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On the spectral residuum of closed operators

WANG SHENGWANG

1. Introduction

The spectral residuum [5], [2] of a linear operator T is a minimal closed subset S of the spectrum $\sigma(T)$, on whose complement T possesses the spectral properties of decomposable operators. It was shown in [2] that for every bounded linear operator there exists a spectral residuum. It is the purpose of the present paper to extend this property to the class of all closed operators which map a Banach space X into itself.

Throughout this paper, T denotes a closed operator with domain D_T and range in a complex Banach space X. $\mathbf{C}$ is the complex plane and $\mathbf{C}_\infty$ denotes its one-point compactification. All topological attributes for sets in $\mathbf{C}_\infty$ will be referred to the topology of $\mathbf{C}_\infty$. If $E\subset\mathbf{C}_\infty$, then $E^c=\mathbf{C}_\infty-E$ and $\bar{E}$ is the closure of E. For all operators involved in this paper, $\sigma(\cdot)$ denotes the extended spectrum. For a linear operator A, $\varrho(A)$ is the resolvent set and $R(\cdot\,;A)$ denotes the resolvent operator. Further notations will be given later.

We recall some basic concepts from [2], [5] and [6]. For $x\in X$ and $\lambda\in\mathbf{C}_\infty$, $\lambda\in\delta_T(x)$ if there exists a neighborhood U of λ and there is a function $f_x\colon U\to D_T$, analytic on U such that

$$(\mu-T)f_x(\mu)=x,\quad \mu\in U\cap\mathbf{C}.$$

Such a function f_x is said to be T-associated with x. Given T, there exists a unique maximal open set $\Omega_T\subset\mathbf{C}_\infty$ such that, for every set $G\subset\Omega_T$ and every analytic function f defined on G, the equation

$$(\mu-T)f(\mu)=0,\quad \mu\in G\cap\mathbf{C}$$

implies that $f(\mu)=0$ on G. Put $S_T=\Omega_T^c$ and for any $x\in X$, let

$$\gamma_T(x)=\delta_T(x)^c,\quad \sigma_T(x)=\gamma_T(x)\cup S_T,\quad \varrho_T(x)=\sigma_T(x)^c.$$

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Given T and $F\subset\mathbf{C}_\infty$, define the linear manifold

$$X_T(F) = \{x\in X:\ \sigma_T(x) \subset F\},$$

which is non void only if $F\supset S_T$ [5].

For a subspace (closed linear manifold) Y of X, we write $Y\in I(T)$ if $T(Y\cap D_T)\subset Y$ and $Y\in I_T$ if $Y\subset D_T$ and $T(Y)\subset Y$. For a closed $F\subset\mathbf{C}_\infty$, define

$$I(T, F) = \{Y\in I(T):\ \sigma(T|Y) \subset F\},\quad I_{T,F} = \{Y\in I_T:\ \sigma(T|Y) \subset F\}.$$

The inclusion $\subset$ defines a partial ordering in the families $I(T, F)$ and $I_{T,F}$. If $I(T, F)$, $(I_{T,F})$ has an upper bound belonging to $I(T, F)$, $(I_{T,F})$, denote it by $X(T, F)$ (resp. $X_{T,F}$).

$Y\in I(T)$ is said to be a spectral maximal space of T if, for every $Z\in I(T)$, the relation $\sigma(T|Z)\subset\sigma(T|Y)$ implies $Z\subset Y$. It follows easily that if $F\subset\mathbf{C}_\infty$ is closed and $X(T, F)$ exists, then $X(T, F)$ is a spectral maximal space of T. Conversely, if Y is a spectral maximal space of T, then $Y=X(T, F)$, with $F=\sigma(T|Y)$.

Let $S\subset\mathbf{C}_\infty$ be closed and let n be a given positive integer. The family of open sets $\{G_S, G_1, \ldots, G_n\}$ is called an (S, n)-covering of a closed set F, if

$$G_S\cup\left(\bigcup_{i=1}^{n} G_i\right) \supset F\cup S,\quad \bar{G}_i\cap S = \emptyset \quad \text{for} \quad i = 1, 2, \ldots, n.$$

1.1. Definition. Given T, suppose $S\subset\sigma(T)$ is closed and n is a positive integer. T is called (S, n)-decomposable if, for any (S, n)-covering $\{G_S, G_1, G_2, \ldots, G_n\}$ of $\sigma(T)$, there exist spectral maximal spaces $X_i\subset D_T$, $(i=1, 2, \ldots, n)$ and X_S of T, such that

$$X = X_S + \sum_{i=1}^{n} X_i,\quad \sigma(T|X_S) \subset \bar{G}_S,\quad \sigma(T|X_i) \subset \bar{G}_i \quad (i = 1, 2, \ldots, n).$$

If T is (S, n)-decomposable for every positive integer n, then T is called S-decomposable.

Next, we list a few known properties that will be used in the subsequent theory.

1.2. Lemma. [3] *Given T, let F be closed such that $S_T\subset F\subset\mathbf{C}_\infty$. If $X_T(F)$ is closed, then $X_T(F)=X(T, F)$.*

1.3. Lemma. [3] *If T is $(S, 1)$-decomposable, then $S_T\subset S$.*

1.4. Lemma. [3] *If T is $(S, 1)$-decomposable and $F\supset S$ is closed, then*

$$X_T(F) = X(T, F) \quad \text{and} \quad \sigma[T|X_T(F)] \subset F.$$

1.5. Lemma. [2, 7] *If T and $Y\in I(T)$ are such that $\sigma(T)\cup\sigma(T|Y)\neq\mathbf{C}$, then the coinduced operator $\hat{T}$ on the quotient space X/Y is closed and $\sigma(\hat{T})\subset\sigma(T)\cup\sigma(T|Y)$, $\sigma(T|Y)\subset\sigma(T)\cup\sigma(\hat{T})$, $\sigma(T)\subset\sigma(\hat{T})\cup\sigma(T|Y)$.*

1.6. Lemma. [8] *Given T, every spectral maximal space Y of T is hyperinvariant under T, in particular, $\sigma(T|Y)\subset\sigma(T)$.*

1.7. Theorem. [1] *Given T, for every $x\in X$ and $\lambda_0\in\mathbf{C}$, the following assertions are equivalent:*

(i) *there is a neighborhood $\delta\subset\mathbf{C}$ of λ_0 and there is a function $f\colon \delta\to D_T$, analytic on δ, satisfying*

$$(\lambda-T)f(\lambda)=x;$$

(ii) *there are numbers $M>0$, $R>0$ and a sequence $\{a_n\}_{n=0}^{\infty}\subset D_T$ with the following properties:*

(a) $(\lambda_0-T)a_0=x$; (b) $(\lambda_0-T)a_{n+1}=a_n$; (c) $\|a_n\|\leqq MR^n$ $(n=0,1,\ldots)$.

2. Some properties of $(S,1)$-decomposable operators

2.1. Theorem. *Suppose that T is $(S,1)$-decomposable, H is closed in $\mathbf{C}_\infty$, $H\cap S=\emptyset$. Then $X_{T,H}$ exists and*

$$X_T(S\cup H)=X_T(S)\oplus X_{T,H}. \tag{2.1}$$

Proof. Put $F=S\cup H$. Lemma 1.4 implies that

$$X_T(F)=X(T,F)\quad\text{and}\quad \sigma[T|X_T(F)]\subset F=S\cup H.$$

Refer to [3, Theorem 1], consider $S_1=S_2=S$ in the hypotheses of Part (2) of the proof, note that the proof holds for $(S_i,1)$-decomposable operators $(i=1,2)$ and conclude that $X_{T,H}$ exists and

$$X_T(F)=Z_S\oplus X_{T,H} \tag{2.2}$$

where $Z_S\in I(T)$ and $\sigma(T|Z_S)\subset S$. It remains to show that $Z_S=X_T(S)$. The existence of $X_T(S)$ follows from Lemma 1.4 and the inclusion $Z_S\subset X_T(S)$ is evident. Since $\sigma[T|X_T(S)]\subset S\subset F$, we have $X_T(S)\subset X_T(F)$. Letting $\sigma_H=\sigma(T|X_{T,H})$, it follows from [3, Theorem 1] that σ_H is bounded. Let D be a bounded Cauchy domain such that $\sigma_H\subset D\subset\bar{D}\subset S^c$, with the positively oriented boundary ∂D. Put

$$P_H=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(F)]^{-1}\,d\lambda.$$

It follows from $X_T(S)\subset X_T(F)$ and $\sigma[T|X_T(S)]\subset S$, that for every $x\in X_T(S)$, we have

$$P_Hx=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(F)]^{-1}x\,d\lambda=\frac{1}{2\pi i}\int_{\partial D}[\lambda-T|X_T(S)]^{-1}x\,d\lambda=0.$$

Therefore, $X_T(S) \subset Z_S$ and hence

$$X_T(S) = Z_S. \tag{2.3}$$

Relations (2.2) and (2.3) conclude the proof.

2.2. Remark. By the method used in the proof of Theorem 2.1, we can actually prove a more general result: If T is $(S, 1)$-decomposable and F, H are disjoint closed sets with $F \supset S$, then

$$X_T(F \cup H) = X_T(F) \oplus X_{T,H}.$$

2.3. Theorem. *Suppose that T is $(S, 1)$-decomposable and F, H are closed sets with $F \supset S$ and $S \cap H = \emptyset$. Then*

$$X_T(F) \cap X_{T,H} = X_{T,F\cap H}. \tag{2.4}$$

Proof. By Theorem 2.1, we have

$$X_T(S \cup H) = X_T(S) \oplus X_{T,H}, \quad X_T[S \cup (F \cap H)] = X_T(S) \oplus X_{T,F\cap H}.$$

Consequently,

$$[X_T(S) \oplus X_{T,H}] \cap X_T(F) = X_T(S \cup H) \cap X_T(F) = X_T[(S \cup H) \cap F] = \tag{2.5}$$
$$= X_T[S \cup (F \cap H)] = X_T(S) \oplus X_{T,F\cap H}.$$

The following evident relations

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(F),$$

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(S) \oplus X_{T,H}$$

imply

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset [X_T(S) \oplus X_{T,H}] \cap X_T(F). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$X_T(S) + [X_T(F) \cap X_{T,H}] \subset X_T(S) \oplus X_{T,F\cap H}. \tag{2.7}$$

Since, evidently

$$X_T(F) \cap X_{T,H} \supset X_{T,F\cap H}, \tag{2.8}$$

(2.7) is actually an equality. Moreover, the left-hand side of (2.7) being a direct sum, we obtain

$$X_T(S) \oplus [X_T(F) \cap X_{T,H}] = X_T(S) \oplus X_{T,F\cap H}. \tag{2.9}$$

Now, (2.8) and (2.9) imply that

$$X_T(F) \cap X_{T,H} = X_{T,F\cap H}$$

and hence (2.4) follows.

2.4. Theorem. *Suppose that T is $(S, 1)$-decomposable, and H_1, H_2 are closed disjoint sets such that $H_i \cap S = \emptyset$, $i=1, 2$. Then*

$$X_{T, H_1 \cup H_2} = X_{T, H_1} \oplus X_{T, H_2}. \tag{2.10}$$

Proof. It follows from the relations

$$X_{T, H_1} \cap X_{T, H_2} \subset X_{T, H_1} \cap [X_T(S) \oplus X_{T, H_2}] = X_{T, H_1} \cap X_T(S \cup H_2)$$

and from Remark 2.2, that

$$X_{T, H_1} \cap X_{T, H_2} = \{0\}. \tag{2.11}$$

Since $X_{T, H_1 \cup H_2} \supset X_{T, H_i}$ $(i=1, 2)$, (2.10) would follow from (2.11) if we could prove

$$X_{T, H_1 \cup H_2} \subset X_{T, H_1} + X_{T, H_2}. \tag{2.12}$$

Let $V = T|X_{T, H_1 \cup H_2}$. Then $\sigma(V) \subset H_1 \cup H_2$. Therefore, the sets $\sigma_{H_i} = \sigma(V) \cap H_i$ are disjoint spectral sets of V. It follows from [4, V. Theorem 9.2] that

$$X_{T, H_1 \cup H_2} = Z_{H_1} \oplus Z_{H_2} \quad \text{and} \quad \sigma(V|Z_{H_i}) = \sigma_{H_i} \quad (i = 1, 2).$$

Since V is bounded, $T|Z_{H_i} = V|Z_{H_i}$ are also bounded and then $Z_{H_i} \in I_{T, H_i}$ $(i=1, 2)$. Hence $Z_{H_i} \subset X_{T, H_i}$ $(i=1, 2)$ and (2.12) follows.

2.5. Theorem. *Suppose that T is $(S, 1)$-decomposable and H is a closed set satisfying $H \cap S = \emptyset$. Then $X(T, H)$ exists and $X(T, H) = X_{T, H}$.*

Proof. By Theorem 2.1, $X_{T, H}$ exists. If S is bounded then T is bounded [6, Proposition 3.1] and the statement of the theorem is evident. So suppose that $\infty \in S$. Then $H \cap S = \emptyset$ implies that H is bounded. As we mentioned in the Introduction, for every operator appearing in this paper, we consider the extended spectrum. Hence, for each $Y \in I(T, H)$, $\sigma(T|Y) \subset H$ implies that the extended spectrum $\sigma(T|Y)$ is bounded. Then $Y \in I_{T, H}$ and hence $I(T, H) \subset I_{T, H}$. On the other hand, we evidently have $I_{T, H} \subset I(T, H)$. Thus,

$$I(T, H) = I_{T, H} \tag{2.13}$$

and the conclusion of the proof follows immediately from (2.13).

2.6. Theorem. *Suppose that T is $(S, 1)$-decomposable and G is open in $\mathbf{C}_\infty$ such that $\bar{G} \cap S = \emptyset$. Then the coinduced operator $T^{\bar{G}}$ on the quotient space $X/X_{T, \bar{G}}$ is closed and $\sigma(T^{\bar{G}}) \subset G^c$.*

Proof. Let $\lambda \in G$ and let $G_S \supset S$ be open in $\mathbf{C}_\infty$ such that $\{G_S, G\}$ is an $(S, 1)$-covering of $\sigma(T)$ and $\lambda \notin \bar{G}_S$. By Lemma 1.4 and Theorem 2.5, $X_T(\bar{G}_S)$ and $X_{T, \bar{G}}$ are spectral maximal spaces of T. Consequently,

$$X = X_T(\bar{G}_S) + X_{T, \bar{G}}. \tag{2.14}$$

Let $\cong$ denote the topological isomorphism between two Banach spaces. In view of (2.14),

$$X/X_{T,\bar{G}} \cong X_T(\bar{G}_S)/X_T(\bar{G}_S)\cap X_{T,\bar{G}}.$$

It follows from Theorem 2.3 that

$$X_T(\bar{G}_S)\cap X_{T,\bar{G}} = X_{T,\bar{G}_S\cap\bar{G}}$$

and hence

$$X/X_{T,\bar{G}} \cong X_T(\bar{G}_S)/X_{T,\bar{G}_S\cap\bar{G}}. \tag{2.15}$$

In view of (2.15), $T^{\bar{G}}$ can be considered as an operator on $X_T(\bar{G}_S)/X_{T,\bar{G}_S\cap\bar{G}}$. Since $\lambda\notin\bar{G}_S$ and $\sigma[T|X_T(\bar{G}_S)]\cup\sigma(T|X_{T,\bar{G}_S\cap\bar{G}})\subset\bar{G}_S$, it follows from Lemma 1.5 that $T^{\bar{G}}$ is closed and $\lambda\notin(T^{\bar{G}})$. As λ is arbitrary in G, we have $\sigma(T^{\bar{G}})\subset G^c$.

3. Equivalence of closed $(S, 1)$-decomposable and S-decomposable operators

3.1. Theorem. *Suppose that T is $(S, 1)$-decomposable and $G\subset\mathbf{C}$ is open in $\mathbf{C}_\infty$ such that $\bar{G}\cap S=\emptyset$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic D_T-valued functions defined on G, with the property*

$$h_n(\lambda) = (\lambda-T)f_n(\lambda) \to 0 \quad as \quad n\to\infty \tag{3.1}$$

in the strong topology of X and uniformly on every bounded subset of G. Then

$$f_n(\lambda) \to 0 \quad as \quad n\to\infty$$

in the strong topology of X and uniformly on every bounded subset of G.

Proof. We may suppose that

$$G = \{\lambda\in\mathbf{C}\colon\ |\lambda| < R,\ R > 0\}.$$

By decreasing R, we may suppose that (3.1) holds uniformly on G. Let R_0 with $0<R_0<R$ be arbitrary. Choose the numbers R_1, R_1', R_2', R_2 such that $R_0<R_1< <R_1'<R_2'<R_2<R$ and put

$$G_j = \{\lambda\in\mathbf{C}\colon\ |\lambda| < R_j\},\quad j = 0, 1;$$

$$H = \{\lambda\in\mathbf{C}\colon\ R_1 \leqq |\lambda| \leqq R_2\};\quad H' = \{\lambda\in\mathbf{C}\colon\ R_1' \leqq |\lambda| \leqq R_2'\}.$$

By Theorem 2.6, the coinduced operator T^H on $X/X_{T,H}$ is closed and

$$\sigma(T^H) \subset (H^0)^c, \tag{3.2}$$

where $H^0=\{\lambda\in\mathbf{C}\colon\ R_1<|\lambda|<R_2\}$.

If $x\in X$ and f is an X-valued function, then we use the notations $\hat{x}=x+X_{T,H}$ and $\hat{f}(\lambda)=f(\lambda)+X_{T,H}$ for the cosets in the quotient space $X/X_{T,H}$. In $\hat{X}/X_{T,H}$, the convergence (3.1) gives rise to

$$\hat{h}_n(\lambda) = (\lambda - T^H)\hat{f}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on G. In view of (3.2), $(\lambda - T^H)^{-1}$ is uniformly bounded on H' and hence

$$\hat{f}_n(\lambda) = (\lambda - T^H)^{-1}\hat{h}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on H'. By the maximum principle,

$$\hat{f}_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of $X/X_{T,H}$ and uniformly on $\bar{G}_1$.

For $\lambda \in G$ and $n=1, 2, \ldots,$ let

$$f_n(\lambda) = \sum_{k=0}^{\infty} a_{nk}\lambda^k$$

be the power series expansion of f_n. Then

$$\hat{f}_n(\lambda) = \sum_{k=0}^{\infty} \hat{a}_{nk}\lambda^k.$$

By the Cauchy inequalities, we have

$$\|\hat{a}_{nk}\| \leqq \varepsilon_n/R_1^k, \quad n = 1, 2, \ldots, \quad k = 0, 1, \ldots,$$

where

$$\varepsilon_n = \max\{\|\hat{f}_n(\lambda)\|: \lambda \in \bar{G}_1\} \to 0 \quad (n \to \infty).$$

For every $\hat{a}_{nk}$, there is $b_{nk} \in \hat{a}_{nk}$ such that $\|b_{nk}\| \leqq 2\|\hat{a}_{nk}\|$. For every n, let

$$g_n(\lambda) = \sum_{k=0}^{\infty} b_{nk}\lambda^k. \tag{3.3}$$

Then

$$\|g_n(\lambda)\| \leqq \sum_{k=0}^{\infty} \|b_{nk}\| \cdot |\lambda|^k \leqq 2\varepsilon_n \sum_{k=0}^{\infty} |\lambda|^k/R_1^k, \quad \lambda \in G_1$$

and hence the series (3.3) is absolutely and uniformly convergent in $\bar{G}_0$, with

$$\|g_n(\lambda)\| \leqq 2\varepsilon_n R_1/(R_1 - R_0) \to 0, \quad \lambda \in \bar{G}_0. \tag{3.4}$$

Since $b_{nk} \in \hat{a}_{nk}$ implies that $\hat{f}_n(\lambda) = \hat{g}_n(\lambda)$ on $\bar{G}_0$, we have

$$k_n(\lambda) = f_n(\lambda) - g_n(\lambda) \in X_{T,H}, \quad \lambda \in \bar{G}_0. \tag{3.5}$$

Next, consider positive numbers $\tilde{R}_1, \tilde{R}_1', \tilde{R}_2', \tilde{R}_2$ related by the inequalities $R_0 < \tilde{R}_1 < \tilde{R}_1' < \tilde{R}_2' < \tilde{R}_2 < R_1$ and put $\tilde{H} = \{\lambda \in \mathbf{C}: \tilde{R}_1 \leqq |\lambda| \leqq \tilde{R}_2\}$. All the above conclusions remain valid for $\tilde{R}_1, \tilde{R}_1', \tilde{R}_2', \tilde{R}_2$ substituting R_1, R_1', R_2', R_2, respectively. Hence, for $n=1, 2, \ldots$, there exists an X-valued analytic function $\tilde{g}_n$ with

(3.6) $$\|\tilde{g}_n(\lambda)\| \leqq 2\tilde{\varepsilon}_n \tilde{R}_1/(\tilde{R}_1 - R_0) \to 0, \quad \lambda \in \bar{G}_0,$$

where $\tilde{\varepsilon}_n$ is the analogue of ε_n. Furthermore, we have

(3.7) $$\tilde{k}_n(\lambda) = f_n(\lambda) - \tilde{g}_n(\lambda) \in X_{T,\tilde{H}}, \quad \lambda \in \bar{G}_0.$$

Now, subtract (3.7) from (3.5) and use (3.4) and (3.6) to obtain

(3.8) $$\|k_n(\lambda) - \tilde{k}_n(\lambda)\| = \|g_n(\lambda) - \tilde{g}_n(\lambda)\| \leqq 2(\varepsilon_n + \tilde{\varepsilon}_n)\tilde{R}_1/(\tilde{R}_1 - R_0) \to 0; \quad \lambda \in \bar{G}_0.$$

Since H and $\tilde{H}$ are disjoint bounded closed sets with $S \cap H = \emptyset$, $S \cap \tilde{H} = \emptyset$, Theorem 2.4 implies that

$$X_{T,H \cup \tilde{H}} = X_{T,H} \oplus X_{T,\tilde{H}}.$$

Hence, there is $M > 0$ so that, for $x \in X_{T,H}$ and $\tilde{x} \in X_{T,\tilde{H}}$,

(3.9) $$\|x\| + \|\tilde{x}\| \leqq M\|x + \tilde{x}\|.$$

It follows from (3.8) and (3.9) that

(3.10) $$\|k_n(\lambda)\| \leqq 2(\varepsilon_n + \tilde{\varepsilon}_n) M\tilde{R}_1/(\tilde{R}_1 - R_0) \to 0, \quad \lambda \in \bar{G}_0.$$

Thus, (3.5), (3.4) and (3.10) imply that

$$\|f_n(\lambda)\| \leqq \|k_n(\lambda)\| + \|g_n(\lambda)\| \to 0$$

uniformly on $\bar{G}_0$. Since $R_0 \in (0, R)$ is arbitrary, the proof is complete.

It is easily seen that if $\{f_n\}$ in Theorem 3.1 is replaced by a double sequence, then the conclusion remains valid.

3.2. Corollary. *Suppose that T is $(S, 1)$-decomposable, $G \subset \mathbf{C}$ is open in $\mathbf{C}_\infty$ such that $\bar{G} \cap S = \emptyset$. If $\{f_{nm}: G \to D_T\}$ is a double sequence of functions, analytic on G such that*

$$(\lambda - T) f_{nm}(\lambda) \to 0 \quad (n, m \to \infty)$$

in the strong topology of X and uniformly on every bounded subset of G, then

$$f_{nm}(\lambda) \to 0 \quad (n, m \to \infty)$$

in the strong topology of X and uniformly on every bounded subset of G.

3.3. Theorem. *Let T be $(S, 1)$-decomposable. If for $x \in X$ there is a sequence $\{f_n: G \to D_T\}$ of analytic functions on an open set $G \subset \mathbf{C}$ with $\bar{G} \cap S = \emptyset$, such that*

(3.11) $$\|x - (\lambda - T) f_n(\lambda)\| \to 0 \quad (n \to \infty)$$

uniformly on every bounded subset of G, then $G \subset \varrho_T(x)$.

Proof. Put $f_{nm}(\lambda)=f_n(\lambda)-f_m(\lambda)$, $\lambda\in G$. Corollary 3.2 implies that $f_{nm}(\lambda)\to 0$ $(n, m\to\infty)$ in the strong topology of X and uniformly on every bounded subset of G. Then the function $f\colon G\to X$, defined by

$$f(\lambda) = \lim_{n\to\infty} f_n(\lambda)$$

is analytic on G. Since T is closed, (3.11) implies that

$$f(\lambda)\in D_T \quad \text{and} \quad (\lambda-T)f(\lambda) = x \quad \text{for} \quad \lambda\in G. \tag{3.12}$$

Since, by Lemma 1.3, $\bar{G}\cap S_T\subset\bar{G}\cap S=\emptyset$, (3.12) implies that $G\subset\sigma_T(x)$.

3.4. Theorem. *Suppose that T is $(S, 1)$-decomposable, $F\subset\mathbf{C}_\infty$ is closed such that $X(T, F)$ (resp. $X_{T,F}$) exists. Then for every (S, m)-covering $\{G_S, G_1, \ldots, G_m\}$ of F, where m is a positive integer, we have*

$$X(T, F) \subset X_T(\bar{G}_S)+\sum_{i=1}^{m} X_{T,\bar{G}_i}, \tag{3.13}$$

respectively,

$$X_{T,F} \subset X_T(\bar{G}_S)+\sum_{i=1}^{m} X_{T,\bar{G}_i}. \tag{3.13'}$$

Proof. We confine the proof to (3.13), that of (3.13′) being similar.

If S is bounded, the statement of the theorem is [2, Theorem 4]. Therefore, we suppose that $\infty\in S$. We divide the proof into four parts.

Part A. Assume that $m=1$. Then $\{G_S, G_1\}$ is an $(S, 1)$-covering of F. Let $H=\overline{G_S\cap G_1}$. Then $H\cap S=\emptyset$ and by Theorems 2.1 and 2.6, $X_{T,H}$ exists, the coinduced operator T^H on $X/X_{T,H}$ is closed and

$$\sigma(T^H) \subset (G_S\cap G_1)^c. \tag{3.14}$$

For the cosets in $X/X_{T,H}$ and for the $X/X_{T,H}$-valued functions we use the notations introduced in the proof of Theorem 3.1.

Let $x\in X(T, F)$ and put $x(\lambda)=[\lambda-T|X(T, F)]^{-1}x$, for $\lambda\in F^c$. It follows from $(\lambda-T)x(\lambda)=x$, that $(\lambda-T^H)\hat{x}(\lambda)=\hat{x}$, $\lambda\in F^c$. In view of (3.14), the resolvent operator $R(\lambda; T^H)$ is defined for $\lambda\in G_S\cap G_1$. Define

$$\hat{f}(\lambda) = \begin{cases} \hat{x}(\lambda), & \text{if} \quad \lambda\in F^c,\\ R(\lambda; T^H)\hat{x}, & \text{if} \quad \lambda\in G_S\cap G_1.\end{cases}$$

Clearly, $\hat{f}$ is well-defined and is analytic on $F^c\cup(G_S\cap G_1)$. Since $\infty\in S\subset G_S$, $F-G_S$ is bounded. Let D be a bounded Cauchy domain such that $F-G_S\subset D$ and $\bar{D}\cap(F-G_1)=\emptyset$. If ∂D is the positively oriented boundary of D, put

$$\hat{x}_0 = \frac{1}{2\pi i}\int_{\partial D}\hat{f}(\lambda)\,d\lambda, \quad \hat{x}_1 = \hat{x}-\hat{x}_0. \tag{3.15}$$

Evidently, $\hat{x}_0$ is independent of the choice of D. Now (3.15) gives rise to the following representation of x:

$$x = x_0 + x_1 + y, \quad \text{with} \quad x_i \in \hat{x}_i \quad (i = 0, 1), \quad y \in X_{T,H}. \tag{3.16}$$

Part B. In this part we prove that $x_0 \in X_T(\bar{G}_S) + X_{T,\bar{G}_1}$.

Let $\lambda_0 \notin S \cup \bar{G}_1$ and let δ be a neighborhood of λ_0 so that $\bar{\delta} \cap (S \cup \bar{G}_1) = \emptyset$. We may choose the Cauchy domain D satisfying $\bar{D} \cap \bar{\delta} = \emptyset$. For $\lambda \in \delta$, we have successively

$$(\lambda - T^H) \frac{1}{2\pi i} \int_{\partial D} \frac{\hat{f}(\mu)}{\lambda - \mu} d\mu = \frac{1}{2\pi i} \int_{\partial D} \frac{(\lambda - T^H)\hat{f}(\mu)}{\lambda - \mu} d\mu =$$

$$= \frac{1}{2\pi i} \int_{\partial D} \frac{\hat{x}}{\lambda - \mu} d\mu + \frac{1}{2\pi i} \int_{\partial D} \hat{f}(\mu)\, d\mu = \hat{x}_0.$$

By Theorem 1.7, there is a sequence $\{\hat{a}_n\}_{n=0}^{\infty} \subset D_{TH}$ and there are numbers $M>0$, $R>0$, such that

$$(\lambda_0 - T^H)\hat{a}_0 = \hat{x}_0, \quad (\lambda_0 - T^H)\hat{a}_{n+1} = \hat{a}_n, \quad \|\hat{a}_n\| \leqq MR^n, \quad n = 0, 1, \ldots. \tag{3.17}$$

By the definition of D_{TH}, $\hat{a}_n \cap D_T \neq \emptyset$. Let $a_n \in \hat{a}_n \cap D_T$. Then $\hat{a}_n = a_n + X_{T,H} \subset D_T$ and hence we may choose a_n to satisfy the inequality $\|a_n\| \leqq 2\|\hat{a}_n\|$, $n = 0, 1, \ldots$. In view of (3.17), we have

$$(\lambda_0 - T)a_0 = x_0 + b_0, \quad (\lambda_0 - T)a_{n+1} = a_n + b_{n+1}, \tag{3.18}$$
$$\|a_n\| \leqq 2MR^n, \quad n = 0, 1, \ldots$$

where $\{b_n\}_{n=0}^{\infty} \subset X_{T,H}$. Let

$$A_n(\lambda) = \sum_{k=0}^{n} a_k(\lambda_0 - \lambda)^k, \quad B_n(\lambda) = \sum_{k=0}^{n} b_k(\lambda_0 - \lambda)^k.$$

Then, it follows from

$$\sigma(T|X_{T,H}) \cap \bar{\delta} \subset H \cap \bar{\delta} \subset \bar{G}_1 \cap \bar{\delta} = \emptyset,$$

that for $\lambda \in \delta$,

$$(\lambda - T)[A_n(\lambda) - (\lambda - T|X_{T,H})^{-1}B_n(\lambda)] = (\lambda - T)A_n(\lambda) - B_n(\lambda) =$$

$$= x_0 - a_n(\lambda_0 - \lambda)^{n+1}.$$

Let $\delta_0 = \{\lambda \in \mathbf{C} \colon |\lambda - \lambda_0| < 1/2R\}$. For $\lambda \in \delta \cap \delta_0$, the last inequality of (3.18), implies

$$\|a_n\| \cdot |\lambda_0 - \lambda|^{n+1} \leqq M/2^n R \to 0 \quad (n \to \infty)$$

and hence

$$\|(\lambda - T)[A_n(\lambda) - (\lambda - T|X_{T,H})^{-1}B_n(\lambda)] - x_0\| \to 0,$$

uniformly on $\delta \cap \delta_0$. By Theorem 3.3, $\delta \cap \delta_0 \subset \varrho_T(x_0)$ and hence $\lambda_0 \in \varrho_T(x_0)$. Since

$\lambda_0 \notin S \cup \bar{G}_1$ is arbitrary, we have $\sigma_T(x_0) \subset S \cup \bar{G}_1$. Thus,

$$x_0 \in X_T(S \cup \bar{G}_1) = X_T(S) \oplus X_{T,\bar{G}_1} \subset X_T(\bar{G}_S) + X_{T,\bar{G}_1}. \tag{3.19}$$

Part C. In this part we show that $x_1 \in X_T(\bar{G}_S)$. Let $\lambda_0 \notin \bar{G}_S$. There exists a neighborhood γ of λ_0 such that $\bar{\gamma} \cap \bar{G}_S = \emptyset$. We can choose a Cauchy domain D such that $D \supset \bar{\gamma} \cup (F - G_S)$. Then for $\lambda \in \gamma$, we obtain successively

$$(\lambda - T^H)\left[-\frac{1}{2\pi i}\int_{\partial D}\frac{\hat{f}(\mu)}{\lambda-\mu}\,d\mu\right] = -\frac{1}{2\pi i}\int_{\partial D}\frac{(\lambda - T^H)\hat{f}(\mu)}{\lambda-\mu}\,d\mu =$$

$$= -\frac{1}{2\pi i}\int_{\partial D}\hat{f}(\mu)\,d\mu + \frac{1}{2\pi i}\int_{\partial D}\frac{\hat{x}}{\mu-\lambda}\,d\mu = \hat{x} - \hat{x}_0 = \hat{x}_1.$$

By repeating the method used in Part B, one obtains

$$x_1 \in X_T(\bar{G}_S). \tag{3.20}$$

Part D. It follows from (3.16), (3.19), (3.20) and $y \in X_{T,H} \subset X_T(\bar{G}_S)$, that

$$X(T, F) \subset X_T(\bar{G}_S) + X_{T,\bar{G}_1}.$$

A subsequent repetition, via induction on m, leads one to (3.13).

3.5. Theorem. *Every closed $(S, 1)$-decomposable operator is S-decomposable.*

Proof. Let $\{G_S, G_1, \ldots, G_n\}$ be an S-covering of $\sigma(T)$. By Theorem 3.4, we have

$$X = X[T, \sigma(T)] \subset X_T(\bar{G}_S) + \sum_{i=1}^{n} X_{T,\bar{G}_i} \subset X$$

and hence T is S-decomposable.

4. The spectral residuum

4.1. Definition. Given T, let $\Sigma(T)$ be the family of all closed sets S such that $S_T \subset S \subset \sigma(T)$ and T is S-decomposable. If there exists $S^* \in \Sigma(T)$ such that $S^* \subset S$ for any $S \in \Sigma(T)$, then S^* is called the spectral residuum of T.

4.2. Theorem. *The spectral residuum exists for every closed operator T.*

Proof. We only sketch the proof because it is similar to that of [2, Theorem 6]. Since $\sigma(T)$ is in $\Sigma(T)$, $\Sigma(T)$ is nonempty. Let $\{S_\alpha\colon \alpha \in A\}$ be a totally ordered subfamily of $\Sigma(T)$ and let $S_0 = \cap\{S_\alpha\colon \alpha \in A\}$. If $H \subset \mathbf{C}_\infty$ is a closed set disjoint from S_0 then, since $\mathbf{C}_\infty$ is compact, there is $\alpha \in A$ such that $H \cap S_\alpha = \emptyset$. Hence an S_0-covering of $\sigma(T)$ is an S_α-covering of $\sigma(T)$ for some $\alpha \in A$. Since T is S_α-decomposable, it is also S_0-decomposable. By Zorn's lemma, there is a minimal element in $\Sigma(T)$. It remains to prove that, for $S_1, S_2 \in \Sigma(T)$, $S = S_1 \cap S_2 \in \Sigma(T)$.

Let $\{G_S, G\}$ be an S-covering of $\sigma(T)$. In view of [3, Theorem 1 (6)] or [2, Theorem 6], we may choose open sets G_{S_i}, G_i $(i=1, 2)$, such that

$$G_{S_i} \supset S_i \cup G_S, \quad i = 1, 2; \tag{4.1}$$

$$\bar{G}_{S_1} \cap \bar{G}_{S_2} = \bar{G}_S, \tag{4.2}$$

$$G_i \subset G, \quad \bar{G}_i \cap S_i = \emptyset, \quad G_i \cup G_{S_i} \supset G, \quad i = 1, 2. \tag{4.3}$$

Thus, $\{G_{S_i}, G_i\}$ $(i=1, 2)$ is an $(S_i, 1)$-covering of $\sigma(T)$. Let G'_{S_2} be open in $\mathbf{C}_\infty$ such that $\bar{G}'_{S_2} \subset G_{S_2}$ and $\{G'_{S_2}, G_2\}$ is an $(S_2, 1)$-covering of $\sigma(T)$. Since T is S_2-decomposable, we have

$$X = X_T(\bar{G}'_{S_2}) + X_{T, \bar{G}_2}. \tag{4.4}$$

Since T is S_i-decomposable $(i=1, 2)$, $X_{T, \bar{G}}$ exists by part 2 of the proof of [3, Theorem 1]. It follows from $G_2 \subset G$ and (4.4), that

$$X = X_T(\bar{G}'_{S_2}) + X_{T, \bar{G}}. \tag{4.5}$$

Put $F = \bar{G}'_{S_2} \cap \sigma(T)$. Since $X_T(\bar{G}'_{S_2})$ is a spectral maximal space of T, by Lemma 1.6,

$$\sigma[T|X_T(\bar{G}'_{S_2})] \subset \sigma(T).$$

Thus, we have

$$\sigma[T|X_T(\bar{G}'_{S_2})] \subset \bar{G}'_{S_2} \cap \sigma(T) = F$$

and it follows easily that $X_T(\bar{G}'_{S_2})$ is the upper bound of $I(T, F)$, i.e.

$$X_T(\bar{G}'_{S_2}) = X(T, F). \tag{4.6}$$

Furthermore, $S_T \subset S_1 \cap S_2 = S \subset \bar{G}_S$ and (4.2) imply that $X_T(\bar{G}_S)$ exists and

$$X_T(\bar{G}_S) = X_T(\bar{G}_{S_1}) \cap X_T(\bar{G}_{S_2}).$$

Hence $X_T(\bar{G}_S)$ is closed. Similarly, $S = S_1 \cap S_2$ implies that

$$X_T(S) = X_T(S_1) \cap X_T(S_2)$$

and hence $X_T(S)$ is closed.

By (4.2), we have $G_{S_1} \cap G_{S_2} \subset G_S$, and hence

$$F = \bar{G}_{S_2} \cap \sigma(T) \subset G_{S_2} \cap (G_{S_1} \cup G_1) \subset (G_{S_2} \cap G_{S_1}) \cup G_1 \subset G_S \cup G_1.$$

Next, we prove

$$X(T, F) \subset X_T(\bar{G}_S) + X_{T, \bar{G}_1}. \tag{4.7}$$

Let $H = \overline{G_S \cap G_1}$. Then $H \cap S_1 \subset \bar{G}_1 \cap S_1 = \emptyset$. Since T is S_1-decomposable, $X_{T,H}$ exists by Theorem 2.1, the coinduced operator T^H on $X/X_{T,H}$ is closed and $\sigma(T^H) \subset \subset (G_S \cap G_1)^c$ by Theorem 2.6. By repeating parts A, B and C of the proof of Theorem 3.4; one obtains that, for every $x \in X(T, F)$, $x = x_0 + x_1 + y$, where $y \in X_{T,H}$, $\sigma_T(x_1) \subset \bar{G}_S$ and $\sigma_T(x_0) \subset S \cup \bar{G}_1$. Hence

$$y \in X_{T,H} \subset X_T(\bar{G}_S), \tag{4.8}$$

$$x_1 \in X_T(\bar{G}_S). \tag{4.9}$$

As for x_0, by repeating the proof of Theorem 2.1, we obtain

$$x_0 \in X_T(S) \oplus X_{T,\bar{G}_1} \subset X_T(\bar{G}_S) + X_{T,\bar{G}_1}. \tag{4.10}$$

Thus (4.7) follows from (4.8), (4.9) and (4.10). In view of (4.3), we have $X_{T,\bar{G}_1} \subset X_{T,\bar{G}}$ and then, with the help of (4.5), (4.6), (4.7), we obtain

$$X = X_T(\bar{G}_S) + X_{T,\bar{G}}.$$

Thus, T is $(S, 1)$-decomposable. Theorem 3.5 concludes the proof.

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DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING, CHINA