Acta Sci. Math., 47 (1984), 85—92

On additive functions satisfying a congruence

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1. Let f, g, u, v be real-valued completely additive functions,

$$L_n = f(n)+g(n+1)+u(n+2)+v(n+3).$$

We shall prove the following

Theorem. *If* $L_n \equiv 0 \pmod 1$ *for every* $n \geqq 1$, *then* f, g, u, v *assume integer values for every* n.

Corollary. *If* $L_n = 0$ *for every* $n \geqq 1$, *then* f, g, u, v *are indentically zero-functions.*

For the proof of the Corollary see [1].

Let A_N denote the assertion:

$$A_N: f(N),\ g(N),\ u(N),\ v(N) \equiv 0 \pmod 1.$$

Let $\mathcal{P}$ denote the set of primes. For the sake of brevity we shall put $a \equiv b$ instead of $a \equiv b \pmod 1$.

We shall prove our Theorem in two steps. First we shall prove Theorem 1′, after then Lemma 1:

Theorem 1′. *Theorem is true if* A_N *is true for* $N \leqq 11$.

Lemma 1. *If* $L_n \equiv 0 \pmod 1$ *for every* $n \geqq 1$, *then* A_N *is true for* $N \leqq 11$.

2. Proof of Theorem 1′. Assume that Theorem 1′ does not hold. Then there exists a smallest N for which A_N does not hold. From $L_{N-3} \equiv 0 \pmod 1$ it follows that $v(N) \equiv 0 \pmod 1$. Furthermore,

$$0 \equiv L_{N-2} \equiv u(N)+v(N+1) \pmod 1.$$

If $N+1 \notin \mathcal{P}$, then $v(N+1) \equiv 0 \pmod 1$, and so $u(N) \equiv 0 \pmod 1$. If $N+1 \in \mathcal{P}$, then N is even, and so $u(N) \equiv 0 \pmod 1$. Hence it follows that $f(N) \not\equiv 0 \pmod 1$,

Received October 6, 1982.

or $g(N)\not\equiv 0 \pmod 1$, and that $N\in\mathcal{P}$. Let $N=P\in\mathcal{P}$. Now we distinguish three cases:

(I) $f(P)\equiv\xi,\quad g(P)\equiv\eta,\quad \xi\not\equiv 0,\quad \eta\not\equiv 0;$

(II) $f(P)\equiv 0,\quad g(P)\equiv\eta,\quad \eta\not\equiv 0;$

(III) $f(P)\equiv\xi;\quad g(P)\equiv 0,\quad \xi\not\equiv 0.$

Lemma 2. (1) *Let* $3P+b=2Z$, $b\equiv 1 \pmod 3$, $Z+1<2P$. *Then* $u(Z)\equiv 0$, $v(Z)\equiv 0$.

(2) *Let* $3P+c=2U$, $c\equiv -1 \pmod 3$, $U+1<2P$. *Then* $f(U)\equiv 0$, $g(U)\equiv 0$.

Proof. (1) We may assume that $Z\in\mathcal{P}$. Since $Z\equiv -1 \pmod 3$, therefore all the prime factors occurring in $Z-3$, $Z-2$, $Z-1$, $Z+1$ are smaller than P. From $L_{Z-2}\equiv 0$, $L_{Z-3}\equiv 0$ we get that $u(Z)\equiv 0$, $v(Z)\equiv 0$.

(2) We may assume that $U\in\mathcal{P}$. Since $U\equiv 1 \pmod 3$, therefore all the prime factors occurring in $U-1, U+1, U+2, U+3$ are smaller than P. From $L_U\equiv 0$, $L_{U-1}\equiv 0$ we get that $f(U)\equiv 0$, $g(U)\equiv 0$.

Case (I). Observing that $P-1$, $P+1$, $P+3$ are even numbers with prime factors $<P$, we get from $L_{P-1}\equiv 0$, $L_P\equiv 0$ that $U(P+2)\equiv -\xi$, $g(P+2)\equiv -\xi$, and so $P+2\in\mathcal{P}$, $\mathcal{P}\equiv -1 \pmod 3$. Similarly, in view of $2P+5\equiv 0 \pmod 3$, $2P+5\leqq 3P$, we see that $g(2P+3)\equiv -\xi$, $2P+3\in\mathcal{P}$. Since $2P-1\equiv 0 \pmod 3$, $2\mid P+1$, therefore $L_{2P-1}\equiv 0$ implies that $u(2P+1)\equiv -\xi$, $2P+1\in\mathcal{P}$.

Now we shall prove that $3P+2\not\equiv 0 \pmod 7$, i.e., $P\not\equiv 4 \pmod 7$. Indeed, if $7\mid 3P+2$, then from $L_{3P-1}\equiv 0$ we infer that

$$0\equiv f(3P-1)+g(3P)+u(3P+1),$$

which gives that $f(3P-1)\not\equiv 0$ or $u(3P+1)\equiv 0$, but this is impossible as it was proved in Lemma 2. Since $P\not\equiv 4 \pmod 7$, and P, $P+2, 2P+1, 2P+3\in\mathcal{P}$, we get that $P\not\equiv 0, 2, 3, 4, 5 \pmod 7$; consequently $P\equiv 1$ or $6 \pmod 7$.

First, by considering $L_{2P-2}\equiv 0$ we deduce that $v(2P+1)\equiv 0$, and hence, by $L_{4P-1}\equiv 0 \pmod 5$, and by taking into account that $5\mid 4P-1$ we get that $g(4P)+u(4P+1)\equiv 0$, i.e., $u(4P+1)\equiv -\xi$. So $4P+1=3R$, $u(R)\equiv -\eta$. It is obvious that $R\in\mathcal{P}$, since in the opposite case all its prime factors would be smaller than P. From $L_{R-2}\equiv 0$, by observing that $(R+1)/2<P$, we deduce that $f(R-2)\equiv\eta$, and so that $R-2\in\mathcal{P}$. Since $3(R-2)=4P-5$, therefore $f(4P-5)\equiv\eta$, and so

$$0\equiv\eta+g(4P-4)+u(4P-3)+v(4P-2)\equiv\eta+u(4P-3).$$

Since $2\nmid 4P-3$, $3\nmid 4P-3$, therefore $P\not\equiv 6 \pmod 7$.

It remains to consider the case $P\equiv 1 \pmod 7$. Then $3R\equiv 5 \pmod 7$, $R\equiv 4 \pmod 7$, $2R-1\equiv 0 \pmod 7$. Let us consider now

$$0\equiv L_{2R-2}\equiv f(2R-2)+g(2R-1)+u(2R)+v(2R+1).$$

Since $R, R-2\in\mathcal{P}$, therefore $R\equiv 1 \pmod 3$, consequently $3 \mid 2R+1$. Furthermore $(2R+1)/3<P+2$, and so $v(2R+1)\equiv 0$. Since $4 \mid 2R-2$, $7 \mid 2R-1$, therefore $f(2R-2)\equiv 0$, $g(2R-1)\equiv 0$, whence $u(R)\equiv 0$, which contradicts $u(R)\equiv -\eta$. So we have proved that Case (I) cannot occur.

Case (II). We get as earlier that $v(P+2)\equiv -\eta$, $P+2\in\mathcal{P}$, and so $P\equiv -1 \pmod 3$. Since $3 \mid 2P+1$, therefore from $L_{2P-1}\equiv 0$ we infer that $u(2P+1)\equiv -\eta$, $2P+1\in\mathcal{P}$. Lemma 2 implies that $f(3P-1)\equiv 0$, $u(3P+1)\equiv 0$, and so from $L_{3P-1}\equiv 0$ we deduce that $v(3P+2)\equiv -\eta$, $3P+2\in\mathcal{P}$. Since $P, P+2, 2P+1, 3P+2\in\mathcal{P}$, therefore $P\equiv -1 \pmod 5$. From $L_{2P-2}\equiv 0$ it follows that $v(2P+1)\equiv 0$, and so by $L_{4P-1}\equiv 0$, $5 \mid 4P-1$, we have

$$0 \equiv L_{4P-1} \equiv f(4P-1)+g(4P)+u(4P+1)+v(4P+2) \equiv 0+\eta+u(4P+1)+0,$$

hence $u(4P+1)\equiv -\eta$.

Thus $4P+1=3R$, $u(R)\equiv -\eta$, and so $R\in\mathcal{P}$. From $L_{R-2}\equiv 0$ we deduce that $f(R-2)\equiv\eta$, $R-2\in\mathcal{P}$. Consequently $R\equiv 1 \pmod 3$. Now we have $f(4P-5)= =f(3(R-2))\equiv\eta$, implying

$$0 \equiv L_{4P-5} = f(4P-5)+g(4P-4)+u(4P-3)+v(4P-2) \equiv \equiv \eta+0+u(4P-3)+v(2P-1). \tag{2.1}$$

Now we shall prove that $v(2P-1)\equiv 0 \pmod 1$. Indeed,

$$0 \equiv L_{2P-4} \equiv f(2P-4)+g(2P-3)+u(2P-2)+v(2P-1),$$

whence by $5 \mid 2P-3$ it follows immediately that $v(2P-1)\equiv 0$, and so from (2.1), $u(4P-3)\equiv -\eta$, $4P-3\in\mathcal{P}$. Since $P, P+2, 2P+1, 3P+2, 4P-3, 4P+3\in\mathcal{P}$, therefore $P\equiv 2 \pmod 7$. From $4P+1=3R$, $R\equiv 1 \pmod 8$ we get that $P\equiv 5 \pmod 9$.

Let us consider now the relation

$$0 \equiv f(5P-1)+g(5P)+u(5P+1)+v(5P+2).$$

We have $7 \mid 5P+2$, $6 \mid 5P-1$, and so $f(5P-1)\equiv 0$, $v(5P+2)\equiv 0$, yielding $u(5P+1)\equiv -\eta$. Thus $5P+1=4X$ or $5P+1=2X$ with a prime $X>P$. First we consider the case $5P+1=4X$. Since $u(X)\not\equiv 0$, therefore from $L_{X-2}\equiv 0$ we get that $f(X-2)\not\equiv 0$. But, from $P\equiv -1 \pmod 3$ we get that $X\equiv -1 \pmod 3$, $3 \mid X-2$, $(X-2)/3<P$, where $f(X-2)\equiv 0$.

It remains to consider the case $5P+1=2X$, $X\in\mathcal{P}$. We have $u(X)\equiv -\eta$. Furthermore $X\equiv 2 \pmod 7$. So

$$0 \equiv L_{X-2} \equiv f(X-2)+g(X-1)+u(X)+v(X+1).$$

Observing that $7 \mid X-2$, $6 \mid X-1$, and that $X/6<P$, we get that

$$v(X+1) \equiv \eta. \tag{2.2}$$

Taking into account that $2X+2=5P+3$, $9|5P+2$, from $L_{5P}\equiv 0$ we deduce that $g(5P+1)\equiv -\eta$, i.e., $g(X)\equiv -\eta$. This, together with

$$0\equiv f(X-1)+g(X)+u(X+1)+v(X+2),$$

$3\mid X+2$, and $(X+2)/3<P$, implies that

$$u(X+1)\equiv \eta. \tag{2.3}$$

Consequently $X+1=2Z$.

From (2.2) and (2.3) we get that $u(Z)\equiv\eta$, $v(Z)\equiv\eta$, $Z\in\mathcal{P}$. Using $(Z+1)/2<P$ and $2\mid Z+1, Z-1, Z-3$, we see immediately that $f(Z-2)\equiv -\eta$, $g(Z-2)\equiv -\eta$, $Z-2\in\mathcal{P}$. Since $2(Z-2)=X-3$, we have $g(X-3)\equiv -\eta$. Let us consider the relation

$$0\equiv f(X-4)+g(X-3)+u(X-2)+v(X-1).$$

In view of $X\equiv 2$ (7), $u(X-2)\equiv 0$. Furthermore $2, 3\mid X-1$, and so $v(X-1)\equiv 0$. Consequently $f(X-4)\equiv\eta$. But this is impossible, since $3\mid X-4$, $(X-4)/3<P$.

Case (III). From $L_P\equiv 0$ we get that $u(P+2)\equiv -\xi$, $P+2\in\mathcal{P}$. Hence $P\equiv -1$ (mod 3). Observing that $3\mid 2P+5$, we get from $L_{2P+2}\equiv 0$, that

$$g(2P+3)\equiv \xi, \quad 2P+3\in\mathcal{P}. \tag{2.4}$$

Let us consider now the relation

$$f(3P+4)+g(3P+5)+u(3P+6)+v(3P+7)\equiv 0.$$

From Lemma 2 we get that $g(3P+5)\equiv 0$, $v(3P+7)\equiv 0$, thus $f(3P+4)\equiv\xi$, $3P+4\in\mathcal{P}$. Since, $P, P+2, 2P+3, 3P+4\in\mathcal{P}$, therefore $P\equiv -1$ (mod 5).

Furthermore $L_{2P+3}\equiv 0$ immediately implies that $f(2P+3)\equiv 0$. Thus, by $5\mid 4P+9$, we get that

$$0\equiv L_{4P+6}\equiv f(4P+6)+g(4P+7)+u(4P+8)+v(4P+9)\equiv$$
$$\equiv 0+g(4P+7)+u(P+2)+0,$$

i.e., $g(4P+7)\equiv\xi$.

Let $4P+7=3E$, $g(E)\equiv\xi$, $E\in\mathcal{P}$. From $L_{E-1}\equiv 0$ we deduce that $v(E+2)\equiv -\xi$. Hence it follows that $E\equiv -1$ (mod 3) and so $P\equiv 2$ (mod 9). Now we prove that $u(E)\equiv 0$. Indeed, in the opposite case from $L_{E-2}\equiv 0$ it would follow that $f(E-2)\not\equiv 0$, but this is impossible since $3\mid E-2$, $(E-2)/3<P$.

So we have that $u(3E)\equiv u(4P+7)\equiv 0$. Then

$$0\equiv f(4P+5)+g(4P+6)+u(4P+7)+v(4P+8)\equiv$$
$$\equiv f(4P+5)+g(2P+3)+0+v(P+2).$$

From $L_{P-1}\equiv 0$ we get that $g(P)+v(P+2)\equiv 0$, and so $v(P+2)\equiv 0$. Using (2.4) we see that $f(4P+5)\equiv -\xi$, $4P+5\in\mathcal{P}$. Since $P\equiv -1$ (mod 5), we get that $E\equiv 1$

(mod 5). Consequently $5 \mid 2E+3$. So

$$0 \equiv f(2E+1)+g(2E+2)+u(2E+3)+v(2E+4) \equiv f(2E+1)+0+0-\xi,$$

i.e., $f(2E+1)\equiv\xi$, $2E+1\in\mathcal{P}$. Similarly, $3 \mid 2E-1$, therefore

$$0 \equiv f(2E-1)+g(2E)+u(2E+1)+v(2E+2) \equiv 0+\xi+u(2E+1)+0,$$

i.e., $u(2E+1)\equiv-\xi$. We have $3(2E+1)=8P+17$, hence

$$0 \equiv f(8P+15)+g(8P+16)+u(8P+17)+v(8P+18) \equiv$$
$$\equiv f(8P+15)+g(P+2)-\xi+v(4P+9).$$

Since $4 \mid P+4$, we get from $L_{P-1}\equiv0$ that $g(P+2)\equiv0$. Also, $5 \mid 4P+9$ implies that $v(4P+9)\equiv0$. Thus we have that $f(8P+15)\equiv\xi$.

Hence $8P+15$ has to be a prime or the product of 7 and K, where $K\in\mathcal{P}$, $f(K)\equiv\xi$. Assume that $8P+15=7K, f(K)\equiv\xi$. Then we get from $L_K\equiv0$ that $u(K+2)\equiv-\xi$, $K+2\in\mathcal{P}$. But $8P+15\equiv7K$, $P\equiv-1 \pmod 3$ imply that $3 \mid K+2$, and hence, by $(K+2)/2<P$, $u(K+2)\equiv0$.

So $8P+15\in\mathcal{P}$. Since $P, P+2, 2P+3, 3P+4, 4P+5, 8\mathcal{P}+15\in\mathcal{P}$, therefore $P\equiv3 \pmod 7$. Let us consider now the relation

$$0 \equiv f(5P+8)+g(5P+9)+u(5P+10)+v(5P+11).$$

Since $9 \mid 5P+8$, $6 \mid 5P+11$, and $u(5P+10)\equiv u(P+2)\equiv-\xi$, therefore $f(5P+8)\equiv0$, $v(5P+11)\equiv0$, and so $g(5P+9)\equiv\xi$. Then $5P+9=2A$, or $5P+9=4A$, where $A\in\mathcal{P}$, $g(A)\equiv\xi$. The second case cannot occur. Let us assume that $5P+9=4A$, $g(A)\equiv\xi$. Then, taking into account that $2 \mid A-1$, $A+1$, $(A+1)/2<P$, we get from $L_{A-1}\equiv0$ that $v(A+2)\equiv-\xi$. But this is impossible since $3 \mid A+2$.

Let us assume that $5P+9=2A$. It follows from $P\equiv3 \pmod 7$ that $A\equiv5 \pmod 7$, i.e., $7 \mid A+2$. Furthermore, $3 \mid A+1$, $(A+1)/3<P$, consequently $u(A+1)\equiv0$, $v(A+2)\equiv0$, and so $L_{A-1}\equiv0$ immediately implies that $f(A-1)\equiv-\xi$. Since $A-1$ is an even number and has a prime divisor greater than P, therefore $A-1=2B$, $B\in\mathcal{P}$, $f(B)\equiv-\xi$. From $L_B\equiv0$ we deduce that $u(B+2)\equiv\xi$. Since $5P+7=4B$, $9 \mid 5P+8$, $v(P+2)\equiv0$, we get

$$0 \equiv f(5P+7)+g(5P+8)+u(5P+9)+v(5P+10) \equiv -\xi+u(2A),$$

i.e., $u(A)\equiv\xi$. So we have

$$f(A-2)+g(A-1)+u(A)+v(A+1) \equiv 0.$$

Since $3 \mid A-2, A+1$, and $(A+1)/3<P$, therefore $f(A-2)\equiv0$, $v(A+1)\equiv0$, and so $g(A-1)\equiv g(B)\equiv-\xi$. In view of $L_{B-1}\equiv0$ this yields that

$$v(B+2) \equiv \xi. \tag{2.5}$$

Since $2B+4=A+3$, we have that $u(A+3)\equiv\xi$, $v(A+3)\equiv\xi$. Let us consider now the relation

$$f(A+1)+g(A+2)+u(A+3)+v(A+4)\equiv 0.$$

Since $7\mid A+2$, therefore $g(A+2)\equiv 0$. Furthermore $3\mid A+1$, $3\mid A+4$. As

$$(A+1)/3=(2A+2)/6=(5P+11)/6<P$$

for $P>11$, we have $f(A+1)\equiv 0$. We know that $v(P)\equiv 0$ and $v(P+2)\equiv 0$. Since $P, P+2\in\mathcal{P}$, therefore $P+4$ is a composite number, and so the smallest integer on which v assumes a nonzero value (mod 1) is $\geqq P+6$. However,

$$(A+4)/3=(2A+8)/6=(5P+17)/6<P+6,$$

therefore $v(A+4)\equiv 0$. Consequently $u(A+3)\equiv 0$, contradicting (2.5).

The proof of Theorem 1′ is finished.

3. Proof of Lemma 1. For an arbitrary completely additive function $h(n)$ we can extend the domain of definition for the set of positive rational numbers by $h(a/b)=h(a)-h(b)$. Let us do it for f, g, u, v. For the sake of brevity the relation

$$f(a)+g(b)+u(c)+v(d)\equiv 0 \pmod 1$$

will be denoted by $\langle a, b, c, d\rangle$, where a, b, c, d are arbitrary positive rational numbers.

From the additivity it follows that

$$\text{if}\quad \langle a, b, c, d\rangle \quad\text{and}\quad \langle A, B, C, D\rangle, \quad\text{then}\quad \langle aA, bB, cC, dD\rangle.$$

We shall say that $\langle aA, bB, cC, dD\rangle$ is the product of $\langle a, b, c, d\rangle$ and $\langle A, B, C, D\rangle$. It is obvious that $\langle 1/a, 1/b, 1/c, 1/d\rangle$ holds if $\langle a, b, c, d\rangle$ holds.

Let now $L_n=\langle n, n+1, n+2, n+3\rangle$. First we shall express the values $f(p)$, $g(p)$, $u(p)$, $v(p)$ for primes $p\leqq 20$ as linear combinations of

$$K=\{f(2),\ g(2),\ u(2),\ v(2),\ f(3),\ g(3),\ u(3),\ v(3)\}.$$

The appropriate formulas will be denoted by $F(p)$, $G(p)$, $U(p)$, $V(p)$. Hence we can get some linear relations between the values listed in K.

$$V(5)=L_2=\langle 2;\ 3;\ 2^2;\ 5\rangle,$$

$$U(5)=L_3=\langle 3;\ 2^2;\ 5;\ 2\cdot 3\rangle,$$

$$F(7)=L_7L_2^{-1}=\langle 7\cdot 2^{-1};\ 2^3\cdot 3^{-1};\ 2^{-2}\cdot 3^2;\ 2\rangle,$$

$$G(7)=L_6=\langle 2\cdot 3;\ 7;\ 2^3;\ 3^2\rangle,$$

$$V(11)=L_8L_3^{-1}=\langle 2^3\cdot 3^{-1};\ 2^{-2}\cdot 3^2;\ 2;\ 11\cdot 2^{-1}\cdot 3^{-1}\rangle,$$

$$V(17)=L_{48}L_3^{-2}L_6^{-2}=\langle 2^2\cdot 3^{-3};\ 2^{-4};\ 2^{-5};\ 17\cdot 2^{-2}\cdot 3^{-5}\rangle,$$

$$G(5)=L_{14}(F(7)V(17))^{-1}=\langle 3^3;\ 2\cdot 3^2\cdot 5;\ 2^{11}\cdot 3^{-2};\ 2\cdot 3^5\rangle,$$

$$V(7) = L_4 G(5)^{-1} = \langle 2^2\cdot 3^{-3};\ 2^{-1}\cdot 3^{-2};\ 2^{-10}\cdot 3^3;\ 7\cdot 2^{-1}\cdot 3^{-5}\rangle,$$

$$F(11) = L_{24}L_4L_{11}^{-1}G(5)^{-3} = \langle 11^{-1}\cdot 2^5\cdot 3^{-8};\ 2^{-5}\cdot 3^{-7};\ 2^{-31}\cdot 3^7;\ 2^{-4}\cdot 3^{-12}\rangle,$$

$$F(5^3) = L_5L_{25}V(5)L_{12}^{-1}V(7)^{-1} = \langle 5^3\cdot 2^{-3}\cdot 3^2;\ 2^3\cdot 3^4;\ 2^{11};\ 2^6\cdot 3^4\rangle,$$

$$V(13) = \frac{L_{54}L_{15}L_{12}^2L_7L_3F(5^3)}{L_{168}L_{10}L_5^3L_2^3G(5)} = \langle 2^{-5}\cdot 3^5;\ 2^8\cdot 3^{-4};\ 2^{-4}\cdot 3^3;\ 13^{-1}\cdot 2^{-1}\cdot 3^3\rangle,$$

$$F(5) = L_{75}L_4L_3V(13)L_5^{-1}L_{18}^{-1}L_9^{-1} = \langle 5\cdot 2^{-4}\cdot 3^3;\ 2^{10}\cdot 3^{-5};\ 2^{-5}\cdot 3^4;\ 2^{-4}\cdot 3^3\rangle,$$

$$U(7) = F(5)L_5^{-1} = \langle 2^{-4}\cdot 3^3;\ 2^9\cdot 3^{-6};\ 7^{-1}\cdot 2^{-5}\cdot 3^4;\ 2^{-7}\cdot 3^3\rangle,$$

$$G(13) = L_{12}U(7)L_2^{-1} = \langle 2^{-3}\cdot 3^4;\ 13\cdot 2^9\cdot 3^7;\ 2^{-6}\cdot 3^4;\ 2^{-7}\cdot 3^4\rangle,$$

$$G(11) = F(5)V(13)^{-1}L_{10}^{-1} = \langle 3^{-4};\ 11^{-1}\cdot 2^2\cdot 3^{-1};\ 2^{-3};\ 2^{-3}\rangle,$$

$$U(17) = F(5)L_{15}^{-1} = \langle 2^{-4}\cdot 3^2;\ 2^6\cdot 3^{-5};\ 17^{-1}\cdot 2^{-5}\cdot 3^4;\ 2^{-4}\cdot 3\rangle,$$

$$U(13) = F(11)L_{11}V(7)^{-1} = \langle 2^3\cdot 3^{-5};\ 2^{-2}\cdot 3^{-4};\ 13\cdot 2^{-21}\cdot 3^4;\ 2^{-2}\cdot 3^{-7}\rangle,$$

$$U(11) = L_9G(5)^{-1} = \langle 3^{-1};\ 3^{-2};\ 11\cdot 2^{-11}\cdot 3^2;\ 2\cdot 3^{-4}\rangle,$$

$$V(19) = L_{54}G(11)U(7)G(5)^{-1} = \langle 2^{-3}\cdot 3^{-1};\ 2^{10}\cdot 3^{-9};\ 2^{-16}\cdot 3^6;\ 19\cdot 2^{-11}\cdot 3^{-1}\rangle,$$

$$G(17) = V19)L_{16}^{-1} = \langle 2^{-7}\cdot 3^{-1};\ 17^{-1}\cdot 2^{10}\cdot 3^{-9};\ 2^{-17}\cdot 3^4;\ 2^{-11}\cdot 3^{-1}\rangle,$$

$$F(17) = \frac{L_{85}L_9L_6L_7L_2}{L_{42}L_{27}F(5)G(5)V(11)} = \langle 17\cdot 2^2\cdot 3^{-6};\ 3^2\cdot 2^{-6};\ 3\cdot 2^{-4};\ 2^9\cdot 3^{-7}\rangle,$$

$$U(19) = F(17)L_2L_{17}^{-1} = \langle 2^3\cdot 3^{-6};\ 2^{-7}\cdot 3;\ 19^{-1}\cdot 2^{-2}\cdot 3;\ 2^7\cdot 3^{-7}\rangle,$$

$$G(19) = L_{18}V(7)^{-1}L_3^{-2} = \langle 2^{-1}\cdot 3^3;\ 19\cdot 2^{-3}\cdot 3^2;\ 2^{11}\cdot 3^{-3};\ 2^{-1}\cdot 3^4\rangle,$$

$$F(19) = L_{19}U(7)V(11)^{-1}G(5)^{-1} = \langle 19\cdot 2^{-5}\cdot 3;\ 2^{12}\cdot 3^{-10};\ 2^{-17}\cdot 3^7;\ 2^{-6}\cdot 3^{-1}\rangle.$$

$$F_1 := L_1 = \langle 1;\ 2;\ 3;\ 2^2\rangle,$$

$$F_2 := \frac{F(5)^3L_1^{12}}{F(5)^3} = \left\langle \frac{2^9}{3^7};\ \frac{3^{19}}{2^{15}};\ 2^{26};\ \frac{2^{42}}{3^5}\right\rangle,$$

$$F_3 := \frac{L_{340}L_2L_8G(11)L_1^7}{L_{17}L_{30}L_3V(7)^3} = \left\langle \frac{3^2}{2};\ 2^9\cdot 3^6;\ 2^{25};\ 2^{11}\cdot 3^{13}\right\rangle,$$

$$F_4 := \frac{L_{133}L_{20}L_{15}L_2G(5)^2L_1^{21}}{L_{66}L_{14}L_9L_3L_7L_6F(19)F(11)F(5)} = \left\langle 2^6;\ 2^7\cdot 3^{27};\ 2^{69};\ \frac{2^{58}}{3^{17}}\right\rangle,$$

$$F_5 := \frac{L_{32}G(11)U(17)}{L_2L_1V(7)} = \left\langle \frac{3}{2^2};\ \frac{2^8}{3^4};\ 2;\ \frac{3^6}{2^9}\right\rangle,$$

$$F_6 := \frac{L_3L_1^{15}}{L_{33}F(11)G(17)U(7)} = \langle 2^6\cdot 3^6;\ 2^2\cdot 3^{22};\ 2^{53};\ 2^{63}\cdot 3^9\rangle,$$

$$F_7 := \frac{L_{31}L_6L_4L_2L_1^8}{L_{62}L_9} = \langle 2^4;\ 2^5\cdot 3^5;\ 2^{19};\ 2^7\cdot 3^8\rangle,$$

$$F_8 := \frac{L_7^2 G(5)^2 L_1^8}{L_{49}L_2^2 U(17)V(13)} = \left\langle \frac{2^7}{3};\ 2\cdot 3^{11};\ 2^{27};\ 2^{24}\cdot 3^6\right\rangle,$$

$$F_9 := \frac{L_{63}L_2L_1^6}{L_8L_7U(13)} = \left\langle \frac{3^7}{2^5};\ 2^{11}\cdot 3^3;\ 2^{22};\ 2^{14}\cdot 3^8\right\rangle,$$

$$F_{10} := \frac{L_{55}L_{13}F(11)U(19)}{L_{26}L_6^3L_3U(7)F(5)L_1^2} = \left\langle \frac{2^{13}}{3^{23}};\ \frac{3^2}{2^{31}};\ \frac{1}{2_{25}};\ \frac{2^{14}}{3^{30}}\right\rangle,$$

$$F_{11} := \frac{L_{37}L_8L_4^2L_1^2}{L_{74}L_{18}L_2U(19)U(13)} = \left\langle \frac{3^9}{2^2};\ 2^{12}\cdot 3^3;\ 2^{20};\ 2^2\cdot 3^{13}\right\rangle,$$

$$F_{12} := \frac{L_{30}L_{23}L_{45}L_2^3V(19)V(13)L_1^7}{L_{92}L_{22}L_8L_3F(11)F(5)^2} = \left\langle \frac{3^{10}}{2^7};\ 2^{12}\cdot 3^5;\ 2^{27};\ 2^{18}\cdot 3^9\right\rangle.$$

Let R denote the column vector $[f(2,f(3), g(2), g(3), u(2), v(2), v(3)]$. The formulas $F_2, \ldots, F_{12}$ lead to the linear equation $MR \equiv 0 \pmod 1$, where

$$M = \begin{bmatrix} 9 & -7 & -15 & 19 & 26 & 42 & -5 \\ -1 & 2 & 9 & 6 & 25 & 11 & 13 \\ 6 & 0 & 7 & 27 & 69 & 58 & -17 \\ -2 & 1 & 8 & -4 & 1 & -9 & 6 \\ 6 & 6 & 2 & 22 & 53 & 63 & 9 \\ 4 & 0 & 5 & 5 & 19 & 7 & 8 \\ 7 & -1 & 1 & 11 & 27 & 24 & 6 \\ -5 & 7 & 11 & 3 & 22 & 14 & 8 \\ 13 & -23 & -31 & 2 & -25 & 14 & -30 \\ -2 & 9 & 12 & 3 & 20 & 2 & 13 \\ -7 & 10 & 12 & 5 & 27 & 18 & 9 \end{bmatrix}.$$

By using the Gaussian elimination over the ring of integers, we get easily that the only solution of it is $R \equiv 0 \pmod 1$. Hence, by the formulas $F(p), \ldots, V(p)$ we get immediately that $f(p), g(p), u(p), v(p) \equiv 0 \pmod 1$ for $p \leqq 19$.

References

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