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On commuting unbounded self-adjoint operators. I

KONRAD SCHMÜDGEN

Dedicated to Professor B. Szőkefalvi-Nagy on the occasion of his 70th birthday

Let A and B be unbounded self-adjoint operators in a Hilbert space $\mathscr{H}$ which are both essentially self-adjoint on a common dense domain $\mathscr{D}\subseteq\mathscr{D}(AB)\cap\mathscr{D}(BA)$ in $\mathscr{H}$ and commute on $\mathscr{D}$. We then write $\{A, B\}\in\mathfrak{N}_1$. It is well known that the spectral projections of A and B may fail to commute for $\{A, B\}\in\mathfrak{N}_1$. The first counter-example was constructed by NELSON [10]; see also [6], [9], [13], [15]. In this paper we begin a study of this phenomenon in terms of commutators of bounded operators. In the present paper we restrict ourselves to the case where the spectra $\sigma(A)$ and $\sigma(B)$ are both different from the real line. A similar approach is possible in the general case if we use the Cayley transforms of A and B. But the methods of construction are somewhat different in that case (we have to deal with commutators of two unitaries).

Suppose that $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ and $\beta\in\mathbf{R}_1\setminus\sigma(B)$. In Section 1 we characterize the couples $\{A, B\}$ in $\mathfrak{N}_1$ in terms of the bounded self-adjoint operators $X:=(A-\alpha)^{-1}$ and $Y:=(B-\beta)^{-1}$. We show that $\{A, B\}\in\mathfrak{N}_1$ if and only if $\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(X)=$ $=\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(Y)=\{0\}$. Probably the simplest example of this kind for which $[X, Y]\neq 0$ is given by $X=\operatorname{Re} S$, $Y=\operatorname{Im} S$, where S is the unilateral shift. Therefore, $\{A:=(\operatorname{Re} S)^{-1}, B:=(\operatorname{Im} S)^{-1}\}\in\mathfrak{N}_1$, but A and B do not commute strongly.

In the remaining sections of the paper we establish pairs of bounded self-adjoint operators X, Y having these properties. We describe three typical situations. All irreducible pairs in $\mathfrak{N}_1$ for which the commutator $[X, Y]$ has rank one are classified in Section 2. Here we use the principal function [11] of the pair X, Y and the tracial bilinear form [8]. Toeplitz operators (mainly with polynomial symbols) are considered in Section 3. In Section 4 we study pairs of the class $\mathfrak{N}_1$ obtained by taking real and imaginary parts of certain one-dimensional "perturbations" of normal operators.

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Let us fix some notation. If T is an operator in a Hilbert space $\mathscr{H}$, then we use $\mathscr{D}(T), \mathscr{N}(T), \mathscr{R}(T)$ and $\sigma(T)$ to denote the domain, the kernel, the range and the spectrum of T, respectively. For a subset $\mathscr{K}$ of $\mathscr{H}, \overline{\mathscr{K}}$ is the closure of $\mathscr{K}$ in the Hilbert space norm. We denote by $\mathbf{N}_0$ and $\mathbf{N}$ the non-negative, resp., positive integers.

1. The class $\mathfrak{N}_1$

Throughout this section, let A and B denote self-adjoint operators in a Hilbert space $\mathscr{H}$.

1.1. Definition 1. We say that the couple $\{A, B\}$ is of the class $\mathfrak{N}_1$ if there exists a linear subspace $\mathscr{D}$ of $\mathscr{H}$ such that

(1) $\mathscr{D}\subseteq\mathscr{D}(AB)\cap\mathscr{D}(BA)$ and $AB\varphi=BA\varphi$ for all $\varphi\in\mathscr{D}$.

(2) $\mathscr{D}$ is dense in $\mathscr{H}$.

(3) $A\restriction\mathscr{D}$ and $B\restriction\mathscr{D}$ are essentially self-adjoint*) (e.s.a.).

Remarks. 1. Suppose that $\{A, B\}\in\mathfrak{N}_1$. If A (or B) is bounded, then A and B *strongly commute* (that is, by definition, the spectral projections $E(\lambda)$ of A and $F(\mu)$ of B commute for all $\lambda, \mu\in\mathbf{R}_1$). We sketch the proof. Since A is bounded and $B\restriction\mathscr{D}$ is e.s.a., (1) extends by continuity on $\mathscr{D}(B)$, i.e., $AB\varphi=BA\varphi$ for all $\varphi\in\mathscr{D}(B)$. Since B is self-adjoint, this gives $[A, F(\mu)]=0$ for $\mu\in\mathbf{R}_1$ and hence $[E(\lambda), F(\mu)]=0$ for $\lambda, \mu\in\mathbf{R}_1$.

2. A pair $\{A, B\}$ in $\mathfrak{N}_1$ is said to be *irreducible* if each decomposition $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, $A=A_1\oplus A_2$, $B=B_1\oplus B_2$, where A_j and B_j are self-adjoint operators in the Hilbert spaces $\mathscr{H}_j$, $j=1, 2$, is trivial, that is, $\mathscr{H}_1=\{0\}$ or $\mathscr{H}_2=\{0\}$. Obviously, this is the case if and only if each projection commuting with A and B is either 0 or I.

1.2. As mentioned above, we restrict ourselves in this paper to the case where $\sigma(A)\neq\mathbf{R}_1$ and $\sigma(B)\neq\mathbf{R}_1$. Suppose that $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ and $\beta\in\mathbf{R}_1\setminus\sigma(B)$. We now reformulate the conditions occurring in Definition 1 in terms of the bounded self-adjoint operators $X:=(A-\alpha)^{-1}$ and $Y:=(B-\beta)^{-1}$.

For let P denote the orthogonal projection of $\mathscr{H}$ on $\overline{\mathscr{R}([X, Y])}$ and let $\mathscr{D}(A, B):=XY(I-P)\mathscr{H}$.

Definition 2. If $\{A, B\}\in\mathfrak{N}_1$, then $d(A, B):=\dim P\mathscr{H}$ is called the defect number of the pair $\{A, B\}$.

*) Recall that a symmetric operator T is called essentially selfadjoint if its closure $\overline{T}$ is selfadjoint. Thus (3) means that $\overline{A\restriction\mathscr{D}}=A$ and $\overline{B\restriction\mathscr{D}}=B$.

It is easy to check that $d(A,B)$ does not depend on the choice of $\alpha\in\mathbf{R}_1\setminus\sigma(A)$, $\beta\in\mathbf{R}_1\setminus\sigma(B)$. Moreover, A and B commute strongly if and only if X and Y commute, that is, $d(A,B)=0$.

Lemma 3. *$\mathscr{D}(A,B)$ is the largest linear subspace of $\mathscr{H}$ satisfying* (1). *Moreover, $\mathscr{D}(A,B)\equiv XY(I-P)\mathscr{H}=YX(I-P)\mathscr{H}\equiv\mathscr{D}(B,A)$.*

Proof. Suppose that $\varphi\in\mathscr{D}(AB)\cap\mathscr{D}(BA)$ and $AB\varphi=BA\varphi$. Then, $\varphi=XY\xi=YX\eta$ for some $\xi,\eta\in\mathscr{H}$. $(A-\alpha)(B-\beta)\varphi=\eta$ and $(B-\beta)(A-\alpha)\varphi=\xi$ imply that $\xi=\eta$. Hence $0=\langle(XY-YX)\xi,\psi\rangle=\langle\xi,-(XY-YX)\psi\rangle$ for all $\psi\in\mathscr{H}$, i.e. $\xi\perp P\mathscr{H}$ and thus $\varphi=XY(I-P)\xi\in\mathscr{D}(A,B)$.

Conversely, let $\varphi=XY(I-P)\xi$ for some $\xi\in\mathscr{H}$. In particular, $\langle(I-P)\xi,-(XY-YX)^2(I-P)\xi\rangle=0=\|(XY-YX)(I-P)\xi\|^2$. Therefore, $\varphi=XY(I-P)\xi=YX(I-P)\xi$ which gives $AB\varphi=BA\varphi$. Moreover, this shows that $XY(I-P)\mathscr{H}\subseteq YX(I-P)\mathscr{H}$. Replacing XY by YX, we get $YX(I-P)\mathscr{H}\subseteq XY(I-P)\mathscr{H}$ thus completing the proof.

Lemma 4. *$\mathscr{D}(A,B)$ is dense in $\mathscr{H}$ if and only if $\overline{\mathscr{R}([X,Y])}\cap\mathscr{R}(XY)=\{0\}$.*

Proof. $\mathscr{D}(A,B)\equiv\mathscr{R}(YX(I-P))$ is dense if and only if $\mathscr{N}((YX(I-P))^*)=\mathscr{N}((I-P)XY)=\{0\}$. Obviously, $\varphi\in\mathscr{N}((I-P)XY)$ is equivalent to $XY\varphi\in\overline{\mathscr{R}([X,Y])}$. This gives the assertion, because $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$.

Lemma 5. *$A\restriction\mathscr{D}(A,B)$ is e.s.a. if and only if*

$$P\mathscr{H}\cap\mathscr{D}(B)=\overline{\mathscr{R}([X,Y])}\cap\mathscr{R}(Y)=\{0\}.$$

$B\restriction\mathscr{D}(A,B)$ is e.s.a. if and only if

$$P\mathscr{H}\cap\mathscr{D}(A)=\overline{\mathscr{R}([X,Y])}\cap\mathscr{R}(X)=\{0\}.$$

Proof. We only prove the first assertion. Since $(A-\alpha)^{-1}=X$ is a bounded self-adjoint operator, $A\restriction\mathscr{D}(A,B)$ is e.s.a. if and only if $(A-\alpha)\mathscr{D}(A,B)\equiv Y(I-P)\mathscr{H}\equiv\mathscr{R}(Y(I-P))$ is dense in $\mathscr{H}$ or equivalently if $\mathscr{N}((I-P)Y)=\{0\}$. Since $\mathscr{N}(Y)=\{0\}$, the latter is equivalent to $P\mathscr{H}\cap\mathscr{R}(Y)=\{0\}$, which completes the proof.

In case that $\mathscr{R}([X,Y])$ is closed. the next Lemma gives a characterization of the class $\mathfrak{N}_1$ only in terms of domains.

Lemma 6. *If $\{A,B\}\in\mathfrak{N}_1$, then $\mathscr{D}(AB)\cap\mathscr{D}(A)=\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(AB)\cap\mathscr{D}(BA)=\mathscr{D}(A,B)$ (and, by definition, this domain is dense in $\mathscr{H}$). Conversely, suppose that $\mathscr{R}([X,Y)]$ is closed. If $\mathscr{D}(A,B)$ is dense in $\mathscr{H}$ and $\mathscr{D}(AB)\cap\mathscr{D}(A)=\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(AB)\cap\mathscr{D}(BA)$, then $\{A,B\}\in\mathfrak{N}_1$.*

Proof. Suppose that $\{A, B\}\in\mathfrak{N}_1$. Since $XY(I-P)\mathscr{H}=YX(I-P)\mathscr{H}$ by Lemma 3, it is clear that $\mathscr{D}(AB)\cap\mathscr{D}(A)\supseteqq\mathscr{D}(AB)\cap\mathscr{D}(BA)=YX\mathscr{H}\cap XY\mathscr{H}\supseteqq$ $\supseteqq XY(I-P)\mathscr{H}=\mathscr{D}(A, B)$. Now let $\varphi=YX\xi=X\eta\in\mathscr{D}(AB)\cap\mathscr{D}(A)$. Then, $X(\eta-Y\xi)=[X, Y](-\xi)$. Since $\{A, B\}\in\mathfrak{N}_1$, $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ by Lemma 5. Hence $X(\eta-Y\xi)=0$ and, since $\mathscr{N}(X)=\{0\}$, $\eta=Y\xi$. As in the proof of Lemma 3, $XY\xi=YX\xi$ implies that $\xi\perp\overline{\mathscr{R}([X, Y])}$. Therefore, $\xi=(I-P)\xi$ and $\varphi=$ $=YX(I-P)\xi\in\mathscr{D}(A, B)$ which proves that $\mathscr{D}(A, B)\supseteqq\mathscr{D}(AB)\cap\mathscr{D}(A)$. Changing the role of A and B, we get $\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(A, B)$.

We now prove the second assertion. Set $\mathscr{D}=\mathscr{D}(A, B)$ in Definition 1. Then (1) and (2) are satisfied by Lemma 3, resp., by assumption. We show that $\mathscr{R}([X, Y])\cap\mathscr{R}(X)=\{0\}$. Suppose that $\varphi:=[X, Y]\xi=X\eta$ for some $\xi, \eta\in\mathscr{H}$. Then $\psi:=XY\xi-X\eta=YX\xi\in\mathscr{D}(A)\cap\mathscr{D}(AB)$. By assumption, $\psi\in\mathscr{D}(BA)$, that is, $\psi=XY\zeta$ for some $\zeta\in\mathscr{H}$. Hence $X(Y\xi-\eta-Y\zeta)=0$ which gives $\eta=Y(\xi-\zeta)$. Therefore, $\eta\in\mathscr{R}(Y)$ and $\varphi\in\mathscr{R}([X, Y])\cap\mathscr{R}(XY)$. Since we assumed that $\mathscr{D}(A, B)$ is dense, Lemma 4 gives $\varphi=0$. This proves $\mathscr{R}([X, Y])\cap\mathscr{R}(X)=\{0\}$. From Lemma 5 (recall that $\mathscr{R}([X, Y])$ is closed!) we conclude that $B\upharpoonright\mathscr{D}(A, B)$ is e.s.a. Similarly, $A\upharpoonright\mathscr{D}(A, B)$ is e.s.a. Thus $\{A, B\}\in\mathfrak{N}_1$.

Remarks. 1. If we do not assume that $\mathscr{D}(A, B)$ is dense, then the equality of the domains in Lemma 6 does not ensure that $\{A, B\}\in\mathfrak{N}_1$ in general. For an example, recall that there are unbounded self-adjoint operators A and B so that $\mathscr{D}(A)\cap\mathscr{D}(B)=\{0\}$ ([17]). Then, $\mathscr{D}(AB)\cap\mathscr{D}(A)=\mathscr{D}(BA)\cap\mathscr{D}(B)=\mathscr{D}(AB)\cap\mathscr{D}(BA)=$ $=\mathscr{D}(A, B)=\{0\}$, but $\{A, B\}\notin\mathfrak{N}_1$.

2. The second assertion in Lemma 6 is no longer true if we replace the density of $\mathscr{D}(A, B)$ by that of $\mathscr{D}(AB)\cap\mathscr{D}(BA)$. To prove this remark, let $\mathscr{H}=L^2(1, 2)$, let X be the multiplication operator by x, and let $Y=I+H/2$, where H is the finite Hilbert transform in $L^2(1, 2)$. Then X and Y have bounded inverses denoted by A resp. B. Moreover, $[X, Y]$ has rank one and $\mathscr{D}(AB)=\mathscr{D}(A)=\mathscr{D}(BA)=$ $=\mathscr{D}(B)=\mathscr{H}$, but $\{A, B\}\notin\mathfrak{N}_1$ by the first remark in 1.1.

3. It follows from the preceding that if $\{A, B\}\in\mathfrak{N}_1$, then we can take $\mathscr{D}=\mathscr{D}(A, B)$ in Definition 1.

1.3. Theorem 7. *Suppose* $\{A, B\}\in\mathfrak{N}_1$ *Suppose also that* $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ *and* $\beta\in\mathbf{R}_1\setminus\sigma(B)$. *If* $X:=(A-\alpha)^{-1}$ *and* $Y:=(B-\beta)^{-1}$, *then*

$$(4)\quad \mathscr{N}(X)=\mathscr{N}(Y)=\{0\} \quad \text{and} \quad \overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(X)=\overline{\mathscr{R}([X, Y])}\cap\mathscr{R}(Y)=\{0\}.$$

Conversely, if X *and* Y *are bounded self-adjoint operators satisfying* (4), *then* $\{A:=X^{-1}+\alpha, B:=Y^{-1}+\beta\}\in\mathfrak{N}_1$ *for* $\alpha, \beta\in\mathbf{R}_1$.

The proof of Theorem 7 follows immediately from the three Lemmas 3, 4 and 5 above.

2. Pairs with defect number one

In this section we classify, up to unitary equivalence, all irreducible pairs $\{A, B\} \in \mathfrak{N}_1$ with defect number one for which $\sigma(A) \neq \mathbf{R}_1$ and $\sigma(B) \neq \mathbf{R}_1$.

2.1. We first collect some facts concerning bounded operators with rank one self-commutators. A very readable account of this theory is given in [3]; see also [11], [2], [8], [4].

A bounded operator T in $\mathscr{H}$ is said to be *completely hyponormal* if T is hyponormal (that is, $[T^*, T] \geqq 0$) and if T has no nontrivial reducing subspace on which it is normal. Suppose that T is completely hyponormal and that $\dim \mathscr{R}([T^*, T]) = 1$. The essential underlying result for our study is the following. There is a function $g(x, y) \in L^1(\mathbf{R}_2)$ with compact support such that

$$\text{(5)} \qquad \operatorname{Tr} i[p(X, Y), q(X, Y)] = \frac{1}{2\pi} \iint \left(\frac{\partial p}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial p}{\partial y} \frac{\partial q}{\partial x} \right) g(x, y)\, dx\, dy$$

for all polynomials p and q in X and Y. Moreover, $0 \leqq g(x, y) \leqq 1$ a.e. on $\mathbf{R}_2$. $g = g_T$ is called the *principal function of* T. It was introduced by PINCUS [11]. g_T is a complete unitary invariant for T, that is, two completely hyponormal operators T and $\tilde{T}$ with rank one self-commutators are unitarily equivalent if and only if their principal functions g_T and $g_{\tilde{T}}$ coincide (considered as elements of $L^1(\mathbf{R}_2)$). Moreover, for each function $g \in L^1(\mathbf{R}_2)$ with compact support satisfying $0 \leqq g \leqq 1$ there exists a completely hyponormal operator T with principal function g such that $\dim \mathscr{R}([X, Y]) = 1$.

We now return to the class $\mathfrak{N}_1$. Let $\{A, B\}$ and $\{\tilde{A}, \tilde{B}\}$ be pairs of the class $\mathfrak{N}_1$ in Hilbert spaces $\mathscr{H}$ and $\tilde{\mathscr{H}}$. $\{A, B\}$ is called *unitarily equivalent* to $\{\tilde{A}, \tilde{B}\}$ if there is an isometry U of $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ such that $A = U^* \tilde{A} U$ and $B = U^* \tilde{B} U$. In that case we clearly have $U\mathscr{D}(A, B) = \mathscr{D}(\tilde{A}, \tilde{B})$. As in Section 1, we assume that $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$ and $\beta \in \mathbf{R}_1 \setminus \sigma(B)$. It is easy to see that $\{A, B\}$ is unitarily equivalent to $\{\tilde{A}, \tilde{B}\}$ if and only if $\alpha \notin \sigma(\tilde{A})$, $\beta \notin \sigma(\tilde{B})$ and $T := X + iY = (A - \alpha)^{-1} + i(B - \beta)^{-1}$ is unitarily equivalent to $\tilde{T} := \tilde{X} + i\tilde{Y} = (\tilde{A} - \alpha)^{-1} + i(\tilde{B} - \beta)^{-1}$. Moreover, $\{A, B\}$ is irreducible if and only if T is irreducible.

Suppose now in addition that $d(A, B) = 1$. Then the self-adjoint operator $D := [T^*, T]$ has rank one and therefore either $D \geqq 0$ or $D \leqq 0$. Obviously, $\operatorname{sign} D$ is a unitary invariant for $\{A, B\} \in \mathfrak{N}_1$. By changing the role of A and B in case $D \leqq 0$, we can restrict ourselves to pairs $\{A, B\} \in \mathfrak{N}_1$ for which $D = [T^*, T] \geqq 0$, that is, T is hyponormal. Since D has rank one, $\{A, B\} \in \mathfrak{N}_1$ is irreducible (or equivalently, T is irreducible) if and only if T is completely hyponormal. Therefore, under the above assumptions (i.e., $d(A, B) = 1$, $\alpha \in \mathbf{R}_1 \setminus \sigma(A)$, $\beta \in \mathbf{R}_1 \setminus \sigma(B)$ and $[T^*, T] \geqq 0$), g_T is a complete unitary invariant for irreducible $\{A, B\} \in \mathfrak{N}_1$.

To proceed in the converse direction, it still remains to decide when a given completely hyponormal operator with rank one self-commutator leads to an (irreducible) pair $\{A, B\}$ in $\mathfrak{N}_1$. The answer is contained in

Theorem 1. *Suppose that T is a completely hyponormal operator in the Hilbert space $\mathscr{H}$ with rank one self-commutator. Let g be the principal function of $T, X = \operatorname{Re} T$ and $Y = \operatorname{Im} T$. Let $x_0, y_0 \in \mathbf{R}_1$. Then the inverses $A := (X-x_0)^{-1}$ and $B := (Y-y_0)^{-1}$ exist. $\{A, B\} \in \mathfrak{N}_1$ if and only if $\iint g(x, y)(x-x_0)^{-2}\, dx\, dy = \iint g(x, y)(y-y_0)^{-2} dx\, dy = +\infty$.*

2.2. In the proof of Theorem we use the following easy

Lemma 2. *Let N be a bounded operator in $\mathscr{H}$ and let $\varphi, \psi \in \mathscr{H}$. Let $\{z_n, n \in \mathbf{N}\}$ be a zero sequence of complex numbers such that $z_n \notin \sigma(N)$ for all $n \in \mathbf{N}$.*

(i) *If $\lim_n (N-z_n)^{-1}\psi = \varphi$, then $N\varphi = \psi$.*

(ii) *Suppose that N is normal and that $\mathscr{N}(N) = \{0\}$. Suppose that*

$$c := \sup_{n \in \mathbf{N}} |z_n| (\operatorname{dist}(z_n, \sigma(N)))^{-1} < \infty.$$

If $N\varphi = \psi$, then $\lim_n (N-z_n)^{-1}\psi = \varphi$.

Proof. (i) $\psi = N(N-z_n)^{-1}\psi - z_n(N-z_n)^{-1}\psi \to N\varphi - 0\varphi$ as $n \to \infty$, i.e., $N\varphi = \psi$.

(ii) Letting $N = \int z\, dG(z)$ be the spectral decomposition for N, we have

$$\|(I - G(\{0\})) z_n (N-z_n)^{-1}\varphi\|^2 = \int_{\sigma(N) \setminus \{0\}} |z_n|^2 |z - z_n|^{-2} d\|G(z)\varphi\|^2 \leqq \int c^2 d\|G(z)\varphi\|^2 < \infty.$$

The dominated Lebesgue theorem yields $\lim_n (I - G(\{0\}))z_n(N-z_n)^{-1}\varphi = 0$. Since $\mathscr{N}(N) = \{0\}$, we have $G(\{0\}) = 0$ and therefore $\lim_n z_n(N-z_n)^{-1}\varphi = 0$. Therefore, $(N-z_n)^{-1}\psi = z_n(N-z_n)^{-1}\varphi + \varphi \to 0 + \varphi$ as $n \to \infty$, which completes the proof.

Proof of Theorem 1. Since T is hyponormal and $\dim \mathscr{R}([T^*, T]) = 1$, there is a vector $\xi \neq 0$ in $\mathscr{H}$ such that $[T^*, T] = 2i[X, Y] = \xi \otimes \xi$. Since T is completely hyponormal, the operators X and Y are absolutely continuous ([12], Theorem 2.2.4 or [3], Theorem 3.2). In particular, $\mathscr{N}(X-x_0) = \mathscr{N}(Y-y_0) = \{0\}$. Hence the inverses A and B exist and are self-adjoint operators in $\mathscr{H}$.

By induction it follows that $2i[X^{n+1}, Y] = \sum_{j=0}^{n} X^j \xi \otimes X^{n-j}\xi$ for $n \in N_0$. Therefore, $\operatorname{Tr} 2i[X^{n+1}, Y] = (n+1)\langle X^n \xi, \xi\rangle$. Applying the tracial functional calculus in case $p(X, Y) = X^{n+1}$, $q(X, Y) = Y$, we get

$$\operatorname{Tr} i[X^{n+1}, Y] = (1/2\pi) \iint (n+1) x^n g(x, y)\, dx\, dy.$$

Putting both together, we conclude that

$$\pi\langle p(X)\xi, \xi\rangle = \iint p(x)g(x, y)\,dx\,dy \tag{6}$$

for each complex polynomial $p(\cdot)$ in X.

Setting $N:=X-x_0$ and $z_n:=2^{-n}i$ for $n\in\mathbf{N}$, the assumptions of Lemma 2 are fulfilled. Therefore, by Lemma 2, $\xi\in\mathscr{R}(X-x_0)$ if and only if the sequence $\varphi_n:=(X-x_0+z_n)^{-1}\xi$, $n\in\mathbf{N}$, converges in $\mathscr{H}$. We prove that

$$\xi\in\mathscr{R}(X-x_0) \quad \text{if and only if} \quad \iint g(x, y)(x-x_0)^{-2}\,dx\,dy < \infty.$$

For simplicity in notation, let $x_0=0$. Assume that $\{\varphi_n\}$ converges. Take a positive number L so that $\operatorname{supp} g\subseteqq[-L, L]\times\mathbf{R}_1$ and $\sigma(X)\subseteqq[-L, L]$. Let $n\in\mathbf{N}$. If we approximate the continuous function $|x+2^{-n}i|^{-2}$ by polynomials uniformly on $[-L, L]$, it follows from (6) that

$$\pi\|\varphi_n\|^2 = \pi\|(X+2^{-n}i)^{-1}\xi\|^2 = \iint g(x, y)|x+2^{-n}i|^{-2}\,dx\,dy.$$

By Fatou's lemma (recall that $g(x, y)\geqq 0$ a.e.), we obtain

$$\iint g(x, y)x^{-2}dx\,dy \leqq \lim_n \iint g(x, y)|x+2^{-n}i|^{-2}\,dx\,dy = \pi \lim_n \|\varphi_n\|^2 < \infty.$$

Conversely, assume that $\iint g(x, y)x^{-2}dx\,dy<\infty$. Again uniform approximation by polynomials yields

$$\pi\|\varphi_n-\varphi_m\|^2 = \iint g(x, y)|(x+2^{-n}i)^{-1}-(x+2^{-m}i)^{-1}|^2\,dx\,dy \tag{7}$$

for $n, m\in\mathbf{N}$. Since $|x+2^{-n}i|^{-1}\leqq|x|^{-1}$ for $n\in\mathbf{N}$ and the above integral is finite, Lebesgue's dominated convergence theorem (in the formulation given in [1], IV, § 3, 7.) applies and gives

$$\lim_n \iint g(x, y)|(x+2^{-n}i)^{-1}-x^{-1}|^2\,dx\,dy = 0.$$

Because of (7), the latter implies that $\{\varphi_n\}$ is a Cauchy sequence in $\mathscr{H}$. Therefore, $\{\varphi_n\}$ converges in $\mathscr{H}$. This completes the proof of (5).

Similarly, $\xi\in\mathscr{R}(Y-y_0)$ if and only if $\iint g(x, y)(y-y_0)^{-2}dx\,dy<\infty$. Theorem 1.7 now gives the result.

2.3. Remarks. 1. Each hyponormal operator on a separable Hilbert space can be represented as a singular integral operator on a direct integral Hilbert space ([16], [11]; see [3], 2.3). This result can be used to obtain concrete realizations for pairs $\{A, B\}\in\mathfrak{N}_1$ with $d(A, B)=1$, $\sigma(A)\neq\mathbf{R}_1$ and $\sigma(B)\neq\mathbf{R}_1$.

2. Given a function $g\in L^1(\mathbf{R}_2)$ with compact support satisfying $0\leqq g\leqq 1$ and $\iint gx^{-2}dx\,dy=\iint gy^{-2}dx\,dy=\infty$, there exists an (unique up to unitary equivalence) irreducible pair $\{A:=X^{-1}, B:=Y^{-1}\}\in\mathfrak{N}_1$ with defect number one such that g is the principal function of the completely hyponormal operator $T=X+iY$.

This follows from Theorem 1 and the discussion before Theorem 1. Especially, we see that there is a large variety of irreducible pairs of the class $\mathfrak{N}_1$ even in the case $d(A, B)=1$.

3. We illustrate Theorem 1 by a well-known general example (see, for example, [3]). Let $\mathcal{J}$ be a Lebesgue measurable bounded subset of $\mathbf{R}_1$. Let $a, b \in L^\infty(\mathcal{J})$, where $b(t) \neq 0$ a.e. on $\mathcal{J}$ and $a(t)$ is real valued. Define an operator T on $\mathcal{H}=L^2(\mathcal{J})$ by

$$(Tf)(t) \equiv ((X+iY)f)(t) := tf(t)+i\left[a(t)f(t)+\frac{b(t)}{\pi i}\int_{\mathcal{J}}\frac{b(s)f(s)}{t-s}\,ds\right].$$

Then, $[T^*, T]=(2/\pi)b\otimes b$. By a result of Xa Dao-xeng [16], T is the most general completely hyponormal operator with rank one self-commutator whose real part has a cyclic vector. The principal function of T is the characteristic function of the set

$$\mathcal{E} = \{(x, y)\in \mathbf{R}_2:\ x\in\mathcal{J} \ \text{ and } \ a(x)-|b(x)|^2 \leq y \leq a(x)+|b(x)|^2\}.$$

If $(x_0, y_0)\in\mathbf{R}_2$ is in the interior of $\mathcal{E}$, then the conditions in Theorem 1 are, of course, satisfied and thus $\{A:=(X-x_0)^{-1}, B:=(Y-y_0)^{-1}\}\in\mathfrak{N}_1$. In that case $\sigma(A)\supseteq(-\infty, -L)\cup(L, +\infty)$ and $\sigma(B)\supseteq(-\infty, L)\cup(L, +\infty)$ for some $L>0$. If (x_0, y_0) is not in the closure of $\mathcal{E}$, then $\{A, B\}\notin\mathfrak{N}_1$.

We now discuss two rather simple (but typical) examples for which (x_0, y_0) is in the boundary of $\mathcal{E}$. First suppose x_0 is in the interior of $\mathcal{J}$ and $a(x_0)-|b(x_0)|^2= =y_0<a(x_0)+|b(x_0)|^2$. Assume that $a+|b|^2$ is continuous at x_0. Assume that $a(x)-|b|(x)|^2=\lambda|x|^\alpha$ for $x_0-\varepsilon<x<x_0$ and that $a(x)-|b(x)|^2=\mu|x|^\beta$ for $x_0<x< <x_0+\varepsilon$ with $\lambda, \mu\in\mathbf{R}_1, \alpha>0, \beta>0$ and $\varepsilon>0$. Then, $\{A, B\}\in\mathfrak{N}_1$ if and only if $\lambda>0, \mu>0, \alpha<1$ and $\beta<1$. Moreover, $\sigma(A)\supseteq(-\infty, -L)\cup(L, +\infty)$ for some $L>0$.

In the second example we assume that $\mathcal{J}\subseteq(x_0, +\infty)$, $(x_0, x_0+\varepsilon)\subseteq\mathcal{J}$ for some $\varepsilon>0$ and $a(x)-|b(x)|^2\geq 0$ on $\mathcal{J}$. Suppose that $a(x)-|b(x)|^2=\lambda|x|^\alpha$ and $a(x)+|b(x)|^2=\mu|x|^\beta$ for $x\in(x_0, x_0+\varepsilon)$ with $\lambda>0, \mu>0, \alpha>0$ and $\beta>0$. Then $\{A, B\}\in\mathfrak{N}_1$ if and only if $\beta\leq 1\leq\alpha$. (Note that $\lambda<\mu$ in case $\alpha=\beta$, since $b(x)\neq 0$ a.e. on $\mathcal{J}$.) In this example A and B are positive operators.

3. Toeplitz operators

3.1. Let $L^2=L^2(\mathbf{T})$ be the L^2-space on the unit circle $\mathbf{T}$ with normalized Lebesgue measure. Let H^2 be the usual Hardy space on $\mathbf{T}$ and let P_+ denote the orthogonal projection on H^2. If $\varphi\in L^\infty(\mathbf{T})$, then the Toeplitz operator T_φ is defined by $T_\varphi f=P_+\varphi f$ for $f\in H^2$. Let $\varphi=\sum_{n=-\infty}^{\infty}\varphi_n e_n$ be the Fourier expansion of φ, where $e_n:=e^{int}$ for $n\in\mathbf{Z}$. The matrix $(d_{nk})_{n,k\in\mathbf{N}_0}$ of the self-commutator $D=[T_\varphi^*, T_\varphi]$ with respect to the orthonormal base $(e_n)_{n\in\mathbf{N}_0}$ of H^2 is given by

$$(8)\qquad d_{nk} = \langle De_k, e_n\rangle = \sum_{j=0}^{\infty} \varphi_{j-k}\overline{\varphi_{j-n}} - \varphi_{n-j}\overline{\varphi_{k-j}}.$$

As usual, we identify H^2 and $l_2(\mathbf{N}_0)$ via Fourier expansion. If $\varphi\in H^\infty$, then $\varphi_n=0$ for all $n<0$ and (8) yields

$$(9)\qquad d_{nk} = \langle (S^*)^{n+1}\varphi, (S^*)^{k+1}\varphi\rangle \quad\text{for all } k, n\in\mathbf{N}_0.$$

Here S^* is the backward shift in $l_2(\mathbf{N}_0)$ which is defined by $S^*(\psi_0, \psi_1, \psi_2, \ldots)= =(\psi_1, \psi_2, \ldots)$.

Theorem 1. *Let $p(z)=\sum a_j z^j$ be a non-constant complex polynomial of degree n. Let $X=\operatorname{Re} T_p$, $Y=\operatorname{Im} T_p$, $A=X^{-1}$ and $B=Y^{-1}$. Then $\{A, B\}\in\mathfrak{N}_1$ if and only if each $z\in\mathbf{C}_1$, $z\neq 0$, satisfying*

$$(10)\qquad \big(\bar p(z)+p(z)\big)\big(\bar p(z)-p(z)\big) = 0$$

lies on $\mathbf{T}$, where $\bar p(z):=\sum \bar a_j z^{-j}$. If this is true, then $d(A, B)=n$.

Proof. Let $\mathscr{K}_n:=\operatorname{Lin}\{e_j: j=0, \ldots, n-1\}$. First we check that $\mathscr{R}([X, Y])\equiv \equiv\mathscr{R}([T_p^*, T_p])\equiv\mathscr{R}(D)=\mathscr{K}_n$. From (9) it is obvious that $DH^2\subseteq\mathscr{K}_n$. Since $a_n\neq 0$, it follows that the n vectors $(S^*)^n p=(a_n, 0, \ldots)$, $(S^*)^{n-1}p=(a_{n-1}, a_n, 0, \ldots)$, ..., $S^*p=(a_1, \ldots, a_n, 0, \ldots)$ are linearly independent. That is, the Gram determinant $\det(d_{nk})_{n,k=0,\ldots,n-1}$ is non-zero. Therefore, the map $D:\mathscr{K}_n\to\mathscr{K}_n$ is one-to-one and $\mathscr{R}([X, Y])\equiv\mathscr{R}(D)=\mathscr{K}_n$.

Clearly, $2X=T_{\bar p+p}$ and $2Y=T_{i(\bar p-p)}$. It is well-known that each nonzero bounded self-adjoint Toeplitz operator has trivial kernel ([7], Ex. 198). Hence $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$ and the operators $A:=X^{-1}$ and $B:=Y^{-1}$ exist.

Set $q(z)=z^n(\bar p(z)+p(z))$. Let $f\in H^2$. Obviously, $Xf\in\mathscr{K}_n$ is equivalent to $0=\langle Xf, e_{n+k}\rangle$ for all $k\in\mathbf{N}_0$. Since $\bar p+p$ is real on $\mathbf{T}$, this is equivalent to $0=\langle T_{\bar p+p}f, e_{n+k}\rangle=\langle P_+(\bar p+p)f, e_{n+k}\rangle=\langle f, (\bar p+p)e_{n+k}\rangle=\langle f, S^k(\bar p+p)e_n\rangle=\langle f, S^k q\rangle$ for $k\in\mathbf{N}_0$. Combined with $\mathscr{N}(X)=\{0\}$, this shows that $\mathscr{R}(X)\cap\mathscr{K}_n=\{0\}$ if and only if $q(z)$ is a cyclic vector for the shift operator, that is, if $q(z)$ is an outer function in H^2. But a polynomial is an outer function in H^2 if and only if it does not vanish in the set $\{z\in\mathbf{C}_1: |z|<1\}$. Since $\mathscr{R}([X, Y])=\mathscr{K}_n$ as we have seen above, it follows that $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\mathscr{R}(X)\cap\mathscr{K}_n=\{0\}$ if and only if $q(z)$ has no zeros in $\{z\in\mathbf{C}_1: |z|<1\}$. Obviously, $q(z)=0$ for a complex number $z\neq 0$ implies that $q(\bar z^{-1})=0$. Moreovei, $q(0)\neq 0$, since $a_n\neq 0$. Therefore, $\mathscr{R}(X)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ if and only if each solution $z\in\mathbf{C}_1$, $z\neq 0$, of $\bar p(z)+p(z)=0$ lies on the unit circle. Similarly, $\mathscr{R}(Y)\cap\overline{\mathscr{R}([X, Y])}=\{0\}$ if and only if $\bar p(z)-p(z)=0$ for some $z\in\mathbf{C}_1$, $z\neq 0$, implies $|z|=1$.

Now the assertion follows immediately from Theorem 1.7.

Corollary 2. *If all zeros of* $p(z)$ *are in* $\{z\in\mathbf{C}_1\colon |z|\leqq 1\}$, *then* $\{A, B\}\in\mathfrak{N}_1$.

Proof. Let $b_1, \ldots, b_n$ be the zeros of $p(z)$. Assume the contrary, that is, $\{A, B\}\notin\mathfrak{N}_1$. Then, by Theorem 1, there is a non-zero $z\in\mathbf{C}_1$, $|z|\neq 1$, satisfying (10). Hence $|p(z)|=|\bar{p}(z)|$. Since $|p(\bar{z}^{-1})|=|\bar{p}(\bar{z}^{-1})|$, we can assume that $|z|<1$. From

$$\left|a_n \prod_{j=1}^{n}(z-b_j)\right| = |p(z)| = |\bar{p}(z)| = \left|\bar{a}_n \prod_{j=1}^{n}\left(\frac{1}{z}-\bar{b}_j\right)\right|$$

we obtain $|z|^n \prod_{j=1}^{n} |(b_j-z)(1-b_jz)^{-1}|=1$. Since $|b_j|\leqq 1$ by assumption and $|z|<1$, it follows that $|(b_j-z)(1-b_jz)^{-1}|\leqq 1$ and $|z|^n<1$, and we have our contradiction.

Examples. 1. Applying Corollary in case $p(z)\equiv z$, we see that the shift operator $S=T_z$ gives a pair $\{A=(\operatorname{Re} S)^{-1}, B=(\operatorname{Im} S)^{-1}\}\in\mathfrak{N}_1$ with defect number one. This could be verified directly or obtained from Section 2 as well.

2. Let $p_1(z)=2(z-\sqrt{2})$ and let $p_2(z)=(1+i)p_1(z)/2=(1+i)(z-\sqrt{2})$. Let $X_j=\operatorname{Re} T_{p_j}$ and $Y_j=\operatorname{Im} T_{p_j}$ for $j=1,2$. Then $X_1=X_2+Y_2$ and $Y_1=Y_2-X_2$. In case $p_2(z)$ the solutions of (10) are given by $z_1=z_2=(1+i)/\sqrt{2}$, $z_3=z_4=(1-i)/\sqrt{2}$. Since they are all of modulus one, $\{X_2^{-1}, Y_2^{-1}\}\in\mathfrak{N}_1$ by Theorem 1. Since the zero of $p_1(z)$ is not contained in $\{|z|\leqq 1\}$, we see that the condition given in Corollary 2 is sufficient, but not necessary. On the other hand, $z_0=\sqrt{2}+1$ is a solution of (10) for $p_1(z)$. Consequently, $\{X_1^{-1}, Y_1^{-1}\}\equiv\{(X_2+Y_2)^{-1}, (Y_2-X_2)^{-1}\}$ does not belong to the class $\mathfrak{N}_1$. This also follows from the fact that X_1^{-1} is bounded, because $\sigma(X_2)=[-2-2\sqrt{2}, 2-2\sqrt{2}]$.

3.2. A further study of the function $\varphi\in L^\infty$ for which $X=\operatorname{Re} T_\varphi$ and $Y=\operatorname{Im} T_\varphi$ satisfy the conditions of Theorem 1.7 seems to be of some interest. As a result in the opposite direction we mention

Proposition 3. *Suppose that* $\varphi\in H^\infty$ *is a cyclic vector for the backward shift operator. If the sequence* $(\|(S^*)^n\varphi\|)_{n\in\mathbf{N}}$ *is in* $l_2(\mathbf{N})$, *then* $\mathscr{R}([T_\varphi^*, T_\varphi])$ *is dense in* H^2 *(and the conditions* (4) *in Theorem* 1.7 *are certainly not fulfilled).*

Proof. Since $D=[T_\varphi^*, T_\varphi]$ is self-adjoint, it suffices to show that D has trivial kernel. Suppose that $Df=[T_\varphi^*, T_\varphi]f=0$ for some function $f=\sum_{j=0}^{\infty} f_j e_j\in H^2$. From $(\|(S^*)^n\varphi\|)\in l_2$ and $(f_n)\in l_2$ it follows that the series $\sum_{k=0}^{\infty} f_k(S^*)^{k+1}\varphi$ converges in the Hilbert space H^2 and thus defines a function $h\in H^2$.

$Df=0$ implies that $\sum_{k=0}^{\infty} d_{nk} f_k=\sum_{k=0}^{\infty}\langle (S^*)^{n+1}\varphi, (S^*)^{k+1}\varphi\rangle f_k=\langle (S^*)^{n+1}\varphi, h\rangle=$ $=\langle (S^*)^n\varphi, Sh\rangle=0$ for all $n\in\mathbf{N}_0$. Because φ is cyclic for S^*, this yields $Sh=0$

and hence $h=0$. This gives $0=\langle e_n, h\rangle=\sum_{k=0}^{\infty} f_k\overline{\varphi_{k+n+1}}=\langle Sf, (S^*)^n_\varphi\rangle$ for $n\in\mathbf{N}_0$. Again by the cyclicity of φ for S^* it follows that $Sf=0$ and $f=0$ thus completing the proof.

Remark. Examples of functions φ as in Proposition 3 are easily obtained by taking lacunary series. Let $(n_k)_{k\in\mathbf{N}}$ be a sequence of positive integers such that $n_{k+1}\geqq\lambda n_k$ for some $\lambda>1$. If $(a_k)_{k\in\mathbf{N}}$ is an l_2-sequence such that $a_k\neq 0$ for all $k\in\mathbf{N}$, then $\varphi:=\sum_{k=1}^{\infty} a_k e_{n_k}\in H^2$ is cyclic for S^* ([5]), Theorem 2.5.5). The assumption $(\|(S^*)^n\varphi\|)\in l_2$ can be fulfilled by choosing a_k sufficiently small for large k.

4. Perturbations of normal operators

4.1. Suppose that $\{A, B\}\in\mathfrak{N}_1$. As in Section 1, we assume that $\alpha\in\mathbf{R}_1\setminus\sigma(A)$ and $\beta\in\mathbf{R}_1\setminus\sigma(B)$ and we set $X=(A-\alpha)^{-1}$ and $Y=(B-\beta)^{-1}$. It is easy to check that A and B commute strongly if and only if the bounded operator $T=X+iY$ is normal. Another method to construct couples of the class $\mathfrak{N}_1$ (with non-zero defect numbers!) is the following. We "perturb" the normality of T by adding an appropriate operator, say R, and we then take the inverses of the real and imaginary parts of $T+R$. We will discuss this method in the case $R=-NE$, where E is a rank one projection.

We denote by $U_r(x, y)$ the closed disk of radius r centered at $(x, y)\in\mathbf{R}_2$, and by K_r the circle of radius r centered at the origin.

Theorem 1. *Let N be a bounded normal operator with spectral resolution $N=\int z\,dG(z)$. Let $X=\operatorname{Re} N$ and $Y=\operatorname{Im} N$. Suppose that $\mathcal{N}(X)=\mathcal{N}(Y)=\{0\}$. Let $\xi\in\mathcal{H}$, $\|\xi\|=1$. Let $\tilde X:=\operatorname{Re} N(I-E_\xi)$ and $\tilde Y:=\operatorname{Im} N(I-E_\xi)$, where $E_\xi:=\xi\otimes\xi$. If ξ can be chosen such that $\tilde X$ and $\tilde Y$ satisfy the condition* (4) *in Theorem* 1.7 *and if $d(\tilde A:=\tilde X^{-1}, \tilde B:=\tilde Y^{-1})\neq 0$, then either $d(\tilde A, \tilde B)=2$ or $d(\tilde A, \tilde B)=3$.*

(i) *There is a $\xi\in\mathcal{H}$ such that $d(\tilde A, \tilde B)=3$ if and only if the (self-adjoint) operators $(X-a)(X-b)Y^{-1}$ and $(Y-a)(Y-b)X^{-1}$ are unbounded for all $a, b\in\mathbf{R}_1$.*

(ii) *There exists a $\xi\in\mathcal{H}$ such that $d(\tilde A, \tilde B)=2$ if and only if there is an $r>0$ such that the (self-adjoint) operators $(X-a)Y^{-1}\restriction\mathcal{H}_r$ and $(Y-a)X^{-1}\restriction\mathcal{H}_r$ are unbounded for all $a\in\mathbf{R}_1$ (or equivalently, the points* $(0, r)$, $(0, -r)$, $(r, 0)$ *and* $(-r, 0)$ *are in the spectrum of $N\restriction\mathcal{H}_r$), where $\mathcal{H}_r:=G(K_r)\mathcal{H}$.*

4.2. Proof. We will denote by $\mathcal{K}$ the linear span of the vectors ξ, $N\xi$ and $N^*N\xi$ in $\mathcal{H}$.

Suppose that $\xi\in\mathcal{H}$, $\|\xi\|=1$, is chosen such that $\tilde X$ and $\tilde Y$ satisfy (4). Suppose that $d(\tilde A=\tilde X^{-1}, \tilde B=\tilde Y^{-1})\neq 0$. By the normality of N, we have $2i[\tilde X, \tilde Y]=$ $=[(N(I-E_\xi))^*, N(I-E_\xi)]=(I-E_\xi)N^*N(I-E_\xi)-N(I-E_\xi)N^*=(-N^*N\xi+\|N\xi\|^2\xi)\otimes$

$\otimes\xi-\xi\otimes N^*N\xi+N\xi\otimes N\xi$. From this formula we see that ξ cannot be an eigenvector of N, because in this case $[\tilde{X}, \tilde{Y}]=0$ and thus $d(\tilde{A}, \tilde{B})=0$. Moreover, $\mathscr{R}([\tilde{X}, \tilde{Y}])\subseteq\mathscr{K}$.

We begin with the case $d(\tilde{A}, \tilde{B})=3$. Since $\mathscr{R}([\tilde{X}, \tilde{Y}])\subseteq\mathscr{K}$, we then have $\mathscr{R}([\tilde{X}, \tilde{Y}])=\mathscr{K}$ and $\dim\mathscr{K}=3$. We show that $(Y-a)(Y-b)\xi\notin\mathscr{R}(X)$ for all $a, b\in\mathbf{R}_1$. Assume the contrary, that is, $(Y-a)(Y-b)\xi=X\eta$ for some $\eta\in\mathscr{H}$. Then, $N^*N\xi+(a+b)iN\xi+ab\xi=(Y-a)(Y-b)\xi+X^2\xi+(a+b)iX\xi=X\psi$ with

$$\psi := \eta+(a+b)i\xi+X\xi$$

and

$$\tilde{X}\psi = X\psi-(1/2)\langle\psi, N\xi\rangle\,\xi-(1/2)\langle\psi, \xi\rangle N\xi = (\ldots)\xi+(\ldots)N\xi+N^*N\xi.$$

Since $\dim\mathscr{K}=3$, $\xi, N\xi$ and $N^*N\xi$ are linearly independent and hence $\tilde{X}\psi\neq 0$. On the other hand, $\tilde{X}\psi\in\mathscr{R}(\tilde{X})\cap\mathscr{K}=\mathscr{R}(\tilde{X})\cap\mathscr{R}([\tilde{X}, \tilde{Y}])$. This contradicts (4). Therefore, $(Y-a)(Y-b)\xi\notin\mathscr{R}(X)$ which implies that $(Y-a)(Y-b)X^{-1}$ is unbounded for all $a, b\in\mathbf{R}_1$. The proof for $(X-a)(X-b)Y^{-1}$ is similar.

Now assume that $0<d(\tilde{A}, \tilde{B})<3$. If the vectors $-N^*N\xi+\|N\xi\|^2\xi$, $-\xi$ and $N\xi$ would be linearly independent, then we could choose vectors $\varphi\in\mathscr{H}$ orthogonal to two of these vectors, but not orthogonal to the third. This would imply that $\dim\mathscr{R}([\tilde{X}, \tilde{Y}])=d(\tilde{A}, \tilde{B})=3$ which contradicts our assumption $d(\tilde{A}, \tilde{B})<3$. Therefore, the vectors $N^*N\xi$, ξ and $N\xi$ are linearly dependent. That is, there is a nontrivial relation $\lambda N^*N\xi+\mu N\xi+\varrho\xi=0$. If $\lambda=0$, then ξ is an eigenvector for N. Because $d(\tilde{A}, \tilde{B})\neq 0$, this is not possible. Thus we can assume without loss of generality that $\lambda=1$. From the spectral theorem (recall that N is normal) we conclude that $\xi\in G(\mathscr{E})$, where $\mathscr{E}:=\{z\in\mathbf{C}_1: |z|^2+\mu z+\varrho=0\}$.

The next step is to show that $\mu=0$. Assume the contrary, that is, $\mu\neq 0$. Then $\mathscr{E}$ is the intersection of a circle and a straight line. Hence $\mathscr{E}$ consists of at most two points. Therefore, $\xi=\xi_1+\xi_2$, where $N\xi_1=z_1\xi_1$ and $N\xi_2=z_2\xi_2$ with $\xi_1, \xi_2\in\mathscr{H}$ and $z_1, z_2\in\mathbf{C}_1$. Since ξ is not an eigenvector as noted above, it follows that $\xi_1\neq 0$, $\xi_2\neq 0$ and $z_1\neq z_2$. From this we conclude that $\mathscr{K}=\text{Lin}\,\{\xi, N\xi\}=\text{Lin}\,\{\xi_1, \xi_2\}$ and $\dim\mathscr{K}=2$. Since ξ_1 and ξ_2 are eigenvectors of $X=\text{Re}\,N$ as well, $\tilde{X}=X-(1/2)N\xi\otimes\xi-(1/2)\xi\otimes N\xi$ leaves $\mathscr{K}$ invariant. Since $\mathscr{N}(X)=\{0\}$ by assumption, $\tilde{X}$ maps $\mathscr{K}$ onto $\mathscr{K}$. Therefore, $\mathscr{R}(\tilde{X})\supseteq\mathscr{K}\supseteq\mathscr{R}([\tilde{X}, \tilde{Y}])$ and $\mathscr{R}(\tilde{X})\cap\mathscr{R}([\tilde{X}, \tilde{Y}])\neq\{0\}$. Because of (4), this is the desired contradiction. This proves that $\mu=0$.

By the preceding, $\mathscr{E}=\{z\in\mathbf{C}_1: |z|^2=-\varrho\}$. Since $\mathscr{E}$ contains more than one point (otherwise $\xi=0$ or ξ is an eigenvector for N), it follows that $-\varrho$ is positive. Let $r:=\sqrt{-\varrho}$. Let $\mathscr{H}_r=G(\mathscr{E})\mathscr{H}\equiv G(K_r)\mathscr{H}$. Since $\xi\in\mathscr{H}_r$, $2i[\tilde{X}, \tilde{Y}]=-\varrho\xi\otimes\xi+N\xi\otimes N\xi$. From this we easily see that the case $d(\tilde{A}, \tilde{B})=1$ is not possible. Indeed, $d(\tilde{A}, \tilde{B})=\dim\mathscr{R}([\tilde{X}, \tilde{Y}])=1$ implies that ξ is an eigenvector for N. But this leads to $d(\tilde{A}, \tilde{B})=0$.

We show that $(Y-a)\xi \notin \mathscr{R}(X)$ for all $a \in \mathbf{R}_1$. Otherwise, $(Y-a)\xi = X\eta$ for some $\eta \in \mathscr{H}$. Then $N\xi - ai\xi = i(Y-a)\xi + X\xi = X\psi$ with $\psi := i\eta + \xi$ and $\tilde{X}\psi = X\psi - (1/2)\langle \psi, N\xi\rangle \xi - (1/2)\langle \psi, \xi\rangle N\xi = (1-(1/2)\langle \psi, \xi\rangle)N\xi - (ai + (1/2)\langle \psi, N\xi\rangle)\xi \in \mathscr{K}$. Because $\xi \in \mathscr{H}_r$, $\dim \mathscr{K} \leqq 2$. Since $\mathscr{R}([\tilde{X}, \tilde{Y}]) \subseteq \mathscr{K}$ and we are in the case $d(\tilde{A}, \tilde{B}) = 2$, we obtain $\mathscr{K} = \mathscr{R}([\tilde{X}, \tilde{Y}])$. Since $\mathscr{R}(\tilde{X}) \cap \mathscr{R}([\tilde{X}, \tilde{Y}]) = \{0\}$ and $\mathscr{N}(\tilde{X}) = \{0\}$ by (4), it follows that $\tilde{X}\psi = 0$ and hence $\psi = 0$. Putting this in the above formula for $\tilde{X}\psi$, we get $\tilde{X}\psi = 0 = N\xi - ai\xi = 0$ which contradicts $d(\tilde{A}, \tilde{B}) \neq 0$. This proves that $(Y-a)\xi \notin \mathscr{R}(X)$. Obviously, $\mathscr{H}_r$ reduces both X and Y. Because $\xi \in \mathscr{H}_r$, $(Y-a)X^{-1} \upharpoonright \mathscr{H}_r$ must be unbounded for all $a \in \mathbf{R}_1$. The proof for $(X-a)Y^{-1} \upharpoonright \mathscr{H}_r$ is almost the same.

We now turn to the proof of the opposite directions. We begin with (i).

Assume that the operators $(X-a)(X-b)Y^{-1}$ and $(Y-a)(Y-b)X^{-1}$ are unbounded for all $a, b \in \mathbf{R}_1$. Suppose for a moment we have proved the existence of a vector $\xi \in \mathscr{H}$ of norm one such that

(11) $\quad (X-z)(X-w)\xi \notin \mathscr{R}(Y)$ and $(Y-z)(Y-w)\xi \notin \mathscr{R}(X)$ for all $z, w \in \mathbf{C}_1$.

Since X and Y are bounded and $XY = YX$, this implies that

(12)
$$(z_1 X^2 + z_2 X + z_3)\xi \notin \mathscr{R}(Y) \quad \text{and} \quad (z_1 Y^2 + z_2 Y + z_3)\xi \notin \mathscr{R}(X)$$
$$\text{for all } z_1, z_2, z_3 \in \mathbf{C}_1, \quad (z_1, z_2, z_3) \neq 0.$$

We check that $\tilde{X} = \operatorname{Re} N(I-E_\xi)$ and $\tilde{Y} = \operatorname{Im} N(I-E_\xi)$ satisfy condition (4). Recall that $\mathscr{R}([\tilde{X}, \tilde{Y}]) \subseteq \mathscr{K}$. Suppose that $\tilde{X}\varphi \in \mathscr{K}$, that is,

$$\tilde{X}\varphi = X\varphi - (1/2)\langle \varphi, N\xi\rangle \xi - (1/2)\langle \varphi, \xi\rangle N\xi = (\lambda + \mu N + \varrho N^* N)\xi$$
$$\text{for some } \lambda, \mu, \varrho \in \mathbf{C}_1.$$

Then $X(\varphi - ((1/2)\langle \varphi, \xi\rangle + \mu)\xi - \varrho X\xi) = (\lambda + (1/2)\langle \varphi, N\xi\rangle)\xi + (\mu + (1/2)\langle \varphi, \xi\rangle)iY\xi + \varrho Y^2\xi =: \psi$. By (12), $\psi = 0$. Again by (12), the vectors ξ, $Y\xi$ and $Y^2\xi$ are linearly independent. Hence $(1/2)\langle \varphi, \xi\rangle + \mu = \varrho = 0$ which gives $X\varphi = 0$. Since $\mathscr{N}(X) = \{0\}$ by assumption, $\varphi = 0$. This proves $\mathscr{R}(\tilde{X}) \cap \overline{\mathscr{R}([\tilde{X}, \tilde{Y}])} = \{0\}$. The same argument shows that $\mathscr{R}(\tilde{Y}) \cap \overline{\mathscr{R}([\tilde{X}, \tilde{Y}])} = \{0\}$ and $\mathscr{N}(\tilde{X}) = \mathscr{N}(\tilde{Y}) = \{0\}$. By Theorem 1.7, $\{\tilde{A} := \tilde{X}^{-1}, \tilde{B} := \tilde{Y}^{-1}\} \in \mathfrak{N}_1$. It remains to prove that $d(\tilde{A}, \tilde{B}) = 3$. As noted in the proof of the necessity part, it suffices to show that ξ, $N\xi$ and $N^*N\xi$ are linearly independent. If $\lambda\xi + \mu N\xi + \varrho N^*N\xi = 0$ for $\lambda, \mu, \varrho \in \mathbf{C}_1$, then $X(\mu\xi + \varrho X\xi) = -\lambda\xi - \mu iY\xi - \varrho Y^2\xi \in \mathscr{R}(X)$. By (12), this leads to $\lambda = \mu = \varrho = 0$.

To complete the proof of (i), we have to prove the existence of a unit vector $\xi \in \mathscr{H}$ satisfying (11). We let $\mathscr{X}$ and $\mathscr{Y}$ denote the x-axis, resp., y-axis. The following preparatory construction will be needed below. Let s be one of the numbers $0, 1, 2$. Let $\gamma \in \mathbf{R}_1$, and let $\varepsilon > 0$. Suppose that $(0, \gamma) \in \sigma(N)$. Since $\mathscr{N}(X) = \mathscr{N}(Y) = \{0\}$, there exists a sequence $(x_n, y_n) \in \sigma(N)$, $n \in \mathbf{N}$, such that $\lim_n (x_n, y_n) = (0, \gamma)$ and $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbf{N}$. Let us assume in addition that

$\lim_n |y_n-\gamma|^s|x_n|^{-1}=\infty$. Let $\mathscr{F}_\gamma:=\bigcup_{n\geqq k} U_{\varepsilon_n}(x_n, y_n)$, where $0<2\varepsilon_n<\min\{|x_n|, |y_n|\}$ and k is chosen so large that $|y_n-\gamma|^s>2|x_n|$ for $n\geqq k$ and $\mathscr{F}_\gamma\subseteq U_\varepsilon(0,\gamma)$. Then $\overline{\mathscr{F}}_\gamma\cap\mathscr{Y}=\{(0,\gamma)\}$. First we consider the case where $\gamma\neq 0$ or $s\neq 0$. Since $(Y-\gamma)^sX^{-1}$ is obviously unbounded on $G(\mathscr{F}_\gamma)$, there is a vector $\varphi_\gamma\in G(\mathscr{F}_\gamma)$ such that $\varphi_\gamma\notin\mathscr{D}((Y-\gamma)^sX^{-1})$, that is, $(Y-\gamma)^s\varphi_\gamma\notin\mathscr{R}(X)$. Now let $\gamma=0$ and $s=0$. Since $\overline{\mathscr{F}}_0\cap\mathscr{X}=\overline{\mathscr{F}}_0\cap\mathscr{Y}=\{(0,0)\}$ by construction and the function $f(x,y)=\min\{1/|x|, 1/|y|\}$ is not essentially bounded on $\mathscr{F}_0$ w.r.t. the spectral measure $G(\cdot)$, we can find a vector $\varphi_0\in G(\mathscr{F}_0)$ such that $\varphi_0\notin\mathscr{D}(f(X,Y))$. By the spectral theorem this implies $Y\varphi_0\notin\mathscr{R}(X)$ and $X\varphi_0\notin\mathscr{R}(Y)$.

In order to construct ξ we consider three cases.

Case 1: $\mathscr{Y}\cap\sigma(N)$ contains at least three different points $(0,\gamma_1), (0,\gamma_2), (0,\gamma_3)$.

We apply the preceding construction to each γ_j in case $s=0$, $0<3\varepsilon<\min\{|\gamma_j-\gamma_1|; j\neq 1\}$ and we obtain vectors $\varphi_{\gamma_1}, \varphi_{\gamma_2}, \varphi_{\gamma_3}$. Set $\xi_1=\varphi_{\gamma_1}+\varphi_{\gamma_2}+\varphi_{\gamma_3}$.

Case 2: $\mathscr{Y}\cap\sigma(N)$ consists of two different points $(0,\gamma_1), (0,\gamma_2)$.

Since $(Y-\gamma_1)(Y-\gamma_2)X^{-1}$ is unbounded and $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$, there is a sequence $(x_n, y_n)\in\sigma(N)$ such that $\lim_n|(y_n-\gamma_1)(y_n-\gamma_2)x_n^{-1}|=\infty$ and $x_n\neq 0, y_n\neq 0$ for all $n\in\mathbf{N}$. By passing to a subsequence if necessary (recall that (y_n) is bounded, since Y is bounded) we can assume that $\lim_n y_n=:\gamma$ exists. Since $(0,\gamma)\in\mathscr{Y}\cap\sigma(N)$, we must have $\gamma=\gamma_1$ or $\gamma=\gamma_2$. Say $\gamma=\gamma_1$. Since $\gamma_1\neq\gamma_2$ and Y is bounded, we have $\lim_n x_n=0$. Let $0<3\varepsilon<|\gamma_1-\gamma_2|$. Setting $s=1$ in case γ_1 and $s=0$ in case γ_2, the above construction yields vectors φ_{γ_1} and φ_{γ_2}. Put $\xi_1=\varphi_{\gamma_1}+\varphi_{\gamma_2}$.

Case 3: $\mathscr{Y}\cap\sigma(N)$ contains only one point $(0,\gamma)$.

Because $(Y-\gamma)^2X^{-1}$ is unbounded, we can find a sequence $(x_n, y_n)\in\sigma(N)$ such that $\lim_n|(y_n-\gamma)^2x_n^{-1}|=\infty$ and $x_n\neq 0, y_n\neq 0$ for $n\in\mathbf{N}$. As in Case 2 we can assume without loss of generality that $\lim_n y_n=\gamma$ and $\lim_n x_n=0$. We apply the above construction in case $s=2$, $\varepsilon=1$ and we set $\xi_1=\varphi_\gamma$.

Since X^{-1} must be unbounded, $\mathscr{Y}\cap\sigma(N)\neq\emptyset$. That is, we have discussed all possible cases.

It is not difficult to see that in each case $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$ for all $z, w\in\mathbf{C}_1$. We check this in Case 2. Recall that, by the spectral theorem, $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$ is equivalent to $\iint|(y-z)(y-w)x^{-1}|^2\, d\|G(x,y)\xi_1\|^2=\infty$. First let $z\neq\gamma_2$ and $w\neq\gamma_2$. Since $\overline{\mathscr{F}}_{\gamma_2}\cap\mathscr{Y}=\{(0,\gamma_2)\}$, we then have $(Y-z)(Y-w)\varphi_{\gamma_2}\notin\mathscr{R}(X)$. Because $U_\varepsilon(0,\gamma_1)\cap U_\varepsilon(0,\gamma_2)=\emptyset$, $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$. Now let $z=\gamma_2$. It is plane from the construction of φ_{γ_1} that $(Y-w)\varphi_{\gamma_1}\notin\mathscr{R}(X)$ for all $w\in\mathbf{C}_1$. Again by $U_\varepsilon(0,\gamma_1)\cap\cap U_\varepsilon(0,\gamma_2)=\emptyset$, this gives $(Y-z)(Y-w)\varphi_{\gamma_1}\notin\mathscr{R}(X)$ and $(Y-z)(Y-w)\xi_1\notin\mathscr{R}(X)$.

We now change the role of X and Y and we repeat the same procedure. The corresponding vectors will be denoted by ψ_{δ_j} and ξ_2. If $(0,0)\in\sigma(N)$ and if the case $s=0, \gamma=\delta=0$ occurs in the first *and* in the second procedure, then we set $\psi_0=\varphi_0$. As in the first part, we have $(X-z)(X-w)\xi_2 \notin \mathscr{R}(Y)$ for all $z, w \in \mathbf{C}_1$. If we take the radii of the circles around $(0, \gamma_j)$, resp., $(\delta_1, 0)$ small enough, then except from the possible case $\psi_0=\varphi_0$ we have just mentioned the vectors φ_{γ_j} and ψ_{δ_t} have disjoint support w.r.t. the spectral measure $G(\cdot)$. Therefore, $\xi:=$ $:=(\xi_1+\xi_2)/\|\xi_1+\xi_2\|$ has the desired properties. Now the proof of (i) is complete.

We only sketch the proof of the sufficiency part of (ii). Assume that the operators $(X-a)Y^{-1} \upharpoonright \mathscr{H}_r$ and $(Y-a)X^{-1} \upharpoonright \mathscr{H}_r$ are unbounded for all $a \in \mathbf{R}_1$. Since $\mathscr{H}_r=G(K_r)\mathscr{H}$ reduces X and Y, we can assume for simplicity in notation that $\mathscr{H}=\mathscr{H}_r$. Then $\sigma(N)\subseteqq K_r$. Since $(X-r)Y^{-1}$ is unbounded, there are points $(x_n, y_n)\in\sigma(N)$, $n\in\mathbf{N}$, so that $\lim_n |(x_n-r)y_n^{-1}|=\infty$. By taking a subsequence if necessary, we can assume that $\lim_n x_n=:\gamma$ exists. Since $x_n^2+y_n^2=r$ for $n\in\mathbf{N}$, it follows that $\gamma=-r, \lim_n y_n=0$ and $(-r, 0)\in\sigma(N)$. Using that $(X+r)Y^{-1}$, $(Y-r)X^{-1}$ and $(Y+r)X^{-1}$ are unbounded, the same argument shows that $(r, 0), (0, -r), (0, r)\in\sigma(N)$. Hence we can take vectors ξ_1, ξ_2, ξ_3 and ξ_4 in $\mathscr{H}(=\mathscr{H}_r)$ supported in the neighbourhood of $(-r, 0), (r, 0), (0, -r)$, resp., $(0, r)$ w.r.t. $G(\cdot)$ such that $\xi_1\notin\mathscr{R}(Y), \xi_2\notin\mathscr{R}(Y)$ and $\xi_3\notin\mathscr{R}(X), \xi_4\notin\mathscr{R}(X)$. Setting $\xi=$ $=\xi_1+\xi_2+\xi_3+\xi_4$, we then have that $(\lambda Y+\mu)\xi\notin\mathscr{R}(X)$ and $(\lambda X+\mu)\xi\notin\mathscr{R}(Y)$ for all $\lambda, \mu\in\mathbf{C}_1$, $(\lambda, \mu)\neq 0$. As in part (i) we can show that X and Y fulfil condition (4) in Theorem 1.7 and $d(\tilde{A}, \tilde{B})=2$.

Now the proof of Theorem 1 is complete.

4.2. Remarks. 1. There are many examples of operators N satisfying the assumptions of Theorem 1. We mention only two of them.

Example 1. Let N be a normal operator such that $\mathscr{N}(\operatorname{Re} N)=\mathscr{N}(\operatorname{Im} N)=$ $=\{0\}$. If $\sigma(N)$ intersects both the x-axis and the y-axis in at least three points, then (as we have seen in the preceding proof) the assumptions of part (i) are fulfilled. Hence, by Theorem 1, ξ can be chosen such that the corresponding pair $\{\tilde{A}, \tilde{B}\}$ is in $\mathfrak{N}_1$ and has defect number three.

Example 2. Let R be an unbounded self-adjoint operator, and let $N=$ $=(R-i)(R+i)^{-1}$ be its Cayley transfoım. N is unitary. Suppose that $\mathscr{N}(R)=$ $=\mathscr{N}(R+I)=\mathscr{N}(R-I)=\{0\}$. Obviously, this is equivalent to $\mathscr{N}(\operatorname{Re} N)=$ $=\mathscr{N}(\operatorname{Im} N)=\{0\}$. If the points $0, -1$ and 1 are in $\sigma(R)$, then the assumptions of part (ii) are satisfied (in case $r=1$) and Theorem 1 (ii), yields a pair $\{\tilde{A}, \tilde{B}\}\in\mathfrak{N}_1$ with $d(\tilde{A}, \tilde{B})=2$. In this case $N(I-E_\xi)$ is a partial isometry with corank one and defect one.

2. We want to interpret the method used in this section from another point of view. Again let $N=X+iY$ be normal and assume that $\mathscr{N}(X)=\mathscr{N}(Y)=\{0\}$. Of course, $\{A:=X^{-1}, B:=Y^{-1}\}\in\mathfrak{N}_1$ and $d(A, B)=0$. Suppose that $\{\tilde{A}=\tilde{X}^{-1}$,

$\tilde{B}=\tilde{Y}^{-1}\}$ is a pair of the class $\mathfrak{N}_1$ constructed as in Theorem 1 (i) or (ii). Then $\tilde{X}=X-(1/2)N\xi\otimes\xi-(1/2)\xi\otimes N\xi$ and $\tilde{Y}=Y-(1/2)iN\xi\otimes\xi+(1/2)i\xi\otimes N\xi$. We denote by F the orthogonal projection on $\mathscr{H}_\xi:=\mathrm{Lin}\,\{\xi, X\xi, Y\xi, NX\xi, NY\xi\}$. Modifying some arguments of the proof of Theorem 1, it can be shown that $\mathscr{R}(XY)\cap\mathscr{H}_\xi=\{0\}$. This implies that $\mathscr{D}_0:=XY(I-F)\mathscr{H}$ is dense in $\mathscr{H}$. Since the vectors ξ and $N\xi$ are orthogonal on $(I-F)\mathscr{H}, X(I-F)\mathscr{H}$ and $Y(I-F)\mathscr{H}$ by construction, it is easily seen that $A\restriction\mathscr{D}_0=\tilde{A}\restriction\mathscr{D}_0, B\restriction\mathscr{D}_0=\tilde{B}\restriction\mathscr{D}_0$ and $A\restriction B\mathscr{D}_0=\tilde{A}\restriction B\mathscr{D}_0$, $B\restriction A\mathscr{D}_0=\tilde{B}\restriction A\mathscr{D}_0$. In other words, the pair $\{\tilde{A},\tilde{B}\}\in\mathfrak{N}_1$ can be considered as an extension of the restriction to the dense domain $\mathscr{D}_0\subseteq\mathscr{D}(AB)\cap\mathscr{D}(BA)\cap\mathscr{D}(\tilde{A}\tilde{B})\cap\cap\mathscr{D}(\tilde{B}\tilde{A})$ of the strongly commuting pair $\{A,B\}\in\mathfrak{N}_1$.

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SEKTION MATHEMATIK
KARL-MARX-UNIVERSITÄT LEIPZIG
KARL-MARX PLATZ
7010 LEIPZIG, GERMAN DEMOCRATIC REPUBLIC