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Steckin-type estimates for locally divisible multipliers in Banach spaces

W. DICKMEIS, R. J. NESSEL, and E. VAN WICKEREN[1])

1. Introduction

Let $X_{2\pi}$ be one of the Banach spaces $C_{2\pi}$ or $L^p_{2\pi}$, $1 \leqq p < \infty$, of 2π-periodic functions, continuous or Lebesgue measurable on the real axis $\mathbf{R}$ with finite norm

$$\|f\|_{C_{2\pi}} := \max_{u\in\mathbf{R}} |f(u)|, \quad \|f\|_{p,2\pi} := \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(u)|^p\, du\right\}^{1/p},$$

respectively. Let $\mathbf{C}$ denote the complex plane and

$$\Pi_n := \left\{p_n \in C_{2\pi};\ p_n(u) := \sum_{k=-n}^{n} c_k e^{iku},\ c_k \in \mathbf{C}\right\} \tag{1.1}$$

the set of trigonometric polynomials of degree at most $n \in \mathbf{N}$ (:=set of natural numbers). For $f \in X_{2\pi}$ the error of best approximation by elements of Π_n is denoted by

$$E(f, n) := E(X_{2\pi};\ f, n) := \inf\{\|f-p_n\|_{X_{2\pi}};\ p_n \in \Pi_n\}. \tag{1.2}$$

Let the rth modulus of continuity of $f \in X_{2\pi}$ be given by ($r \in \mathbf{N}$)

$$\omega_r(X_{2\pi};\ f, t) := \sup_{|h|\leqq t} \left\|\sum_{k=0}^{r} (-1)^k \binom{r}{k} f(u+kh)\right\|_{X_{2\pi}}. \tag{1.3}$$

In these terms, Steckin [15] proved in 1951 the following (weak-type, cf. [5]) inequality ($f \in X_{2\pi}$, $n \in \mathbf{N}$)

$$\omega_r(X_{2\pi};\ f, n^{-1}) \leqq A_r n^{-r} \sum_{k=0}^{n} (k+1)^{r-1} E(X_{2\pi};\ f, k) \tag{1.4}$$

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which immediately furnishes the classical Bernstein inverse approximation theorem (cf. [19, p. 331 ff.]). Ten years later, STECKIN [16] considered the Fejér means

$$(F_{n-1}f)(u) := \sum_{k=-n}^{n} (1-|k|/n) f^{\wedge}(k) e^{iku}$$

of the Fourier series of $f \in X_{2\pi}$ where for $k \in \mathbf{Z}$ (:= set of integers) the kth Fourier coefficient is given by $2\pi f^{\wedge}(k) := \int_{-\pi}^{\pi} f(u) e^{-iku} du$. Corresponding to (1.4) he established the inequality $(f \in X_{2\pi}, n \in \mathbf{N})$

$$\|F_{n-1}f - f\|_{X_{2\pi}} \leq An^{-1} \sum_{k=0}^{n-1} E(X_{2\pi}; f, k). \tag{1.5}$$

In both cases, Steckin essentially used the same technique, namely Bernstein's classical telescope argument, employing the (unique) polynomial of best approximation in 1951 and the delayed means of de La Vallée Poussin in 1961, respectively. Moreover, in [16, p. 271] he pointed out that it would be interesting to obtain estimates, analogous to (1.5), for other methods of summation of Fourier series.

It is the purpose of this paper to derive inequalities of type (1.4, 5) for quite a general class of processes within the abstract framework of Banach spaces, admissible with respect to some Riesz-bounded spectral measure (see also the general approach given in [1a]).

To this end, multipliers are defined in Section 2 for Banach spaces which are generated via closure by some orthonormal structure given in terms of a spectral measure in a Hilbert space. If the spectral measure is Riesz-bounded, then a uniform bound can be derived for families of radial multipliers of Hardy-type (cf. Theorem 2.1). This enables one in Section 3 to introduce polynomials, potential spaces, and de La Vallée Poussin means, a basic tool. In fact, Sections 2 and 3 represent a brief outline of a general framework within which one may successfully treat a number of classical problems of approximation theory and numerical analysis (for details see [4], [12], [21] and the literature cited there).

To derive Steckin-type estimates, the concept of locally (globally) divisible multipliers is introduced in Section 4. Here we are heavily influenced by work of H. S. SHAPIRO [14] concerned with local divisibility within the Wiener ring of Fourier—Stieltjes transforms (see [14] and the literature cited there). In fact, whereas the present approach is finally based upon some Hilbert space structure (e.g., L^2), one may consult [1] for a different type of extension which deals with the local divisibility of Gelfand transforms in commutative Banach algebras (e.g., L^1). In any case, a first application of the present concepts leads to the Jackson-type inequality (4.2) and thus to the global Jackson-type theorem given in (4.3). The actual Steckin-type estimates are derived in Section 5. It is interesting to note that the Bernstein—

Steckin telescoping technique indeed extends to the present abstract situation. In this connection, let us mention that similar arguments in the setting of Besov spaces may also be found in [2], [10], 13], [23].

The sharpness of the classical estimates (1.4, 5) was already discussed by Steckin (cf. [16]) via concrete methods. But this kind of problems can also be dealt with in the present abstract setting. To this end, Section 6 first recalls a general theorem, established in [6], [7], which in fact does not need any orthogonal structure. Based upon some rather mild conditions upon the spectral measure and the locally divisible family of multipliers, Corollary 6.2 then shows that the assumptions of the theorem are indeed satisfied in the present context.

Finally, Section 7 is devoted to some first illustrating applications, emphasizing the unifying approach to the subject.

2. Multipliers

For a complex Hilbert space H let E be a (countably additive, selfadjoint, bounded, linear) spectral measure in $\mathbf{R}^N$, the Euclidean N-space ($N\in\mathbf{N}$) with inner product $xy:=\sum_{j=1}^{N} x_j y_j$ and norm $|x|:=(xx)^{1/2}$. Thus, E maps the family Σ of all Borel measurable sets in $\mathbf{R}^N$ into the set of all self-adjoint, bounded, linear projections of H such that ($B, B_j\in\Sigma$, $\emptyset$ being the void set, I the identity mapping)

$$\text{(2.1)}\qquad \begin{aligned} &\text{(i)}\quad E(B_1\cap B_2) = E(B_1)E(B_2),\\ &\text{(ii)}\quad E(\emptyset) = 0, \quad E(\mathbf{R}^N) = I,\\ &\text{(iii)}\quad E\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} E(B_j) \quad (B_i\cap B_j = \emptyset \quad \text{for} \quad i\neq j). \end{aligned}$$

Let $L^\infty(\mathbf{R}^N, E)$ be the space of complex-valued, E-essentially bounded functions τ with norm

$$\text{(2.2)}\qquad \|\tau\|_{\infty,E} := \inf\{\sup_{u\in B}|\tau(u)|;\ B\in\Sigma,\ E(B) = I\}.$$

For each $\tau\in L^\infty(\mathbf{R}^N, E)$ the integral $T^\tau:=\int_{\mathbf{R}^N}\tau(u)dE(u)$ is a bounded, linear operator of H into itself (for basic properties and further details see [8, pp. 900, 1930, 2186]).

For a given orthonormal structure (H, E) let X be a complex Banach space with norm $\|\cdot\|$ such that H and X are continuously embedded in some linear Hausdorff space (this hypothesis should be added in [4], see [23, p. 116]) and such that

$$\text{(2.3)}\qquad \overline{H\cap X}^{\|\cdot\|_H} = H, \quad \overline{H\cap X}^{\|\cdot\|} = X,$$

i.e., $H \cap X$ is dense in H and X. Then (cf. [4]) $\tau \in L^\infty(\mathbf{R}^N, E)$ is called a multiplier on X if for each $f \in H \cap X$

$$(2.4) \qquad T^\tau f := \int_{\mathbf{R}^N} \tau(u)\, dE(u) f \in H \cap X, \quad \|T^\tau f\| \leqq A \|f\|.$$

In view of (2.3, 4), the closure of T^τ (represented by the same symbol) belongs to $[X]$, the space of bounded, linear operators of X into itself. The set of all multipliers τ on X is denoted by $M = M(X) = M(X, H, E)$, the corresponding set of multiplier operators T^τ by $[X]_M$. Setting

$$(2.5) \qquad \|\tau\|_M := \|T^\tau\|_{[X]} := \sup\{\|T^\tau f\|;\ f \in H \cap X,\ \|f\| \leqq 1\},$$

M is a commutative Banach algebra with unit under the natural vector operations and pointwise multiplication, isometrically isomorphic to the subspace $[X]_M \subset [X]$.

Let $D^{(j)}, j \in \mathbf{N}$, be the set of real-valued, continuous, strictly increasing functions ψ on $[0, \infty)$ with $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$, which are $(j+1)$ times differentiable on $(0, \infty)$ with

$$(2.6) \qquad \text{(i)} \quad t^k |\psi^{(k+1)}(t)| \leqq K\psi'(t) \quad (0 \leqq k \leqq j,\ t > 0),$$

$$\text{(ii)} \quad \lim_{t\to 0+} t\psi'(t) = 0.$$

For $j = 0$ set $D^{(0)} := D^{(1)}$. In view of (2.6) one has

$$(2.7) \qquad t\psi'(t) \leqq \int_0^t [\psi'(u) + u|\psi''(u)|]\, du \leqq (K+1)\psi(t),$$

$$t^k \psi^{(k)}(t) \leqq Kt\psi'(t) \leqq K(K+1)\psi(t) \quad (0 \leqq k \leqq j+1).$$

Thus ψ satisfies

$$(2.8) \qquad \psi(st) \leqq s^{K+1}\psi(t) \quad (s \geqq 1,\ t \geqq 0),$$

since (2.7) implies

$$\log \frac{\psi(st)}{\psi(t)} = \int_t^{st} \frac{\psi'(u)}{\psi(u)}\, du \leqq \int_t^{st} \frac{K+1}{u}\, du = \log s^{K+1}.$$

Note that $\psi(t) = t^\gamma, \gamma > 0$, and $\psi(t) = \log(1+t)$ are admissible choices but not $\psi(t) = e^t$.

Let $\varphi(\varrho)$ be a real-valued, positive function on an index set $\mathscr{I}$. For $\psi \in D^{(j)}$ and a function σ, defined on $[0, \infty)$, the family $\{\sigma^\psi_{\varphi(\varrho)}\}_{\varrho \in \mathscr{I}}$ with $\sigma^\psi_{\varphi(\varrho)}(x) := \sigma(\varphi(\varrho)\psi(|x|))$ is said to be of Hardy-type (φ, ψ) if $\sigma^\psi_{\varphi(\varrho)}$ belongs to M, uniformly for $\varrho \in \mathscr{I}$ (cf. [21] and the literature cited there).

To formulate a criterion for multipliers of this type, assume that for a Banach space X, satisfying (2.3), the spectral measure E is Riesz-(R, j)-bounded for some $j\in\mathbf{P}$ (:= set of non-negative integers), i.e.,

$$r_{j,\varrho}(x) := \begin{cases} (1-|x|/\varrho)^j & \text{for} \quad |x| \leqq \varrho \\ 0 & \text{for} \quad |x| > \varrho \end{cases}$$

belongs to M and $\|r_{j,\varrho}\|_M \leqq C_j$, uniformly for all $\varrho>0$.

Theorem 2.1. *For some $j\in\mathbf{P}$ let E be a (R,j)-bounded spectral measure for X. Let $\psi\in D^{(j)}$ and let $\varphi(\varrho)$ be a positive function on $\mathscr{I}$. If $\sigma\in BV_{j+1}$, the class of (sufficiently smooth) functions satisfying*

$$\|\sigma\|_{BV_{j+1}} := \frac{1}{j!}\int_0^\infty t^j |d\sigma^{(j)}(t)| + \lim_{t\to\infty} |\sigma(t)| < \infty,$$

then the family $\{\sigma^{\psi}_{\varphi(\varrho)}\}_{\varrho\in\mathscr{I}}$ is of Hardy-type (φ,ψ). In fact, $\|\sigma^{\psi}_{\varphi(\varrho)}\|_M \leqq A\|\sigma\|_{BV_{j+1}}$ where A is independent of σ and $\varrho\in\mathscr{I}$.

Note that for $\psi(t)=t$, $\varphi(\varrho)=1/\varrho$, and $\mathscr{I}=(0,\infty)$ Theorem 2.1 covers the multipliers of Fejér-type. For further details, including the fractional extension of the class BV_{j+1}, however, see [4], [21], [22] and the literature cited there.

3. Polynomials, de La Vallée Poussin means, potential spaces

For some $j\in\mathbf{P}$ let E be (R,j)-bounded for a Banach space X. Let $\{\mu_\varepsilon\}_{\varepsilon>0}$ be a family of real-valued, infinitely differentiable functions on $[0,\infty)$ satisfying

$$0 \leqq \mu_\varepsilon(t) \leqq 1, \quad \mu_\varepsilon(t) = \begin{cases} 1, & 0 \leqq t \leqq 1+\varepsilon/2, \\ 0, & t \geqq 1+\varepsilon. \end{cases}$$

It follows from Theorem 2.1 that the family $\{T_{\varepsilon,\varrho}\}$,

$$T_{\varepsilon,\varrho} := T^{\mu_{\varepsilon,\varrho}}, \quad \mu_{\varepsilon,\varrho}(x) := \mu_\varepsilon(|x|/\varrho) \quad (\varepsilon,\, \varrho > 0),$$

is well-defined in $[X]_M$. The set of polynomials in X (of radial degree $\varrho>0$) is then defined by (cf. [11], [12])

$$\Pi := \bigcup_{\varrho>0} \Pi_\varrho, \quad \Pi_\varrho := \{f\in X;\ T_{\varepsilon,\varrho} f = f \quad \text{for all} \quad \varepsilon > 0\}.$$

In the following we shall call a Banach space X *admissible* (with respect to (H,E)) if X satisfies (2.3), E is (R,j)-bounded for some $j\in\mathbf{P}$, and if the polynomials are dense in X, i.e., $\overline{\Pi}^{\|\cdot\|}=X$. Obviously, the latter condition is equivalent to $(\varrho\to\infty)$

$$E(f,\varrho) := E(X;\, f,\varrho) := \inf\{\|p-f\|;\ p\in\Pi_\varrho\} = o(1),$$

where the error of best approximation $E(f, \varrho)$ (cf. (1.2)) is a decreasing function in $\varrho > 0$.

Basic for the present treatment will be the family $\{L_\varrho\}_{\varrho>0}$ of de La Vallée Poussin (or delayed) means. For a real-valued, infinitely differentiable function λ on $[0, \infty)$ satisfying

$$0 \leqq \lambda(t) \leqq 1, \quad \lambda(t) = \begin{cases} 1, & 0 \leqq t \leqq 1, \\ 0, & t \geqq 2, \end{cases} \tag{3.1}$$

set $\lambda_\varrho(x) := \lambda(|x|/\varrho)$ and $L_\varrho := T^{\lambda_\varrho}$. It follows from Theorem 2.1 that the operators L_ϱ are well-defined on each admissible space X. In fact, one has (cf. [11], [12])

Proposition 3.1. *Let X be an admissible Banach space. Then*

$$\|L_\varrho\|_{[X]} \leqq A \quad (\varrho > 0), \tag{3.2}$$

$$L_\varrho f \in \Pi_{2\varrho} \quad (f \in X, \varrho > 0), \tag{3.3}$$

$$L_\varrho p = p \quad (p \in \Pi_\varrho, \varrho > 0), \tag{3.4}$$

$$\|L_\varrho f - f\| \leqq CE(f, \varrho) \quad (f \in X, \varrho > 0). \tag{3.5}$$

Lemma 3.2. *Let X be an admissible Banach space (for $j \in \mathbf{P}$) and $\psi \in D^{(j)}$. Then $\beta_\varrho^\psi(x) := \psi(|x|)\lambda_\varrho(x) \in M$ and*

$$\|\beta_\varrho^\psi\|_M \leqq C\psi(\varrho) \quad (\varrho > 0). \tag{3.6}$$

Proof. Obviously, $v(t) := t\lambda(t) \in BV_{j+1}$ so that Theorem 2.1 yields $v^\psi_{1/\psi(2\varrho)} \in M$, uniformly for $\varrho > 0$. In view of the identity

$$\frac{\psi(|x|)}{\psi(2\varrho)} \lambda_\varrho(x) = v^\psi_{1/\psi(2\varrho)}(x)\lambda_\varrho(x),$$

(3.6) holds true since (cf. (2.8)) $\|\beta_\varrho^\psi\|_M \leqq \psi(2\varrho)\|v^\psi_{1/\psi(2\varrho)}\|_M \|\lambda_\varrho\|_M \leqq C\psi(\varrho)$.

Setting $B_\varrho^\psi := T^{\beta_\varrho^\psi}$ one may now define via $B^\psi g := \lim_{\varrho\to\infty} B_\varrho^\psi g$ the potential operator B^ψ as a closed, linear operator on the domain

$$X^\psi := \{g \in X;\ \lim_{\varrho\to\infty} \|B_\varrho^\psi g - h\| = 0 \quad \text{for some} \quad h \in X\}, \tag{3.7}$$

called potential space (see [12]). It follows that $\Pi \subset X^\psi$ and

$$B_\varrho^\psi = B^\psi L_\varrho, \quad \|B_\varrho^\psi\|_{[X]} = \|\beta_\varrho^\psi\|_M \leqq C\psi(\varrho). \tag{3.8}$$

In particular, X^ψ may be equipped with the seminorm $|g|_\psi := \|B^\psi g\|$ so that the K-functional

$$K_\psi(f, t) := K(X, X^\psi; f, t) := \inf_{g \in X^\psi} \{\|f-g\| + t|g|_\psi\} \tag{3.9}$$

is well-defined for all $f\in X$, $t\geqq 0$. It defines a seminorm on X for each $t\geqq 0$ and satisfies

(3.10) $$K_\psi(f,t)\leqq\begin{cases}\|f\| & (f\in X)\\ t|f|_\psi & (f\in X^\psi).\end{cases}$$

4. Locally divisible multipliers and Jackson-type inequalities

Throughout the next sections, X is an admissible Banach space for a (fixed) $j\in\mathbf{P}$.

Definition 4.1. Let $\psi\in D^{(j)}$ with inverse function ψ^{-1} and let $\varphi(\varrho)$ be a real-valued, positive function on $\mathcal{I}$. A family of uniformly bounded multipliers $\{\tau_\varrho\}_{\varrho\in\mathcal{I}}$ is called locally divisible (at the origin) of order (φ,ψ) if there exists some $\delta>0$ and a uniformly bounded family $\{\theta_\varrho\}_{\varrho\in\mathcal{I}}$ of multipliers such that

(4.1) $$\tau_\varrho(x)=\varphi(\varrho)\psi(|x|)\theta_\varrho(x)\quad\bigl(|x|\leqq\psi^{-1}(\delta/\varphi(\varrho))\bigr).$$

If (4.1) holds true for all $x\in\mathbf{R}^N$, $\varrho\in\mathcal{I}$, then the family $\{\tau_\varrho\}$ is said to be globally divisible.

Proposition 4.2. *Local divisibility implies the global one of the same order.*

Proof. Let $\{\tau_\varrho\}$ satisfy (4.1). Since $1-\lambda(t)=0$ for $0\leqq t\leqq 1$ (see (3.1)), the function $\sigma(t):=(1-\lambda(t))/t$ belongs to BV_{j+1}. Thus $\chi_\varrho:=\sigma^\psi_{\tilde\varphi(\varrho)}$ and $\nu_\varrho:=\lambda^\psi_{\tilde\varphi(\varrho)}$ with $\tilde\varphi(\varrho)=(2/\delta)\varphi(\varrho)$ belong to M, uniformly for $\varrho\in\mathcal{I}$ (see Theorem 2.1). Moreover, for all $x\in\mathbf{R}^N$,

$$1-\nu_\varrho(x)=\tilde\varphi(\varrho)\psi(|x|)\chi_\varrho(x)=(2/\delta)\varphi(\varrho)\psi(|x|)\chi_\varrho(x),$$

$$\tau_\varrho(x)\nu_\varrho(x)=\varphi(\varrho)\psi(|x|)\theta_\varrho(x)\nu_\varrho(x),$$

$$\tau_\varrho(x)=\tau_\varrho(x)\nu_\varrho(x)+\tau_\varrho(x)(1-\nu_\varrho(x))=\varphi(\varrho)\psi(|x|)[\theta_\varrho(x)\nu_\varrho(x)+\tau_\varrho(x)(2/\delta)\chi_\varrho(x)].$$

Hence the assertion follows since the terms in [...] are bounded in M, uniformly for $\varrho\in\mathcal{I}$.

The global and therefore also the local divisibility immediately implies that for any $g\in X^\psi$, $\varrho\in\mathcal{I}$ (cf. (3.7, 8))

$$T^{\tau_\varrho}g=\lim_{r\to\infty}T^{\tau_\varrho}L_rg=\varphi(\varrho)T^{\theta_\varrho}\lim_{r\to\infty}B^\psi_rg=\varphi(\varrho)T^{\theta_\varrho}B^\psi g.$$

Thus one obtains (cf. [3], [4])

Theorem 4.3. *Let $\{\tau_\varrho\}_{\varrho\in\mathcal{I}}$ be locally divisible of order (φ,ψ). Then there holds true the Jackson-type inequality*

(4.2) $$\|T^{\tau_\varrho}g\|\leqq A_1\varphi(\varrho)|g|_\psi\quad(g\in X^\psi,\ \varrho\in\mathcal{I}),$$

and therefore the (global) Jackson-type theorem

$$\|T^{\tau_\varrho}f\| \leqq A_2 K_\psi(f, \varphi(\varrho)) \quad (f\in X, \varrho\in\mathscr{I}). \tag{4.3}$$

Indeed, the estimate (4.3) is an immediate consequence of the definition (3.9) since for any $g\in X^\psi$,

$$\|T^{\tau_\varrho}f\| \leqq \|T^{\tau_\varrho}(f-g)\| + \|T^{\tau_\varrho}g\| \leqq A_0\|f-g\| + A_1\varphi(\varrho)|g|_\psi.$$

A first application yields for the error of the best approximation

Corollary 4.4. *Let* $\psi\in D^{(j)}$. *Then* $E(f, \varrho)\leqq CK_\psi(f, 1/\psi(\varrho))$ $(f\in X, \varrho>0)$.

Proof. Let $\mathscr{I}=(0, \infty)$ and set $v_\varrho:=\lambda^\psi_{\varphi(\varrho)}$, $\varphi(\varrho):=2/\psi(\varrho)$, and $\tau_\varrho:=1-v_\varrho$. Since $\theta(t):=(1-\lambda(t))/t\in BV_{j+1}$ and $\tau_\varrho(x)=(1/\psi(\varrho))\psi(|x|)[2\theta^\psi_{\varphi(\varrho)}(x)]$, the family $\{\tau_\varrho\}$ is globally divisible of order $(1/\psi, \psi)$. Thus Theorem 4.3 implies

$$\|f-T^{v_\varrho}f\| = \|T^{\tau_\varrho}f\| \leqq CK_\psi(f, 1/\psi(\varrho)).$$

Since $v_\varrho(x)=0$ for $|x|\geqq\varrho$, one has $T^{v_\varrho}f\in\Pi_\varrho$ for any $f\in X$ (cf. (3.3)) so that the assertion follows by the definition of $E(f, \varrho)$.

5. Steckin-type inequalities

First observe that in view of (3.4) one has for $p_\varrho\in\Pi_\varrho$, $\varrho>0$,

$$B^\psi_\varrho p_\varrho = B^\psi L_\varrho p_\varrho = B^\psi p_\varrho.$$

Thus Lemma 3.2 implies the following Bernstein-type inequality for polynomials in admissible Banach spaces $(p_\varrho\in\Pi_\varrho, \varrho>0)$

$$|p_\varrho|_\psi = \|B^\psi_\varrho p_\varrho\| \leqq \|B^\psi_\varrho\|_{[X]}\|p_\varrho\| \leqq C\psi(\varrho)\|p_\varrho\|. \tag{5.1}$$

Theorem 5.1. *For* $\psi\in D^{(j)}$ *one has the Steckin-type inequality*

$$K_\psi(f, 1/\varrho) \leqq (C_1/\varrho)\int_0^\varrho E(f, \psi^{-1}(u))\,du \quad (f\in X, \varrho>0), \tag{5.2}$$

thus for a locally divisible family $\{\tau_\varrho\}_{\varrho\in\mathscr{I}}$ *of order* (φ, ψ)

$$\|T^{\tau_\varrho}f\| \leqq C_2\varphi(\varrho)\int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\,du. \tag{5.3}$$

Proof. Obviously, (5.3) follows by (5.2) and Theorem 4.3. To show (5.2), set $P_k := L_{\psi^{-1}(2^k)} - L_{\psi^{-1}(2^{k-1})}$, $k \in \mathbf{Z}$. By (3.5) one has

$$\|P_k f\| \leqq \|L_{\psi^{-1}(2^k)} f - f\| + \|L_{\psi^{-1}(2^{k-1})} f - f\| \leqq A_1 E(f, \psi^{-1}(2^{k-1})) \leqq$$
$$\leqq A_1 2^{-k+2} \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du.$$

Since $P_k f \in \Pi_{2\psi^{-1}(2^k)}$ by (3.3), the Bernstein-type inequality (5.1) yields by (2.8)

$$|P_k f|_\psi \leqq A_2 \psi(2\psi^{-1}(2^k)) \|P_k f\| \leqq A_3 \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du.$$

In view of (3.8) one has $(k \to -\infty)$

$$\|B^\psi L_{\psi^{-1}(2^k)} f\| = \|B^\psi_{\psi^{-1}(2^k)} f\| \leqq A_4 2^k \|f\| = o(1)$$

so that for $m \in \mathbf{Z}$

$$\sum_{k=-\infty}^{m} B^\psi P_k f = \sum_{k=-\infty}^{m} (B^\psi L_{\psi^{-1}(2^k)} f - B^\psi L_{\psi^{-1}(2^{k-1})} f) = B^\psi L_{\psi^{-1}(2^m)} f.$$

Therefore it follows that

$$|L_{\psi^{-1}(2^m)} f|_\psi \leqq \sum_{k=-\infty}^{m} |P_k f|_\psi \leqq A_3 \sum_{k=-\infty}^{m} \int_{2^{k-2}}^{2^{k-1}} E(f, \psi^{-1}(u))\, du =$$
$$= A_3 \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du.$$

Now, let $\varrho > 0$ be arbitrary and $m \in \mathbf{Z}$ be such that $2^m \leqq \varrho < 2^{m+1}$. Then by (3.10)

$$K_\psi(f, 1/\varrho) \leqq K_\psi(f - L_{\psi^{-1}(2^m)} f, 1/\varrho) + K_\psi(L_{\psi^{-1}(2^m)} f, 1/\varrho) \leqq$$
$$\leqq A_4 E(f, \psi^{-1}(2^m)) + (1/\varrho) |L_{\psi^{-1}(2^m)} f|_\psi \leqq$$
$$\leqq A_4 2^{-(m-1)} \int_{2^{m-1}}^{2^m} E(f, \psi^{-1}(u))\, du + (A_3/\varrho) \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du \leqq$$
$$\leqq 4(A_4/\varrho) \int_{2^{m-1}}^{\varrho} E(f, \psi^{-1}(u))\, du + (A_3/\varrho) \int_0^{2^{m-1}} E(f, \psi^{-1}(u))\, du.$$

This establishes (5.2) completely.

Let us illustrate Theorem 5.1 in connection with the multiplier criterion of Theorem 2.1.

Corollary 5.2. *Let σ be a complex-valued function on* $[0, \infty)$, *locally divisible (of order $\psi_1(t) = t$) in BV_{j+1}, i.e., there exists an element $\chi \in BV_{j+1}$ satisfying*

$\chi(t)=1$ for $0\leqq t\leqq\delta$ and some $\delta>0$ such that

$$\theta(t) := \sigma(t)t^{-1}\chi(t)\in BV_{j+1}. \tag{5.4}$$

Suppose that $\{\sigma^{\psi}_{\varphi(\varrho)}\}_{\varrho\in\mathcal{J}}$ is of Hardy-type (φ, ψ). Then $\{\sigma^{\psi}_{\varphi(\varrho)}\}$ is globally divisible (in M) of order (φ, ψ). Moreover, there hold true the Jackson- and Steckin-type inequalities ($f\in X$, $\varrho\in\mathcal{J}$)

$$\|T^{\sigma^{\psi}_{\varphi(\varrho)}}f\| \leqq C_1K_{\psi}(f, \varphi(\varrho)) \leqq C_2\varphi(\varrho)\int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\,du. \tag{5.5}$$

Proof. In view of Theorem 4.3 and 5.1 it is sufficient to prove the local divisibility of $\{\sigma^{\psi}_{\varphi(\varrho)}\}$ (in M) of order (φ, ψ). By (5.4) and Theorem 2.1 the family $\{\theta^{\psi}_{\varphi(\varrho)}\}$ belongs to M, uniformly for $\varrho\in\mathcal{J}$. Moreover,

$$\sigma^{\psi}_{\varphi(\varrho)}(x) = \varphi(\varrho)\psi(|x|)\theta^{\psi}_{\varphi(\varrho)}(x) \quad (\varphi(\varrho)\psi(|x|)\leqq\delta)$$

since $\sigma(t)=t\theta(t)$ for $0\leqq t\leqq\delta$. Hence the assertion follows in view of (4.1).

6. Sharpness of Steckin-type inequalities

Let X^* be the class of bounded, *sub*linear functionals on the Banach space X, endowed with the usual operator norm $\|\cdot\|_{X^*}$. Let ω denote an abstract modulus of continuity, thus a function, continuous on $[0, \infty)$ such that

$$0 = \omega(0) < \omega(t) \leqq \omega(s+t) \leqq \omega(s)+\omega(t) \quad (s, t > 0). \tag{6.1}$$

Additionally, we assume that $\omega(t)\neq O(t)$, i.e.,

$$\lim_{t\to 0+} \omega(t)/t = \infty. \tag{6.2}$$

Moreover, let $\mathcal{J}$ be an unbounded subset of $(0, \infty)$ and φ a positive, monotonically decreasing function on $\mathcal{J}$ satisfying

$$\lim_{\varrho\to\infty} \varphi(\varrho) = 0. \tag{6.3}$$

In these terms one has the following result (see [6], [7]).

Theorem 6.1. *Let φ satisfy* (6.3). *Suppose that for $U_\varrho, V_\varrho\in X^*$ there exist constants C and elements $h_\varrho\in X$ with $(r, \varrho\in\mathcal{J})$*

$$\|h_\varrho\| \leqq C_1, \tag{6.4}$$

$$\|V_\varrho\|_{X^*} \leqq C_2, \tag{6.5}$$

$$|V_\varrho h_r| \leqq C_3\varphi(\varrho)/\varphi(r), \tag{6.6}$$

$$|U_\varrho h_r| \leqq C_{4,r}\varphi(\varrho), \tag{6.7}$$

$$\liminf_{\varrho\to\infty} |U_\varrho h_\varrho| \geqq C_5 > 0. \tag{6.8}$$

Then for each modulus ω *satisfying* (6.1, 2) *there exists a counterexample* $f_\omega \in X$ *such that* $(\varrho \to \infty)$

$$|V_\varrho f_\omega| = O(\omega(\varphi(\varrho))), \quad |U_\varrho f_\omega| \neq o(\omega(\varphi(\varrho))).$$

Suppose that the embedding $M(X) \subset L^\infty(\mathbf{R}^N, E)$, as assumed by the definition, is in fact continuous, i.e.,

$$\|\tau\|_{\infty, E} \leqq C\|\tau\|_M \quad (\tau \in M). \tag{6.9}$$

Corollary 6.2. *Let* φ *satisfy* (6.3). *Consider a locally divisible family* $\{\tau_\varrho\}_{\varrho \in \mathscr{I}} \subset M$ *of order* (φ, ψ) *for which there exist constants* K *and Borel sets* $\{B_\varrho\}_{\varrho \in \mathscr{I}} \subset \Sigma$ *with*

$$E(B_\varrho) \neq 0 \quad (\varrho \in \mathscr{I}), \tag{6.10}$$

$$\varphi(\varrho)\psi(|x|) \leqq K_1 \quad (x \in B_\varrho, \varrho \in \mathscr{I}), \tag{6.11}$$

$$|\tau_\varrho(x)| \geqq K_2 > 0 \quad (x \in B_\varrho, \varrho \in \mathscr{I}). \tag{6.12}$$

Then for each modulus (6.1, 2) *there exists a counterexample* $f_\omega \in X$ *such that* $(\varrho \to \infty)$

$$\varphi(\varrho) \int_0^{\varphi(\varrho)} E(f_\omega, \psi^{-1}(u))\, du = O(\omega(\varphi(\varrho))), \quad \|T^{\tau_\varrho} f_\omega\| \neq o(\omega(\varphi(\varrho))).$$

Proof. Let $\alpha(\varrho) := \psi^{-1}(K_1/\varphi(\varrho))$. For any $B \in \Sigma$ with $E(B) = I$ one has by (2.1) (i), (6.10) that $E(B \cap B_\varrho) = E(B)E(B_\varrho) = E(B_\varrho) \neq 0$, thus $B \cap B_\varrho \neq \emptyset$ by (2.1) (ii). Since $\lambda_{\alpha(\varrho)}(x) = 1$ for $x \in B_\varrho$ (cf. (3.1), (6.11)), it follows by (6.12) that

$$\sup_{x \in B} |\tau_\varrho(x)\lambda_{\alpha(\varrho)}(x)| \geqq \sup_{x \in B \cap B_\varrho} |\tau_\varrho(x)\lambda_{\alpha(\varrho)}(x)| = \sup_{x \in B \cap B_\varrho} |\tau_\varrho(x)| \geqq \inf_{x \in B_\varrho} |\tau_\varrho(x)| \geqq K_2.$$

In view of (2.2) this implies $\|\tau_\varrho \lambda_{\alpha(\varrho)}\|_{\infty, E} \geqq K_2$ and hence $\|\tau_\varrho \lambda_{\alpha(\varrho)}\|_M \geqq K_3 > 0$ by (6.9). Therefore, by the definition of the operator norm (see (2.5)) there exists $f_\varrho \in X$, $\|f_\varrho\| \leqq 1$, such that

$$\|T^{\tau_\varrho} L_{\alpha(\varrho)} f_\varrho\| = \|T^{\tau_\varrho \lambda_{\alpha(\varrho)}} f_\varrho\| \geqq K_4 > 0. \tag{6.13}$$

In order to apply Theorem 6.1 set

$$h_\varrho = L_{\alpha(\varrho)} f_\varrho, \quad V_\varrho f = \varphi(\varrho) \int_0^{1/\varphi(\varrho)} E(f, \psi^{-1}(u))\, du, \quad U_\varrho f = \|T^{\tau_\varrho} f\|.$$

Then $\|h_\varrho\| \leqq K_5$ by (3.2) so that (6.4) is fulfilled. Moreover, (6.5) follows with $C_2 = 1$, and (6.8) coincides with (6.13). It remains to show (6.6) since then (6.7) would also follow by Theorem 5.1 with $C_{4,r}=K_6/\varphi(r)$. But $E(h_r, \psi^{-1}(u))=0$ for $u\geqq\psi(2\alpha(r))$ since $h_r\in\Pi_{2\alpha(r)}$. Thus by (2.8)

$$V_\varrho h_r \leqq \varphi(\varrho)\int_0^{\psi(2\alpha(r))} E(h_r, \psi^{-1}(u))\,du \leqq \psi(2\alpha(r))\varphi(\varrho)\|h_r\| \leqq K_6\varphi(\varrho)/\varphi(r).$$

7. Applications

In this section some applications to the previous abstract results are given by studying concrete examples of spaces H, X, spectral measures E, and processes $\{\tau_\varrho\}$. In Section 7.1 we consider spaces of 2π-periodic functions in connection with one-dimensional trigonometric expansions. It is shown how the present approach covers, in a unified way, those classical results of S. B. Steckin mentioned in Section 1 as well as related material of R. Taberski and M. F. Timan on Abel—Poisson and typical means. In fact, the treatment of this example of a discrete expansion may easily be transferred to other discrete orthogonal systems (Jacobi, Hermite, Laguerre, etc.; for some details see [1a], [4], [8a], [12], [21] and the literature cited there). In Section 7.2 we consider the Abel—Cartwright means in connection with the continuous Fourier spectral measure on the Euclidean N-space, subsuming e.g. results of B. I. Golubov. Finally, Section 7.3 is concerned with a semidiscrete difference scheme for the numerical solution of the heat equation, the results being related to work of G. W. Hedstrom, J. Löfström, J. Peetre, V. Thomée, and others.

7.1. Classical results in spaces of periodic functions. Concerning the spaces $X_{2\pi}$, set $N=1, f_k(x):=e^{ikx}$, and for $B\in\Sigma, f\in L^2_{2\pi}$

$$E(B)f=\sum_{k\in B\cap\mathbf{Z}} f^\wedge(k)f_k.$$

Then E is a spectral measure for the Hilbert space $H=L^2_{2\pi}$. Obviously, $E(B)\neq 0$ iff $B\cap\mathbf{Z}\neq\emptyset$, so that $L^\infty(\mathbf{R}, E)$ may be identified with l^∞, the set of bounded sequences $\{\tau(k)\}_{k\in\mathbf{Z}}\subset\mathbf{C}$. Moreover, $L^2_{2\pi}\cap X_{2\pi}$ is dense in $L^2_{2\pi}$ as well as in $X_{2\pi}$, and the definition (2.4) of a multiplier $\tau=\{\tau(k)\}_{k\in\mathbf{Z}}\subset M(X_{2\pi})$ coincides with the classical one, i.e., for each $f\in X_{2\pi}$ there exists $f^\tau\in X_{2\pi}$ such that $\tau(k)f^\wedge(k)=(f^\tau)^\wedge(k)$ for every $k\in\mathbf{Z}$. Since

$$\|\tau\|_{\infty,E}=\sup_{k\in\mathbf{Z}}|\tau(k)|=\sup_{k\in\mathbf{Z}}\|f_k^\tau\|_{X_{2\pi}}\leqq\|\tau\|_{M(X_{2\pi})},$$

$M(X_{2\pi})$ is continuously embedded in l^∞. By Fejér's theorem, E is (R, j)-bounded on any $X_{2\pi}$ for $j=1$ (at least). Moreover, Π_ϱ coincides with (1.1), and Π is dense in $X_{2\pi}$ so that all the Banach spaces $X_{2\pi}$ are admissible (for details cf. [11], [12]).

Concerning the rth modulus (1.3) of continuity ($r\in\mathbf{N}$ even), Theorem 5.1 delivers Steckin's result (1.4).

Corollary 7.1. *Let* $r\in\mathbf{N}$ *be even. Then there holds true the inequality* ($f\in X_{2\pi}$, $n\in\mathbf{N}$)

$$\omega_r(X_{2\pi};\, f,\, 1/n) \leqq C_1 n^{-r}\int_0^{n^r} E(X_{2\pi};\, f,\, u^{1/r})\, du \leqq \tag{7.1}$$

$$\leqq C_2 n^{-r}\sum_{j=0}^{n-1}(j+1)^{r-1}E(X_{2\pi};\, f,\, j).$$

Proof. Let $\mathscr{I}=\mathbf{N}$ and $\tau_n^r(k):=(1-e^{ik/n})^r$. Since $\sigma^r(t)=(1-e^{it})^r/t^r$ is infinitely differentiable on $\mathbf{R}$, the multipliers $\theta_n(k):=\sigma^r(k/n)\lambda_n(k)$ belong to $M(X_{2\pi})$, uniformly for $n\in\mathbf{N}$. Now $\tau_n^r(k)=(k/n)^r\theta_n(k)$ for $|k|\leqq n$, so that $\{\tau_n^r\}$ is locally divisible of order (φ_r, ψ_r) with $\psi_r(u)=u^r$, $\varphi_r(n)=n^{-r}$. Then the first inequality of (7.1) is a consequence of (5.3), whereas the second one follows by substituting $u^{1/r}=t$ and using the monotonicity of $E(f, t)$.

Note that Theorem 5.1 is not applicable for odd $r\in\mathbf{N}$ since the corresponding potential multiplier $(ik)^r$ is not radial.

To reproduce Steckin's second result on Fejér sums, let us introduce a more general class of operators, the typical means

$$Z_n^r f := \sum_{k=-n}^{n}\left(1-\left(\frac{|k|}{n+1}\right)^r\right) f^\wedge(k) f_k.$$

Corollary 7.2. *For* $r\in\mathbf{N}$, $n\in\mathbf{P}$, *and* $f\in X_{2\pi}$

$$\|Z_n^r f-f\|_{X_{2\pi}} \leqq C_1(n+1)^{-r}\int_0^{(n+1)^r} E(X_{2\pi};\, f,\, u^{1/r})\, du \leqq \tag{7.2}$$

$$\leqq C_2(n+1)^{-r}\sum_{j=0}^{n}(j+1)^{r-1}E(X_{2\pi};\, f,\, j).$$

On the other hand, for each modulus (6.1, 2) *there exists an element* $f_\omega\in X_{2\pi}$ *such that* ($t\to 0+$, $n\to\infty$)

$$t\int_0^{1/t} E(X_{2\pi};\, f_\omega,\, u^{1/r})\, du = O(\omega(t)), \tag{7.3}$$

$$\|Z_n^r f_\omega - f_\omega\|_{X_{2\pi}} \neq o(\omega((n+1)^{-r})). \tag{7.4}$$

Proof. For an application of Corollary 5.2, set $\mathscr{I}=\mathbf{P}$, and $\sigma(t)=1-(1-t)_+$, $\psi(t)=t^r$, $\varphi(n)=(n+1)^{-r}$. Then the multiplier $\sigma_{\varphi(n)}^{\psi}$ of Hardy-type (φ, ψ) corresponds to the remainder $I-Z_n^r$. Since $\theta(t):=\sigma(t)\lambda(t)/t\in BV_2$, condition (5.4) is fulfilled so that (7.2) follows by (5.5). Concerning the sharpness of (7.2), apply Corollary 6.2 with $B_n=\{n+1\}$. Since (6.10—12) follow with $K_1=K_2=1$, one obtains (7.3, 4) at once.

Obviously, (7.2) for $r=1$ regains Steckin's result (1.5) on the Fejér means whereas for $r>1$ inequality (7.2) was established in [20]. Concerning the sharpness, it was shown in [6], [7] that even

$$\limsup_{n\to\infty} \|F_n f_\omega - f_\omega\|_{X_{2\pi}}/E(X_{2\pi};\, f_\omega, n) = \infty$$

for some element f_ω satisfying (7.3, 4).

Concerning the Abel—Poisson means, given by $(r\in(0,1)=\mathscr{I}, f\in X_{2\pi})$

$$P_r f := \sum_{k=-\infty}^{\infty} r^{|k|} f^\wedge(k) f_k;$$

consider the multiplier $p_r(k):=r^{|k|}=e^{-|k|\,|\log r|}$ of Hardy-type (φ, ψ) with $\psi(u)=u$, $\varphi(r)=|\log r|$. Since $(1-e^{-t})$, $(1-e^{-t})\lambda(t)/t\in BV_2$, Corollary 5.2 delivers (cf. [18], [20])

Corollary 7.3. *For the Abel—Poisson means P_r one has the Steckin-type inequality* $(f\in X_{2\pi}, 0<r<1)$

$$\|P_r f - f\|_{X_{2\pi}} \leqq C|\log r| \int_0^{1/|\log r|} E(X_{2\pi};\, f, u)\, du \leqq C\frac{1-r}{r} \sum_{0\leqq j\leqq 1/(1-r)} E(X_{2\pi};\, f, j).$$

7.2. Abel—Cartwright means in $L^p(\mathbf{R}^N)$. Let $L^p=L^p(\mathbf{R}^N)$, $1\leqq p\leqq\infty$, $N\in\mathbf{N}$, be the space of Lebesgue measurable functions on $\mathbf{R}^N$ for which the norm

$$\|f\|_p := \begin{cases} \Big\{(2\pi)^{-N/2} \int_{\mathbf{R}^N} |f(u)|^p\, du\Big\}^{1/p} & (1\leqq p<\infty) \\ \operatorname{ess\,sup}_{u\in\mathbf{R}^N} |f(u)| & (p=\infty), \end{cases}$$

respectively, is finite. For $f\in L^2$ let $\mathscr{F}f:=f^\wedge$ be the Fourier—Plancherel transform of f:

$$\lim_{\varrho\to\infty} \|(2\pi)^{-N/2} \int_{|u|\leqq\varrho} f(u) e^{-ivu}\, du - f^\wedge(v)\|_2 = 0,$$

and $\mathscr{F}^{-1}$ the inverse operator. For $B\in\Sigma$ let $\mathscr{P}_B$ be the multiplication projection:

$$\mathscr{P}_B f := \varkappa_B f, \quad \varkappa_B(u) := \begin{cases} 1, & u\in B, \\ 0, & u\notin B. \end{cases}$$

Then $E(B):=\mathscr{F}^{-1}\mathscr{P}_B\mathscr{F}$ is a spectral measure for the Hilbert space $H=L^2$, and $L^\infty(\mathbf{R}^N, E)=L^\infty$ (cf. [8, p. 1989]). Furthermore, $X=L^p$ satisfies (2.3) for $1\leqq p<\infty$, and (2.4) coincides with the classical definition of Fourier multipliers, i.e., $\tau\in M_p:=$

$:=M(L^p)$ iff $T^\tau f:=\mathscr{F}^{-1}(\tau f^\wedge)\in L^p$, $\|T^\tau f\|_p\leq A\|f\|_p$ for any $f\in L^2\cap L^p$ (cf. [17, p. 94]). Moreover, $M_p\subset L^\infty$ continuously. Note that E is (R,j)-bounded for L^p if, e.g., $j>(N-1)|1/p-1/2|$ (cf. [17, p. 114]), and that the polynomials are dense in L^p, where $\Pi_\varrho=\Pi_{\varrho,p}$ is the set of entire functions on $\mathbf{C}^N$ of (radial) exponential type ϱ the restriction to $\mathbf{R}^N$ of which belongs to L^p. Thus, the spaces L^p are admissible for $1\leq p<\infty$.

Let $\psi\in D^{(j)}$ for some $j>(N-1)|1/p-1/2|$ and $\varphi(t)>0$ for $t>0$. Consider the (generalized) Abel—Cartwright means $W^\psi_{\varphi(t)}$, corresponding to the multiplier $w^\psi_{\varphi(t)}$, $w(u)=e^{-u}$, of Hardy-type (φ,ψ). Since $w\in BV_{j+1}$ for every $j\in\mathbf{P}$, the approximation process $W^\psi_{\varphi(t)}$ is well-defined in $[L^p]$, uniformly bounded for $t>0$. In particular, $\psi(u)=\varphi(u)=u^\alpha$, $\alpha>0$, yields the standard Abel—Cartwright means $W_\alpha(t)$ which subsume for $\alpha=1$ the Abel—Poisson and for $\alpha=2$ the Gauss—Weierstrass means.

Corollary 7.4. *Let $1\leq p<\infty$ and $j>(N-1)|1/p-1/2|$. Suppose that $\psi\in D^{(j)}$, and let $\varphi(t)$ be a positive function, tending monotonically to zero for $t\to 0+$* (cf. (6.3)). *Then* $(f\in L^p(\mathbf{R}^N),\ t>0)$

$$\text{(7.5)}\qquad \|W^\psi_{\varphi(t)}f-f\|_p\leq C_1K_\psi(f,\varphi(t))\leq C_2\varphi(t)\int_0^{1/\varphi(t)}E(L^p;\,f,\psi^{-1}(u))\,du.$$

On the other hand, for each modulus (6.1, 2) *there exists a counterexample $f_\omega\in L^p$ such that* $(t\to 0+)$

$$\text{(7.6)}\qquad \varphi(t)\int_0^{1/\varphi(t)}E(L^p;\,f_\omega,\psi^{-1}(u))\,du=O(\omega(\varphi(t))),$$

$$\text{(7.7)}\qquad \|W^\psi_{\varphi(t)}f_\omega-f_\omega\|_p\neq o(\omega(\varphi(t))).$$

Proof. Obviously, (7.5) follows by Corollary 5.2 since $(1-e^{-u})/u\in BV_{j+1}$ for every $j\in\mathbf{P}$. Concerning the sharpness of (7.5), set $\varrho=1/t$, $B_\varrho=\{x\in\mathbf{R}^N;\ 1\leq\varphi(1/\varrho)\psi(|x|)\leq 2\}$, and $\tau_\varrho=1-w^\psi_{\varphi(1/\varrho)}$. Then (6.10—12) follow with $K_1=2$, $K_2=1-e^{-1}$ so that Corollary 6.2 delivers (7.6, 7).

Let us consider the rate of convergence of the standard Abel—Cartwright means $W_\alpha(t)f$ for elements $f\in L^p$ belonging to the (radial (cf. (1.3))) Lipschitz classes $(k\in\mathbf{N},\ 0<\beta\leq 2k)$

$$\mathrm{Lip}_{2k}(L^p(\mathbf{R}^N);\,\beta):=\{f\in L^p(\mathbf{R}^N);\ \omega_{2k}(L^p(\mathbf{R}^N);\,f,t)=O(t^\beta),\ t\to 0+\}.$$

Since one has $(\psi_{2k}(u)=u^{2k}$; cf. [24])

$$\text{(7.8)}\qquad K_{\psi_{2k}}(f,t^{2k}):=K(L^p,(L^p)^{\psi_{2k}};\,f,t^{2k})\leq C_k\omega_{2k}(L^p(\mathbf{R}^N);\,f,t),$$

Corollary 7.4 delivers (cf. [9])

Corollary 7.5. *Let $k\in\mathbf{N}$, $0<\alpha,\beta\leq 2k$, and $f\in\mathrm{Lip}_{2k}(L^p;\beta)$.*

(i) *If* $0<\alpha<2k$, *then* $(t\to 0+)$

(7.9)
$$\|W_\alpha(t)f-f\|_p = \begin{cases} O(t^\beta), & 0<\beta<\alpha,\\ O(t^\alpha|\log t|), & \beta=\alpha,\\ O(t^\alpha), & \beta>\alpha.\end{cases}$$

(ii) *For* $\alpha=2k$ *one has*

(7.10)
$$\|W_\alpha(t)f-f\|_p = O(t^\beta).$$

Proof. Obviously, (7.10) follows by (7.5, 8). Concerning (7.9), Corollary 7.4 implies (cf. [19a] for $a=1$)

(7.11)
$$\|W_\alpha(t)f-f\|_p = O\Big(t^\alpha \int_0^{t^{-\alpha}} E(L^p;\, f, u^{1/\alpha})\, du\Big).$$

By Corollary 4.4 and (7.8) the assumption yields $(u\to\infty)$

$$E(f, u^{1/\alpha}) = O\big(K_{\psi_{2k}}(f, u^{-2k/\alpha})\big) = O\big(\omega_{2k}(L^p;\, f, u^{-1/\alpha})\big) = O(u^{-\beta/\alpha}),$$

and the assertion follows by (7.11).

7.3. A semidiscrete difference scheme for the heat equation. In the frame of Section 7.2, let $N=1$ and $1\leq p<\infty$. In order to approximate the exact solution of the heat equation $(x\in\mathbf{R},\, t>0)$

$$d/dt\, u(x,t) = d^2/dx^2\, u(x,t), \quad u(x,0)=f(x)\in L^p,$$

given by the Gauss—Weierstrass means

$$W_2(t^{1/2})f(x) := (4\pi t)^{-1/2} \int_{-\infty}^{\infty} f(x-u)e^{-u^2/4t}\, du,$$

consider the initial value problem for $h>0$

$$d/dt\, u_h(x,t) = h^{-2}[u_h(x+h,t)-2u_h(x,t)+u_h(x-h,t)],\ u_h(x,0)=f(x).$$

This leads to the semidiscrete difference scheme (cf. [2])

$$u_h(\cdot,t) := D_h(t)f := T^{d_{h,t}}f, \quad d_{h,t}(x) := e^{-2(t/h^2)(1-\cos(xh))}.$$

Thus the multiplier $\tau_{h,t}$ of the remainder $D_h(t)-W_2(t^{1/2})$ has the representation

(7.12)
$$\tau_{h,t}(x) := g_{t/h^2}(xh), \quad g_r(u) := e^{-2r(1-\cos u)} - e^{-ru^2}.$$

Lemma 7.6. *The family* $\{g_r\}_{r>0}$ *is globally divisible of order* (φ_1, ψ_2) *with* $\varphi_1(r)=r$, $\psi_2(u)=u^2$ *and satisfies the (local) condition*

(7.13)
$$g_r(u) = ru^4 e^{-aru^2}\theta_r(u) \quad (|u|\leq\delta)$$

for $a=2/\pi^2$, $\delta=\pi/2$, *where the family* $\{\theta_r\}\subset M_p$ *is uniformly bounded for* $r>0$. *Moreover, there exists a constant* $c>0$ *such that*

(7.14)
$$g_r(u) \geq c \quad \big(r(u-2\pi)^2 \leq 1,\ (2\pi)^2 r \geq 9\big).$$

Proof. Since $e^{-ru^2}\in M_p$, uniformly in $r>0$ (cf. Theorem 2.1), and

$$\|e^{-2r(1-\cos u)}\|_{M_p} \leqq e^{-2r}\sum_{k=0}^{\infty}\frac{1}{k!}\|2r\cos u\|_{M_p}^{k} \leqq 1, \tag{7.15}$$

the family $\{g_r\}$ is uniformly bounded in M_p for $r>0$. To show (4.1), the Fejér-kernel $\sigma(u):=2u^{-2}(1-\cos u)$ belongs to M_p as well as $\chi(u):=(1-\sigma(u))/u^2$ (cf. Theorem 2.1). Consider the identity

$$g_r(u) = ru^2(1-\sigma(u))\int_0^1 e^{-(1-s)ru^2}e^{-2sr(1-\cos u)}\,ds. \tag{7.16}$$

Since the integral is uniformly bounded in M_p (cf. (7.15)), one has global divisibility of $\{g_r\}$ of order (φ_1, ψ_2). Concerning (7.13), set $e_r(x):=\exp\left[r(ax^2-2(1-\cos x))\right]$. Since $1-\cos x\geqq ax^2$ for $|x|\leqq 2\delta$, one has for $|x|\leqq 2\delta$:

$$0\leqq e_r(x)\leqq 1,\quad |e_r'(x)|\leqq C_1 r|x|e^{-arx^2},\quad |e_r''(x)|\leqq C_2 r(1+rx^2)e^{-arx^2}.$$

In view of (3.1) it follows that

$$\|e_r\lambda_\delta\|_{BV_2}\leqq \int_0^{2\delta} x[|e_r(x)\lambda_\delta''(x)|+2|e_r'(x)\lambda_\delta'(x)|+|e_r''(x)\lambda_\delta(x)|]\,dx \leqq$$

$$\leqq \int_0^{2\delta} x|\lambda_\delta''(x)|\,dx+2C_1\sup_{x\geqq 0}|x\lambda_\delta'(x)|\int_0^{2\delta} rxe^{-arx^2}\,dx+$$

$$+C_2\int_0^{2\delta} rx(1+rx^2)e^{-arx^2}\,dx \leqq C_3<\infty.$$

Thus $e_r\lambda_\delta\in M_p$, uniformly for $r>0$ (cf. Theorem 2.1). Therefore one obtains by (7.16) that for $|u|\leqq\delta$

$$g_r(u) = ru^4\chi(u)e^{-aru^2}\int_0^1 e^{-r(1-s)(1-a)u^2}e_{sr}(u)\lambda_\delta(u)\,ds.$$

Hence (7.13) follows since

$$\left\|\int_0^1 e^{-r(1-s)(1-a)u^2}e_{sr}(u)\lambda_\delta(u)\,ds\right\|_M \leqq C\|e^{-u^2}\|_{BV_2}\sup_{r>0}\|e_r\lambda_\delta\|_{BV_2}.$$

Finally, let $(2\pi)^2 r\geqq 9$ and $(u-2\pi)^2\leqq 1/r$. Then $u^2\geqq 4/r$ and

$$g_r(u) = e^{-2r(1-\cos(u-2\pi))}-e^{-ru^2}\geqq e^{-r(u-2\pi)^2}-e^{-4}\geqq e^{-1}-e^{-4}=:c>0.$$

Corollary 7.7. *For $f\in L^p(\mathbf{R})$, $1\leqq p<\infty$, and $h, t>0$*

$$\|D_h(t)f-W_2(t^{1/2})f\|_p\leqq C_1K(L^p,(L^p)^{\psi_2};f,h^2)\leqq C_2h^2\int_0^{h^{-2}} E(L^p;f,u^{1/2})\,du. \tag{7.17}$$

On the other hand, for each (fixed) $t>0$ *and each modulus* (6.1, 2) *there exists a counterexample* $f_\omega \in L^p$ *such that* $(h\to 0+)$

$$h^2\int_0^{h^{-2}} E(L^p;\, f_\omega, u^{1/2})\,du = O(\omega(h^2)),\quad \|D_h(t)f_\omega - W_2(t^{1/2})f_\omega\|_p \neq o(\omega(h^2)).$$

Proof. Set $\mathscr{I}=\{(h,t);\, h,t>0\}$ and $\varphi(h,t)=h^2$. Since M_p is dilation-invariant, i.e., $\|\chi(xh)\|_M=\|\chi\|_M$, the uniform boundedness of $\{g_r\}$ implies that of $\{\tau_{h,t}\}$ (cf. (7.12)). In view of (7.13) one has

$$\tau_{h,t}(x) = g_{t/h^2}(xh) = h^2x^2tx^2e^{-atx^2}\theta_{t/h^2}(xh)\quad (|xh|\leqq\delta),$$
$$\|tx^2e^{-atx^2}\theta_{t/h^2}(xh)\|_{M_p}\leqq \|tx^2e^{-atx^2}\|_{M_p}\|\theta_{t/h^2}\|_{M_p}\leqq K$$

(cf. Theorem 2.1). This implies the local divisibility of order (φ,ψ_2), and thus (7.17) by Theorems 4.3, 5.1.

In order to apply Corollary 6.2, set $\varrho=1/h$, $\varphi(\varrho)=1/\varrho^2$, and $B_\varrho=\{x\in\mathbf{R};\, t(x-2\pi\varrho)^2\leqq 1\}$. Then (6.10, 11) follow at once, and (6.12) by (7.14) for $\varrho\geqq 3/2\pi t^{1/2}$.

In view of (7.8, 17) one has

$$\|D_h(t)f-W_2(t^{1/2})f\|_p\leqq C\omega_2(L^p(\mathbf{R});\, f, h);$$

uniformly for $t>0$. This estimate can be improved to the following one which reflects the behaviour for e.g. $t\to 0+$ more precisely.

Corollary 7.8. *For* $f\in L^p(\mathbf{R})$ *and* $h, t>0$

$$\|D_h(t)f-W_2(t^{1/2})f\|_p\leqq C\begin{cases}(h^2/t)\omega_4(L^p;\, f, t^{1/2})+E(L^p;f,\delta/2h),\\ \omega_2(L^p;\, f, t^{1/2}).\end{cases}\tag{7.18}$$

Proof. Apply Theorem 4.3 to $\sigma_t(x)=a^2t^2x^4e^{-atx^2}$ which belongs to M_p, uniformly for $t>0$, since $u^4\exp(-u^2)\in BV_2$. Obviously, it is globally divisible of order (φ_2,ψ_4) so that Theorem 4.3 and (7.8) imply

$$\|T^{\sigma_t}f\|_p\leqq A_1K_{\psi_4}(f, t^2)\leqq A_2\omega_4(L^p;\, f, t^{1/2}).\tag{7.19}$$

In view of (7.13) one has for all $x\in\mathbf{R}$

$$\tau_{h,t}(x) = \tau_{h,t}(x)\lambda_{\delta/2h}(x)+\tau_{h,t}(x)(1-\lambda_{\delta/2h}(x)) =$$
$$= (h^2/t)\sigma_t(x)[a^{-2}\theta_{t/h^2}(xh)\lambda_{\delta/2h}(x)]+\tau_{h,t}(x)(1-\lambda_{\delta/2h}(x)).$$

Hence the first inequality follows by (3.5) and (7.19).

Since $\{g_r\}$ is globally divisible of order (φ_1,ψ_2) (cf. Lemma 7.6), there exists $\{v_r\}_{r>0}\subset M_p$, uniformly bounded for $r>0$, such that $g_r(u)=ru^2v_r(u)$. Hence $\tau_{h,t}(x)=g_{t/h^2}(xh)=tx^2v_{t/h^2}(xh)$ so that the second part of (7.18) follows by Theorem 4.3 and (7.8).

In Chapter IV of [2] (see also the literature cited there), the fundamental telescoping technique was also used for a parallel treatment of the present example in order to obtain error bounds on Besov spaces. The approach of this paper, however, uses the same technique only in the abstract setting in order to derive the estimates of Theorem 5.1. Consequently, for the concrete example one only needs to verify the basic divisibility assumptions. This procedure in fact delivers a comparison of the processes on the whole space.

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LEHRSTUHL A FÜR MATHEMATIK
RWTH AACHEN
TEMPLERGRABEN 55
5100 AACHEN, GERMANY