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Basic permutation groups on infinite sets

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1. Introduction

An algebra $\mathfrak{A}=\langle A;\ F\rangle$ is said to be (locally) complete if every finitary operation on A is a (local) term function of $\mathfrak{A}$, and $\mathfrak{A}$ is (locally) functionally complete if every finitary operation on A is a (local) algebraic function of $\mathfrak{A}$. Following SALOMAA [11], a permutation group G on a finite set A is said to be basic if for any surjective finitary operation f on A depending on at least two variables, the algebra $\langle A; G\cup\{f\}\rangle$ is complete. For example the full symmetric group on A if $|A|\geqq 5$ (SALOMAA [10]), the triply transitive permutation groups G on A if $|A|\geqq 4$ and G is affine with respect to no elementary abelian 2-group (SCHOFIELD [12]), and the doubly transitive permutation groups G on A if $|A|\geqq 3$ and G is affine with respect to no abelian elementary p-group, p prime (KNOEBEL [5]), are basic groups.

There is an interesting analogy between the above mentioned examples and some results on the functional completeness of finite algebras with large automorphism groups. The first example is the counterpart of the result of CSÁKÁNY [1] stating that almost every nontrivial finite algebra whose automorphism group is the full symmetric group is functionally complete; up to equivalence there are only six exceptions. The second and third examples correspond to the results of [14] and [6], respectively, stating that except for some affine spaces every nontrivial finite algebra whose automorphism group is triply, resp., doubly transitive, is functionally complete. A result in [7] completes this series stating that every nontrivial finite algebra whose automorphism group is a basic group, is functionally complete.

For infinite sets the full symmetric groups are not basic groups in Salomaa's sense (replacing the completeness by local completeness) as it was shown in [9] and [5], but if f is a nontrivial idempotent operation on an infinite set A and S_A is the full symmetric group on A, then the algebra $\langle A;\ S_A\cup\{f\}\rangle$ is locally complete ([9]). Therefore in this paper a permutation group G on an infinite set A is said

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to be basic if for any nontrivial idempotent operation f on A, the algebra $\langle A; G\cup\{f\}\rangle$ is locally complete. (Remark that for finite sets this definition yields exactly the basic groups in Salomaa's sense.) The best result for basic groups on infinite sets was given in [9]: every triply transitive permutation group on an infinite set is basic if it is affine with respect to no elementary abelian 2-group.

As in the case of finite sets, we can observe the same analogy between the above mentioned basic permutation groups on infinite sets and some results on locally functionally complete algebras whose automorphism group is basic. Namely, if the automorphism group of an infinite algebra $\mathfrak{A}$ is the full symmetric group or a triply transitive permutation group which is affine with respect to no elementary abelian 2-group, then $\mathfrak{A}$ is locally functionally complete (Fried, Kaiser and Márki [2]; Kaiser and Márki [4]).

In [4] the following is proved: If $\mathfrak{A}=\langle A; F\rangle$ is an infinite algebra whose automorphism group G is doubly primitive, the stabilizer G_a of any element $a\in A$ admits no partial order on the set $A\setminus\{a\}$, and G is affine with respect to no elementary abelian 2-group, then $\mathfrak{A}$ is locally functionally complete. Observing these results the following question is naturally arising: Is every permutation group G on an infinite set having the properties mentioned above a basic group? The aim of this paper is to give an affirmative answere to this question (Theorem 8 and Corollary 9).

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2. Preliminaries

Let A be a nonempty set. The set of n-ary operations (or functions) on A will be denoted by $O_A^{(n)}$ $(n\geqq 1)$ and we set $O_A=\bigcup_{n=1}^{\infty} O_A^{(n)}$. An operation $f\in O_A$ is idempotent if we have $f(a, \dots, a)=a$ for every $a\in A$. f is trivial if it is a projection. A ternary operation $d\in O_A^{(3)}$ is called a *majority function* if for all $x, y\in A$ we have $d(x, x, y)=d(x, y, x)=d(y, x, x)=x$. An operation $t\in O_A^{(3)}$ is said to be a *minority function* if $t(x, y, y)=t(y, x, y)=t(y, y, x)=x$ for all $x, y\in A$. By an n-ary i-th semi-projection $(3\leqq n,\ 1\leqq i\leqq n)$ we mean an operation $q\in O_A^{(n)}$ which has the following property: $q(x_1, \dots, x_n)=x_i$ whenever at least two elements among $x_1, \dots, x_n$ are equal.

We adopt the terminology of [3] except that polynomials will be called *term functions*. For an algebra $\mathfrak{A}=\langle A; F\rangle$, the sets of term functions and of algebraic functions of $\mathfrak{A}$ will be denoted by $T(\mathfrak{A})$ and $A(\mathfrak{A})$, respectively. An operation

$f\in O_A^{(n)}$ is said to be a *local term (algebraic) function* of $\mathfrak{A}$ if for every finite $B\subseteq A^n$ there is a $g\in T(\mathfrak{A})\cap O_A^{(n)}$ $(g\in A(\mathfrak{A})\cap O_A^{(n)})$ such that $f|B=g|B$. The sets of local term functions and of local algebraic functions of $\mathfrak{A}$ will be denoted by $\hat{T}(\mathfrak{A})$ and $\hat{A}(\mathfrak{A})$, respectively. An algebra $\mathfrak{A}=\langle A; F\rangle$ is termed *locally complete (locally functionally complete)* if $\hat{T}(\mathfrak{A})=O_A$ $(\hat{A}(\mathfrak{A})=O_A)$. $\mathfrak{A}$ is called *trivial* if $T(\mathfrak{A})$ contains projections only.

If ϱ is an h-ary relation on A, i.e., $\varrho\subseteq A^h$, then $\operatorname{Pol}\varrho$ denotes the set of all operations from O_A preserving ϱ. A binary relation ϱ on A is *locally bounded* if for every finite $B\subseteq A$ we have $B\times\{u\}\subseteq\varrho$ and $\{v\}\times B\subseteq\varrho$ for some $u, v\in A$. For $2\leq h$ let

$$\sigma_h=\{(a_1,\dots,a_h)\in A^h\colon a_i\neq a_j,\quad 1\leq i<j\leq h\}.$$

Furthermore, we set $\iota_h=A^h\setminus\sigma_h$. An h-ary relation ϱ on A $(h\geq 2)$ is *totally reflexive* if $\iota_h\subseteq\varrho$. ϱ is *totally symmetric* if $(a_1,\dots,a_h)\in\varrho$ implies $(a_{1\pi},\dots,a_{h\pi})\in\varrho$ for every permutation π of $\{1,\dots,h\}$. If ϱ is totally reflexive, totally symmetric, $\varrho\neq A^h$, and to every finite $B\subseteq A$ there is a $u\in A$ such that $B^{h-1}\times\{u\}\subseteq\varrho$ then ϱ is called *locally central.* The proper unary relations are also referred to as locally central relations. A binary relation ϱ is *reflexive, areflexive, symmetric,* and *asymmetric* if $\iota_2\subseteq\varrho$, $\iota_2\cap\varrho=\emptyset$, $\varrho=\varrho^{-1}$ and $\varrho\cap\varrho^{-1}=\emptyset$. If $f\in O_A^{(n)}$ then $f^\bullet=\{(a_1,\dots,a_n,f(a_1,\dots,a_n))\in A^{n+1}: a_1,\dots,a_n\in A\}$.

Now we are ready to formulate the local completeness criterion from [9].

Theorem 1. *Let $\mathfrak{A}=\langle A; F\rangle$ be an algebra. Then $\mathfrak{A}$ is locally complete if $T(\mathfrak{A})\subseteq\operatorname{Pol}\varrho$ for no relation ϱ of one of the following types:*

(1) *locally bounded partial orders,*

(2) *nontrivial equivalence relations,*

(3) *binary relations $s^\bullet$ where s is a fixed point free permutation of A whose cycles are either all of the same prime length or all infinite,*

(4) *reflexive, symmetric relations ϱ with $\bigcup_{n=1}^{\infty}\varrho^n=A^2$ and $\varrho^n\neq A^2$, $n=1,2,\dots$ $(\varrho^1=\varrho$, $\varrho^{n+1}=\varrho^n\circ\varrho)$,*

(5) *binary locally bounded reflexive antisymmetric relations ϱ with $\varrho^2=A^2$,*

(6) *binary locally bounded areflexive symmetric relations,*

(7) *binary locally bounded areflexive asymmetric relations,*

(8) *ternary relations $\varrho=\sigma\cup\Delta_{12}$ with $\emptyset\neq\sigma\subseteq\sigma_3$ and $\Delta_{12}=\{(a,a,b)\in A^3\colon a,b\in A\}$ such that for all $x,y,z,t\in A$, $(x,y,z)\in\varrho$ implies $(y,x,z)\in\varrho$, $(x,t,z)\in\varrho$ and $(y,t,z)\in\varrho$ imply $(x,y,z)\in\varrho$, and for every finite $B\subseteq A$ we have $B^2\times\{u\}\subseteq\varrho$ for some $u\in A$,*

(9) *quaternary relations $m^\bullet$ where $m(x,y,z)=x-y+z$ for all $x,y,z\in A$ and $\langle A; +\rangle$ is an abelian group which is either an elementary p-group or a torsion-free divisible group,*

(10) *locally central relations,*

(11) *totally reflexive and totally symmetric h-ary relations* ϱ *with* $\varrho \neq A^h$ *and* $h \geqq 3$, *that are not locally central.*

We also need the following result from [9]:

Proposition 2. $\mathrm{Pol}\, \sigma_2$ *and* $\mathrm{Pol}\,(\sigma_3 \cup \Delta_{12})$ *contain no nontrivial idempotent operations.* (Here $\sigma_3 \cup \Delta_{12}$ is the greatest relation of type (8).)

An operation $f \in O_A^{(n)}$ is said to be *affine* with respect to an abelian group $\langle A; +\rangle$ if for all $x_1, y_1, \ldots, x_n, y_n \in A$ we have

$$f(x_1+y_1, \ldots, x_n+y_n) = f(x_1, \ldots, x_n)+f(y_1, \ldots, y_n)-f(0, \ldots, 0).$$

It is easy to check that this means exactly that $f \in \mathrm{Pol}\, m^{\bullet}$ where $m(x, y, z) = x-y+z$ for all $x, y, z \in A$. A set of operations $F \subseteq O_A$ as well as the algebra $\langle A; F\rangle$ is said to be affine with respect to $\langle A; +\rangle$ if every operation in F is affine with respect to $\langle A; +\rangle$.

The full symmetric group acting on A will be denoted by S_A. For a permutation group $G \leqq S_A$, a subset B of A is called a *block* of G if for every $\pi \in G$, either $B\pi = B$ or $B\pi \cap B = \emptyset$. The one element subsets $\{a\}, a \in A$, and A are *trivial blocks* of G. A transitive permutation group $G \leqq S_A$ is said to be *primitive* if it has trivial blocks only. (Notice that except for the case $|A|=2$, the latter condition implies the transitivity of G.) We shall often use the following equivalent definition: for $|A| \geqq 3$, a permutation group $G \leqq S_A$ is primitive if $G \subseteq \mathrm{Pol}\, \varrho$ for no nontrivial equivalence relation ϱ on A, and for $|A|=2$ if $G = S_A$. G is *doubly primitive* if it is doubly transitive and the stabilizer G_a of any element $a \in A$ is primitive on the set $A \setminus \{a\}$.

A permutation group G on an infinite set A is said to be a *basic group* if for every nontrivial idempotent operation f on A, the algebra $\langle A; G \cup \{f\}\rangle$ is locally complete.

We shall use the following well-known fact for primitive permutation groups:

Proposition 3 ([13; Theorem 10.5.7]). *If* $G \leqq S_A$ *is transitive and* $a \in A$, *then* G *is primitive if and only if* G_a *is a maximal proper subgroup of* G.

Finally we need the following well-known result.

Proposition 4 (see e.g. [6]). *If an algebra* $\mathfrak{A}$ *has a nontrivial idempotent term function then it has a majority function or a minority function or a nontrivial first semi-projection or a nontrivial binary idempotent operation among its term functions.*

3. Results

From now on A is supposed to be an infinite set.

Lemma 5. *If G is a doubly primitive permutation group on A and $G \subseteq \operatorname{Pol} \varrho$ where ϱ is one of the relations of types* (1)—(11), *then $\varrho = \sigma_2$ or $\varrho = \sigma_3 \cup \Delta_{12}$ or ϱ is an at least ternary totally reflexive totally symmetric relation or a relation $m^\bullet$ where $m(x, y, z) = x - y + z$ for all $x, y, z \in A$ and $\langle A; + \rangle$ is an elementary 2-group.*

Proof. Let ϱ be one of the relations of types (1)—(11) and suppose that $G \subseteq \operatorname{Pol} \varrho$. Clearly ϱ cannot be a unary relation (of type (10)). If ϱ is binary then $\varrho = \sigma_2$ since G is doubly transitive. If ϱ is an at least ternary relation of type (10) or a relation of type (11) then it is totally reflexive and totally symmetric. Now let ϱ be a relation of type (8) and let $a \in A$ be an arbitrary element. Define a binary relation θ_a on $A \setminus \{a\}$ as follows: $(x, y) \in \theta_a$ if and only if $(x, y, a) \in \varrho$. Taking into consideration the properties of ϱ we immediately get that θ_a is an equivalence relation. Furthermore, since $G \subseteq \operatorname{Pol} \varrho$, every permutation in G_a preserves θ_a. Therefore, owing to the primitivity of G_a on $A \setminus \{a\}$, the relation θ_a is either the equality or the full relation on $A \setminus \{a\}$. By the definition of ϱ, $\varrho \cap \sigma_3 \neq \emptyset$. Therefore there are pairwise distinct elements $a_1, a_2, a_3 \in A$ such that $(a_1, a_2, a_3) \in \varrho$. Choose a permutation $\pi \in G$ such that $a_3\pi = a$. Then $(a_1, a_2, a_3) \in \varrho$ implies $(a_1\pi, a_2\pi, a) = (a_1\pi, a_2\pi, a_3\pi) \in \varrho$ and $(a_1\pi, a_2\pi) \in \theta_a$, showing that θ_a is not the equality relation. Hence θ_a is the full relation, i.e., for all $x, y \in A \setminus \{a\}$ we have $(x, y, a) \in A$. Since a was chosen arbitrarily, it follows that $\varrho = \sigma_3 \cup \Delta_{12}$.

Finally, let $\varrho = m^\bullet$ where $m(x, y, z) = x - y + z$ for all $x, y, z \in A$ and $\langle A; + \rangle$ is an abelian group which is either an elementary p-group or a torsionfree divisible group. Let 0 be the unit element of $\langle A; + \rangle$. If $\pi \in G \subseteq \operatorname{Pol} m^\bullet$ then for all $x \in A$ we have $x\pi = (x\pi - 0\pi) + 0\pi$ and the map $x \to x\pi - 0\pi$ from A to A is an automorphism of $\langle A; + \rangle$. Indeed, if $a, b \in A$ then $(a, 0, b, a+b) \in m^\bullet$ implies $(a\pi, 0\pi, b\pi, (a+b)\pi) \in m^\bullet$. Thus $a\pi - 0\pi + b\pi = (a+b)\pi$ and $(a+b)\pi - 0\pi = (a\pi - 0\pi) + (b\pi - 0\pi)$. Hence we get that every permutation in G is of form $xr + c$ where $r\colon A \to A$ is an automorphism of $\langle A; + \rangle$ and $c \in A$. Therefore, the stabilizer G_0 consists of permutations of the form xr, $r \in \operatorname{Aut} \langle A; + \rangle$.

First let $\langle A; + \rangle$ be a torsionfree divisible group and define a binary relation θ on $A \setminus \{0\}$ as follows: $(x, y) \in \theta$ if and only if $mx = ny$ for some natural numbers m, n. Then θ is an equivalence relation preserved by every permutation in G_0. Therefore, owing to the primitivity of G_0, θ is the full relation on $A \setminus \{a\}$. Therefore $\langle A; + \rangle$ is isomorphic to the additive group of rational numbers. Furthermore it is easy to check that the automorphism group of the additive group of rational numbers is not primitive on the set of nonzero rational numbers. Thus G_0 is not primitive, contrary to our assumption.

Now let $\langle A; +\rangle$ be an elementary p-group (p prime), and define a binary relation Ψ on $A\setminus\{0\}$ as follows: $(x, y)\in\Psi$ if and only if $x=my$ for some natural number m. Then Ψ is an equivalence relation and every permutation in G_0 preserves Ψ. Therefore, since G_0 is primitive, Ψ is either the equality or the full relation on $A\setminus\{a\}$. In the first case $\langle A; +\rangle$ is an elementary 2-group. In the second case we have $A=\{0, a, 2a, ..., (p-1)a\}$ where $a\in A$ with $a\neq 0$. This contradicts our assumption on A (A is infinite).

Lemma 6. *If G is a doubly transitive permutation group on A and $d\in O_A^{(3)}$ is a majority function then the algebra $\langle A; G\cup\{d\}\rangle$ is locally complete.*

Proof. Taking into consideration Theorem 2, we have to prove that $G\cup\{d\}\subseteq\operatorname{Pol}\varrho$ for no relation ϱ of types (1)—(11). Let ϱ be one of the relations listed in Theorem 1, and suppose $G\cup\{d\}\subseteq\operatorname{Pol}\varrho$. Making use of the fact that G is doubly transitive, we immediately get that ϱ cannot be a relation of types (1), (2), (3), (4), (5), (7) or a unary or binary relation of type (10). Furthermore, if ϱ is of type (6) then $\varrho=\sigma_2$ and, by Proposition 2, $d\notin\operatorname{Pol}\sigma_2$. If ϱ is of type (8), then, by definition, $(a, b, c)\in\varrho$ for some pairwise distinct elements $a, b, c\in A$ and $(a, a, b)\in\varrho$, $(b, b, b)\in\varrho$. It follows $(a, b, b)=(d(a, a, b),\ d(b, a, b),\ d(c, b, b))\in\varrho$, contrary to the definition of ϱ. If ϱ is an h-ary ($h\geqq 3$) relation of type (10) or (11) and $(a_1, ..., a_h)\in A^h\setminus\varrho$ then $(a_2, a_2, a_3, a_4, ..., a_h)\in\varrho$, $(a_1, a_1, a_3, a_4, ..., a_h)\in\varrho$ and $(a_1, a_2, a_2, a_4, ..., a_h)\in\varrho$ imply that $(a_1, a_2, a_3, a_4, ..., a_h)=(d(a_2, a_1, a_1), d(a_2, a_1, a_2), d(a_3, a_3, a_2), d(a_4, a_4, a_4), ..., d(a_h, a_h, a_h))\in\varrho$, a contradiction.

Finally, let $\varrho=m^{\bullet}$ where $m(x, y, z)=x-y+z$ for all $x, y, z\in A$ and $\langle A; +\rangle$ is an abelian group. Let $0\in A$ be the unit element of $\langle A; +\rangle$ and choose a non-unit element $a\in A$. Then $(0, 0, a, a)\in m^{\bullet}$, $(0, -a, 0, a)\in m^{\bullet}$ and $(a, 0, 0, a)\in m^{\bullet}$ imply $(0, 0, 0, a)=(d(0, 0, a),\ d(0, -a, 0),\ d(a, 0, 0),\ d(a, a, a))\in m^{\bullet}$ and $0=0-0+0=a$, contrary to the choice of a.

Lemma 7. *If G is a primitive permutation group on A and $t\in O_A^{(3)}$ is a minority function then the algebra $\langle A; G\cup\{t\}\rangle$ is either locally complete or affine with respect to an elementary 2-group.*

Proof. If $\langle A; G\cup\{t\}\rangle$ is locally incomplete then, by Theorem 2, $G\cup\{t\}\subseteq\operatorname{Pol}\varrho$ where ϱ is one of the relations of types (1)—(11). Our proof will be complete if we show that ϱ is of type (9) determined by an elementary 2-group.

If ϱ is a binary reflexive relation then ϱ is an equivalence relation. Indeed, $(a, b)\in\varrho$ implies $(b, a)=(t(b, a, a),\ t(b, a, b))\in\varrho$ and $(a, b)\in\varrho$, $(b, c)\in\varrho$ imply $(a, c)=(t(a, b, b),\ t(b, b, c))\in\varrho$. Since G is primitive, $G\subseteq\operatorname{Pol}\varrho$ implies that $\varrho=\iota_2$ or $\varrho=A^2$, a contradiction. If ϱ is a locally bounded areflexive relation and $(a, b)\in\varrho$ then there is a $c\in A$ such that $(a, c)\in\varrho$ and $(b, c)\in\varrho$. Therefore $(b, b)=(t(a, a, b),\ t(b, c, c))\in\varrho$, contrary to the areflexivity of ϱ. If $\varrho=s^{\bullet}$ where

s is a permutation on A then $G \in \operatorname{Pol} s^{\bullet}$ implies that every cycle of any power s^n of s $(n=1, 2, \ldots)$ is a block of G, showing that s is the identity permutation. Hence ϱ cannot be a relation of types (1)—(7) or a unary or binary relation of type (10). (The unary relations are excluded by the transitivity of G.)

If ϱ is a relation of type (8) then, by definition, there are pairwise different elements $a, b, c \in A$ such that $(a, b, c) \in \varrho$. Now $(a, b, c) \in \varrho$, $(a, a, c) \in \varrho$ and $(b, b, b) \in \varrho$ imply $(b, a, b) = (t(a, a, b),\ t(b, a, b),\ t(c, c, b)) \in \varrho$, contrary to the definition of ϱ. If ϱ is an h-ary $(h \geqq 3)$ relation of type (10) or (11) and $(a_1, \ldots, a_h) \in A^h \setminus \varrho$ then $(a_1, a_1, a_3, \ldots, a_h) \in \varrho$, $(a_3, a_2, a_3, \ldots, a_h) \in \varrho$ and $(a_3, a_1, a_3, \ldots, a_h) \in \varrho$ imply $(a_1, a_2, a_3, \ldots, a_h) = (t(a_1, a_3, a_3), t(a_1, a_2, a_1), t(a_3, a_3, a_3), \ldots, t(a_h, a_h, a_h)) \in \varrho$, contrary to the choice of $(a_1, \ldots, a_h)$.

Hence $\varrho = m^{\bullet}$ where $m(x, y, z) = x - y + z$ for all $x, y, z \in A$ and $\langle A; + \rangle$ is an abelian group. Now for every $a \in A$ we have $(a, 0, a, 2a) \in m^{\bullet}$, $(a, a, 0, 0) \in m^{\bullet}$ and $(0, a, a, 0) \in m^{\bullet}$, which imply that $(0, 0, 0, 2a) = (t(a, a, 0),\ t(0, a, a),\ t(a, 0, a),\ t(2a, 0, 0)) \in m^{\bullet}$, i.e., $2a = 0 - 0 + 0 = 0$. Therefore $\langle A; + \rangle$ is an elementary 2-group, which completes the proof.

Theorem 8. *Let G be a doubly primitive permutation group on A such that for every $a \in A$, the stabilizer G_a admits no nontrivial partial order on the set $A \setminus \{a\}$. Then for every nontrivial idempotent function $f \in O_A$, the algebra $\mathfrak{A} = \langle A; G \cup \{f\} \rangle$ is either locally complete or affine with respect to an elementary 2-group.*

Proof. If $\mathfrak{A}$ has a majority function or a minority function among its term functions then our claim follows from Lemma 6 or Lemma 7. Therefore, suppose that $\mathfrak{A}$ has neither majority functions nor minority functions among its term functions. Then, by Proposition 4, $\mathfrak{A}$ has a nontrivial first semi-projection or a nontrivial binary idempotent operation among its term functions. If $\mathfrak{A}$ is locally incomplete then, by Theorem 1, $G \cup \{f\} \subseteq \operatorname{Pol} \varrho$ where ϱ is one of the relations of types (1)—(11). Taking into consideration Lemma 5 and Proposition 2, we immediately get that ϱ is an at least ternary totally reflexive and totally symmetric relation or $\varrho = m^{\bullet}$ where $m(x, y, z) = x - y + z$ for all $x, y, z \in A$ and $\langle A; + \rangle$ is an elementary 2-group. Therefore our proof will be complete if we show that $G \cup \{f\} \subseteq \operatorname{Pol} \varrho$ for no h-ary $(h \geqq 3)$ totally reflexive and totally symmetric relation different from A^h. In order to prove this, consider the algebra $\mathfrak{B} = \langle A; I \rangle$ where I is the set of all idempotent term functions of $\mathfrak{A}$. We show that $\mathfrak{B}$ has the $1^1/_2$-interpolation property, i.e., for all $a, b, c \in A$, $a \neq b$, there is a unary algebraic function g of $\mathfrak{B}$ such that $g(a) = a$ and $g(b) = c$. Then, clearly, $\mathfrak{B}$ also has the 2-interpolation property, i.e., for all $a, b, c, d \in A$, $a \neq b$, there is a unary algebraic function g of $\mathfrak{B}$ such that $g(a) = c$ and $g(b) = d$.

For $a, b \in A$, $a \neq b$, let $B(a, b)$ denote the set of elements $c \in A$ for which there is a unary algebraic function g of $\mathfrak{A}$ such that $g(a) = a$ and $g(b) = c$. We have to show that $B(a, b)=A$ for all $a, b\in A, a\neq b$. It is easy to check that $a, b\in B(a, b)$ and $c\in B(a, b)$ imply $B(a, c)\subseteq B(a, b)$. Furthermore, for every $\pi\in G$, $B(a, b)\pi= =B(a\pi, b\pi)$. To show the second statement let $c\in B(a, b)$ and let g be a unary algebraic function of $\mathfrak{B}$ such that $g(a)=a$ and $g(b)=c$. Then, clearly, there is an n-ary term function t of $\mathfrak{B}$ and there are elements $a_2, \ldots, a_n\in A$ such that $g(x)=t(x, a_2, \ldots, a_n)$ for all $x\in A$. Then, by definition, the n-ary operation $t'(x_1, \ldots, x_n)=(t(x_1\pi^{-1}, \ldots, x_n\pi^{-1}))\pi$ is a term function of $\mathfrak{B}$ and for the unary algebraic function $g'(x)=t'(x, a_2\pi, \ldots, a_n\pi)$ we have $g'(a\pi)=a\pi$ and $g'(b\pi)=c\pi$. It follows that $B(a, b)\pi\subseteq B(a\pi, b\pi)$. Repeating this reasoning for $a\pi, b\pi$, and π^{-1} instead of a, b and π we have $B(a\pi, b\pi)\pi^{-1}\subseteq B(a, b)$ and $B(a\pi, b\pi)\subseteq \subseteq B(a, b)\pi$. Hence $B(a, b)\pi=B(a\pi, b\pi)$. Taking into consideration the double transitivity of G, we immediately get that $B(a, b)$ has the same cardinality for all $a, b\in A$, $a\neq b$.

Now we show that $|B(a, b)|\geqq 3$ for all $a, b\in A$, $a\neq b$. First suppose that $\mathfrak{B}$ has an n-ary ($n\geqq 3$) nontrivial first semi-projection q among its term functions. Since q is not the first projection, there are pairwise different elements $x_1, \ldots, x_n\in A$ such that $q(x_1, \ldots, x_n)=x_0\neq x_1$. We may suppose that $x_0\neq x_2$. Then for the unary algebraic function $g(x)=q(x_1, x, x_3, \ldots, x_n)$ we have $g(x_1)=x_1$ and $g(x_2)=x_0$. Thus $x_0, x_1, x_2\in B(x_1, x_2)$, showing that $|B(x_1, x_2)|\geqq 3$. Now suppose that $\mathfrak{B}$ has a nontrivial binary idempotent term function and denote it by juxtaposition. If $ab=c\notin \{a, b\}$ for some $a, b\in A, a\neq b$ then for the unary algebraic function $g(x)=ax$ we have $g(a)=a$ and $g(b)=c$. Thus $a, b, c\in B(a, b)$ and $|B(a, b)|\geqq 3$. Suppose that $xy\in \{x, y\}$ for all $x, y\in A$. First consider the case when there is a zero element, i.e., there is an element $a\in A$ such that $a=ax=xa$ for all $x\in A$. Then choose two different elements $b, c\in A\setminus\{a\}$. We may assume that $bc=c$. Thus the unary algebraic function $g(x)=xc$ shows that $a, b, c\in B(a, b)$ and $|B(a, b)|\geqq 3$. Finally consider the case when there is no zero element and let $a, b\in A$, $a\neq b$, $ab=a$. If $ac=c$ for some $c\in A$, $c\neq a$, then $g(x)=ax$ shows that $c, b, a\in B(c, b)$ and $|B(c, b)|\geqq 3$. If $ax=a$ for all $x\in A$, then since our operation is not the first projection, $bc=c$ for some $b, c\in A$, $b\neq c$. If $xc=c$ for all $x\in A$ then we have $a=ac=c$ and a is a zero element, contrary to our assumption. Therefore $dc=d$ for some $d\in A$, $d\neq c$. Thus $g(x)=xc$ shows $d, b, c\in B(d, b)$ and $|B(d, b)|\geqq 3$.

Let a and b be two arbitrary different elements and define a binary relation σ on the set $A\setminus\{a\}$ as follows: $x\,\sigma\, y$ if and only if $B(a, x)\subseteq B(a, y)$. Then σ is a reflexive transitive relation. Furthermore, if $\pi\in G_a$ and $x\,\sigma\, y$ then $B(a, x)\subseteq \subseteq B(a, y)$ implies $B(a, x\pi)=B(a, x)\pi\subseteq B(a, y)\pi=B(a, y\pi)$ and $x\pi\,\sigma\, y\pi$. Hence every permutation in G_a preserves σ. Clearly, σ is not the equality relation on $A\setminus\{a\}$. Indeed, if $c\in B(a, b)$, $c\neq a, b$, then $B(a, c)\subseteq B(a, b)$ shows that $c\,\sigma\, b$. By the assumption on G_a, σ cannot be a partial order, therefore we have $b_1\,\sigma\, b_2$ and $b_2\,\sigma\, b_1$ for some $b_1, b_2\in A\setminus\{a\}$, $b_1\neq b_2$. Thus, by definition, $B(a, b_1)=$

$=B(a, b_2)$. Choose a permutation $\varphi \in G_a$ such that $b_1\varphi = b$. Then we have $B(a, b) = B(a\varphi, b_1\varphi) = B(a, b_1)\varphi = B(a, b_2)\varphi = B(a, b_2\varphi)$. Put $b' = b_2\pi$. Clearly $b \neq b'$. If $\pi \in G_{a,b}$ then we have $B(a, b)\pi = B(a\pi, b\pi) = B(a, b)$. Let $\tau \in G_a$ be a permutation such that $b\tau = b'$. Then $B(a, b)\tau = B(a\tau, b\tau) = B(a, b') = B(a, b)$ and $B(a, b)\tau^{-1} = B(a, b')\tau^{-1} = B(a\tau^{-1}, b'\tau^{-1}) = B(a, b)$. Let us denote by G' the subgroup of G_a generated by the set $G_{a,b} \cup \{\tau, \tau^{-1}\}$. The above argument shows that $B(a, b)\pi = B(a, b)$ for all $\pi \in G'$. Since G_a is a primitive permutation group on $A \setminus \{a\}$, by Proposition 3, $G_{a,b}$ is a maximal subgroup in G_a. Therefore, $G_{a,b} \subset G' \subseteqq G_a$ implies $G' = G_a$. Thus we get that the set $B(a, b) \setminus \{a\}$ is a block of G_a. Again by the primitivity of G_a we have $B(a, b) \setminus \{a\} = \{b\}$ or $B(a, b) \setminus \{a\} = A \setminus \{a\}$. The first case cannot occur since $|B(a, b)| \geqq 3$. Hence $B(a, b) = A$. Since a and b were chosen arbitrarily, this means by definition exactly that the algebra $\mathfrak{B}$ has the $1^1/_2$-interpolation property and consequently the 2-interpolation property.

Now we are ready to prove that $G \cup \{f\} \subseteqq \operatorname{Pol} \varrho$ for no h-ary ($h \geqq 3$) totally reflexive and totally symmetric relation ϱ different from A^h. Let ϱ be an h-ary ($h \geqq 3$) totally reflexive and totally symmetric relation different from A^h such that $G \cup \{f\} \subseteqq \operatorname{Pol} \varrho$. Define a ternary relation $\hat{\varrho}$ as follows: $(a_1, a_2, a_3) \in \hat{\varrho}$ if and only if $(a_1, a_2, a_3, x_4, \ldots, x_h) \in \varrho$ for all $x_4, \ldots, x_h \in A$. Then, clearly, $\hat{\varrho}$ is a totally reflexive and totally symmetric relation with $\hat{\varrho} \neq A^3$. We show that if $g \in \operatorname{Pol} \varrho$ is a surjective operation then $g \in \operatorname{Pol} \hat{\varrho}$. Let n be the arity of g and let $(a_{i1}, a_{i2}, a_{i3}) \in \hat{\varrho}, i = 1, \ldots, n$. We have to show that $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3})) \in \hat{\varrho}$, i.e., $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3}), x_n, \ldots, x_h) \in \varrho$ for all $x_4, \ldots, x_h \in A$. Let $a_4, \ldots, a_h$ be arbitrary elements. Since g is surjective, $a_j = g(a_{1j}, \ldots, a_{nj})$ for some $a_{1j}, \ldots, a_{nj} \in A$, $j = 4, \ldots, h$. By definition $(a_{i1}, a_{i2}, a_{i3}, a_{i4}, \ldots, a_{ih}) \in \varrho$, $i = 1, \ldots, n$. Therefore we have $(g(a_{11}, \ldots, a_{n1}), g(a_{12}, \ldots, a_{n2}), g(a_{13}, \ldots, a_{n3}), a_4, \ldots, a_h) = (g(a_{11}, \ldots, a_{n1}), \ldots, g(a_{1h}, \ldots, a_{nh})) \in \varrho$ as required. Making use of this fact we immediately get that $G \cup \{f\} \subseteqq \operatorname{Pol} \hat{\varrho}$ and $I \subseteqq \operatorname{Pol} \hat{\varrho}$. Then $\hat{\varrho} \neq \iota_3$ since $\operatorname{Pol} \iota_3$ is known to consist of all unary operations and all operations taking on at most two values (see e.g. [8]) and I contains nontrivial idempotent operations.

Choose an element a from A arbitrarily and define a binary relation $\hat{\varrho}_a$ on A as follows: $(x, y) \in \hat{\varrho}_a$ if and only if $(x, y, a) \in \hat{\varrho}$. Taking into consideration the idempotency of the operations in I, it is easy to check that $I \subseteqq \operatorname{Pol} \hat{\varrho}_a$. Furthermore, $\hat{\varrho}_a \neq \iota_2$. Indeed, since $\hat{\varrho} \neq \iota_3$, there are pairwise different elements $c, d, e \in A$ such that $(c, d, e) \in \hat{\varrho}$. Choose a permutation $\pi \in G$ such that $e\pi = a$. Then we have $(c\pi, d\pi, a) = (c\pi, d\pi, e\pi) \in \hat{\varrho}$ and $(c\pi, d\pi) \in \hat{\varrho}_a$. Since $\hat{\varrho}_a$ is reflexive, every algebraic function of the algebra $\mathfrak{B} = \langle A; I \rangle$ preserves $\hat{\varrho}_a$. Making use of the fact that $\mathfrak{B}$ has the 2-interpolation property, we immediately get that $\hat{\varrho}_a = A^2$ and $A^2 \times \{a\} \subseteqq \hat{\varrho}$. Since a was chosen arbitrarily, it follows $\hat{\varrho} = A^3$, which is a contradiction, completing the proof of the theorem.

Corollary 9. *Let G be a doubly primitive permutation group on A such that for every $a \in A$ the stabilizer G_a admits no nontrivial partial order on the set $A \setminus \{a\}$. If G is affine with respect to no elementary 2-group then G is a basic group.*

Finally we give an example for a permutation group satisfying the assumption given in Theorem 8, which is not triply transitive.

Example. Let C be the set of all complex numbers and put $A = C \cup \{\infty\}$. Let G consist of all permutations π of A of the form $x\pi = \frac{ax+b}{cx+b}$ $(x \in A)$ where $a, b, c, d \in C$ with $|ad-bc| = 1$. Then G is doubly transitive and the stabilizer G_∞ consists of all permutations π of the form $x\pi = ax+b$ where $a, b \in C$ and $|a| = 1$. Thinking of the geometrical meaning of the permutations in G_∞, it is routine to show that G_∞ admits no nontrivial equivalence relation and nontrivial partial order on C.

References

[1] B. Csákány, Homogeneous algebras are functionally complete, *Algebra Universalis*, **11** (1980), 149—158.

[2] E. Fried, H. K. Kaiser and L. Márki, An elementary way for polynomial interpolation in universal algebras, *Algebra Universalis*, **15** (1982), 40—57.

[3] G. Grätzer, *Universal algebra*, 2nd ed., Birkhäuser Verlag, (Basel, 1979).

[4] H. K. Kaiser and L. Márki, Remarks on a paper of L. Szabó and Á. Szendrei, *Acta Sci. Math.*, **42** (1980), 95—98.

[5] R. A. Knoebel, Further conclusions on functional completeness, *Fund. Math.*, **99** (1978), 93—112.

[6] P. P. Pálfy, L. Szabó and Á. Szendrei, Algebras with doubly transitive automorphism groups, in: *Finite Algebra and Multiple-valued Logic* (Proc. Conf. Szeged, 1979), Colloq. Math. Soc. J. Bolyai, vol. 28, North-Holland (Amsterdam, 1981); pp. 321—335.

[7] P. P. Pálfy, L. Szabó and Á. Szendrei, Automorphism groups and functional completeness, *Algebra Universalis*, **15** (1982), 385—400.

[8] I. G. Rosenberg, Completeness properties of multiple-valued logic algebras, in: *Computer Science and Multiple-valued Logic. Theory and Applications* (ed. D. C. Rine), North Holland (Amsterdam, 1977).

[9] I. G. Rosenberg and L. Szabó, Local completeness, I, *Algebra Universalis*, to appear.

[10] A. A. Salomaa, A theorem concerning the composition of functions of several variables ranging over a finite set, *J. Symb. Logic*, **25** (1960), 203—208.

[11] A. A. Salomaa, On basic groups for the set of functions over a finite domain, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **338** (1963), 1—15.

[12] P. Schofield, Complete subsets of mappings over a finite domain, *Proc. Cambridge Phil. Soc.*, **62** (1966), 597—611.

[13] R. Scott, *Group Theory*, Prentice-Hall, Inc. (Englewood Cliffs, N. J., 1964).

[14] L. Szabó, Interpolation in algebras with doubly primitive automorphism groups, *Elektron. Informationsverarb. und Kybernet.*, **19** (1983), 603—610.

[15] L. Szabó and Á. Szendrei, Almost all algebras with triply transitive automorphism groups are functionally complete, *Acta Sci. Math.*, **41** (1979), 391—402.

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