

(iii_1)⇒(iv_1). If a and b are non-zero elements of S such that $ab=0$, then $(a\cup Sa)(b\cup bS)=0$. Put $Q=(a\cup Sa)\cap(b\cup bS)$, then $(a\cup Sa)Q=0$. If $Q=S$ then $S^2=Q^2\subseteq(a\cup Sa)Q=0$ whence our first assertion holds true (the second one is trivial anyway). If $Q\neq S$ then $Q^2=0$ and the second condition in (iii_1) implies $Q=0$. Hence

$$(b\cup bS)(a\cup Sa)\subseteq Q=0.$$

This means that (0) is not a prime ideal in S, in fact, it is easily seen that $(b)(a)=0$.

(iv_1)⇒(i_1). Let $0\neq J_1, J_2\lhd S, J_1J_2=0$. If J_1 had a non-trivial subsemigroup A (i.e. $A\neq J_1$, $A\neq 0$), then $A\cup J_2$ were a proper subsemigroup of S such that (0) is not prime in $A\cup J_2$, as $J_1\cap J_2=0$ follows from the second part of (iv_1). Hence J_1 is either a two-element zero semigroup or a group of prime order with 0, and, by analogy, the same holds for J_2. However, if J_1 (or J_2) is a zero semigroup, it cannot be a proper subsemigroup of S. Thus, either S is a two-element zero semigroup or both J_1 and J_2 are as stated in (i_1). This completes the proof of the theorem.

The following result can be considered as a semigroup theoretical analogue of a theorem due to R. WIEGANDT [5].

Theorem 2. *Let S be a semigroup with* 0. *Then the following conditions are equivalent:*

(i_2) *S is either the* 0*-direct union of two subgroups with* 0 *of S or a two-element zero semigroup;*

(ii_2) (0) *is not a prime ideal of S, but every proper quasi-ideal of S is a subgroup with* 0 *of S;*

(iii_2) *S contains divisors of* 0 *but every proper quasi-ideal of S is free from divisors of* 0;

(iv_2) (0) *is not a prime ideal of S, but it is prime in every proper quasi-ideal of S.*

Remark 2. Let $S=\{0, a, e\}$ be a subsemigroup with 0 having the following Cayley table:

	0	a	e
0	0	0	0
a	0	a	a
e	0	a	e

Then S is not a group with 0 but every proper quasi-ideal of S ($\{0\}$ and $\{0, a\}$) is either a subgroup or a subgroup with 0 of S. This example shows that neither condition (ii_1) nor R. Wiegandt's condition (d) in the "Satz" of [5] has a word-for-word analogue in Theorem 2.

Proof. The implications (i_2)⇒(ii_2) and (ii_2)⇒(iii_2) are obvious.

(iii_2)⇒(iv_2). The same proof as for (iii_1)⇒(iv_1) goes through, because $a\cup Sa$ is not only a subsemigroup but also a quasi-ideal (in fact, a left ideal) of S.

(iv_2) ⇒ (i_2). Let $J_1, J_2 \lhd S, J_1J_2 = 0, J_1, J_2 \neq 0$. An argument analogous to that used in the proof of (iv_1)⇒(i_1) shows that J_1 (and J_2) cannot have non-trivial quasi-ideals whence it is either a two-element zero semigroup or a group, and the first case can hold only if $J_1=S$ ($J_2=S$, resp.). This completes the proof.

Our next theorem shows that conditions (c) and (g) in the mentioned "Satz" of Wiegandt are not equivalent to the other ones in the case of semigroups. However, the corollary exhibits the way of repairing the matter.

Theorem 3. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i_3) *S is either the 0-direct union of two left 0-simple semigroups which are not zero semigroups or S itself is a two-element zero semigroup;*

(ii_3) *(0) is not a prime ideal of S but every proper left ideal of S is left 0-simple;*

(iii_3) *S contains divisors of 0 but every proper left ideal of S is free from divisors of 0;*

(iv_3) *(0) is not a prime ideal of S but it is prime in every proper left ideal of S.*

Proof. (i_3)⇒(ii_3) is obvious, and so is (ii_3)⇒(iii_3) because a left 0-simple semigroup is a left simple semigroup with 0 adjoined.

(iii_3)⇒(iv_3). The same as (iii_1)⇒(iv_1).

(iv_3) ⇒ (i_3). Let $J_1, J_2 \lhd S, J_1J_2 = 0, J_1, J_2 \neq 0$. Analogously to Theorems 1 and 2, here it follows that J_1 and J_2 cannot have non-trivial left ideals, and the same proof as there goes through with the obvious modifications.

Corollary 1. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i_2) *S is either the 0-direct union of two subgroups with 0 of S or a two-element zero semigroup;*

(ii_c) (0) *is not a prime ideal of S but every proper one-sided ideal of S is a group;*

(iii_c) *S contains divisors of 0 but every proper one-sided ideal of S is free from divisors of 0;*

(iv_c) (0) *is not a prime ideal of S but it is prime in every proper one-sided ideal of S.*

This Corollary can be obtained by simply putting together Theorem 3 and its dual.

Remark 3. Corollary 1 yields a trivial proof of Theorem 2 according to the following scheme:

$$(i_2) \left\{\begin{matrix} \Rightarrow (ii_2) \Rightarrow (ii_c) \Rightarrow \\ \Rightarrow (iii_2) \Rightarrow (iii_c) \Rightarrow \\ \Rightarrow (iv_2) \Rightarrow (iv_c) \Rightarrow \end{matrix}\right\} (i_2)$$

Nevertheless, we preferred to give there an independent proof, in order to preserve the natural order of the results.

Condition (i) in Wiegandt's "Satz" determines an even larger class of semigroups.

Theorem 4. *Let S be a semigroup with 0. Then the following conditions are equivalent:*

(i_4) *S is either the 0-direct union of two simple semigroups with 0 adjoined or 0-simple with divisors of 0;*

(iii_4) *S contains divisors of 0 but every proper ideal of S is free from divisors of 0.*

Proof. $(i_4)\Rightarrow(iii_4)$ is obvious.

$(iii_4)\Rightarrow(i_4)$. Let $a, b\in S$, $a\neq 0\neq b$, $ab=0$. Then $(a)\cup(b)=S$. Suppose S has a non-trivial ideal J. Then either $Ja=0$ or $bJ=0$ because $Ja\cdot bJ=0$ and J is free from divisors of 0. Let e.g. $Ja=0$. Then $a\notin J$ and $(a)\cup J=S$. Thus, either $b\in(a)$ or $b\in J$. In the first case $(a)=(a)\cup(b)=S$ whence $JS=Ja\cup JaS=0$ and $J=0$, contrary to the assumption. In the second case $Ja=0$ implies $ba=0$, and either $Jb=0$ or $aJ=0$. But $Jb\neq 0$ since $b\in J$. Thus, $aJ=Ja=0$, and also $(a)J=J(a)=0$ whence $((a)\cap J)^2=0$ and therefore $(a)\cap J=0$. We have obtained that S is the 0-direct union of (a) and J. Both must be 0-simple, else a proper ideal containing divisors of 0 could be found. However, (a) and J themselves do not contain divisors of 0, and so they are simple semigroups with 0 adjoined; q. e. d.

As in [5], the condition of type (iv) leads to a different class of semigroups. Here the analogy with the ring case is again complete.

Theorem 5. *Let* S *be a semigroup with* 0. *The following conditions are equivalent:*

(i_5) S *is either the* 0-*direct union of two simple semigroups which are not zero semigroups or* S *itself is the two-element zero semigroup;*

(iv_5) (0) *is not a prime ideal of* S *but it is prime in every proper ideal of* S.

The proof is essentially the same as that of (iv_3)$\Rightarrow$(i_3).

It is easy to find the corresponding conditions (ii_4) and (ii_5) which can be inserted in Theorems 4 and 5, respectively.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
P. O. BOX 127
1364 BUDAPEST, HUNGARY