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Amalgamated free product of lattices. III. Free generating sets

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1. Introduction

In G. GRÄTZER and A. P. HUHN [4] it was proved that for a finite lattice Q any two Q-free products have a common refinement. This means that, whenever L, A_0, A_1, B_0, B_1 are lattices such that $L=A_0 *_Q A_1=B_0 *_Q B_1$, then

$$L = (A_0 \cap B_0) *_Q (A_0 \cap B_1) *_Q (A_1 \cap B_0) *_Q (A_1 \cap B_1)$$

$$A_i = (A_i \cap B_0) *_Q (A_i \cap B_1), \quad i = 0, 1,$$

and

$$B_j = (A_0 \cap B_j) *_Q (A_1 \cap B_j), \quad j = 0, 1.$$

It is still an open question whether there is any lattice Q not having this property. In this paper, we shall prove a related weaker statement.

By a *free generating set* of a lattice L we mean any relative sublattice freely generating L. The following question arises:

Is it true, that a free generating set of an amalgamated free product always contains free generating sets of the components?

In case of an affirmative answer it would follow that, for arbitrary Q, any two Q-free products have a common refinement, thus the above property is, indeed, stronger than the Common Refinement Property. In fact, assume that $L=A_0 *_Q A_1= =B_0 *_Q B_1$. Then $B_0 \cup B_1$ is a free generating set of L. Hence $A_i \cap (B_0 \cup B_1)= =(A_i \cap B_0) \cup (A_i \cap B_1)$ is a generating set of A_i. Thus, by Section 5 of [4], $A_i= =(A_i \cap B_0) *_Q (A_i \cap B_1)$, $i=0, 1$, whence, by the Main Theorem of [4], it follows that the two Q-free products have a common refinement.

We shall give a negative answer by proving the following theorem.

Theorem 1. *There exist lattices L, A_0, A_1, Q with $L=A_0 *_Q A_1$ and a free generating set G of L such that $[G \cap A_i]$ is a proper part of A_i, $i=0, 1$.*

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In fact, in our example $[G\cap A_i]=Q$, and $\bar{Q}$ is a proper part of A_i. The generating set $\bar{G}$ will be of the form $\bar{B}_0\cup\bar{B}_1$, where $\bar{B}_0$, $\bar{B}_1$ are relative sublattices of L, and $L=B_0*_Q B_1$ with $B_i=[\bar{B}_i]$. Therefore, it is natural to ask whether Theorem 1 can be developed into a counterexample showing that the two Q-free products $A_0*_Q A_1$ and $B_0*_Q B_1$ have no common refinement. Theorem 2 in Section 5 shows that this is not the case.

2. The construction of Q, A_i and G

First we shall define a partial lattice P and relative sublattices $\bar{Q}$, $\bar{A}_i$, and $\bar{B}_i$ of P, which will serve as generating sets of Q, A_i, and B_i, respectively. For a set X, let $S_e(X)$ denote the free semigroup on X with unit element e. P is defined as a subset of $S_e(\{0, 1, l, r\})$:

$$P = S_e(\{0,1\})\cup\{sl \mid s\in S_e(\{0,1\})\}\cup\{sr\mid s\in S_e(\{0,1\})\}.$$

The elements of P will be referred to as words, the elements $0, 1, r, l$ will be called letters. The last letter of a word s will be denoted by $\breve{s}$. $|s|$ will denote the number of letters in s. e will be considered the empty word. We shall use the convention that $\breve{e}=0$. Now we start defining joins and meets in P.

(i) For any $s\in S_e(\{0,1\})$, define $s=s0\vee s1=s1\vee s0$.

(ii) For any $s\in S_e(\{0,1\})$, define

$$s = sr\vee sl = sl\vee sr,$$
$$sl = s00\vee s10 = s10\vee s00,$$
$$sr = s01\vee s11 = s11\vee s01.$$

(iii) For any s, $p_0, p_1\in P$ with $\breve{s}=0$, define

$$sr = s01p_0\vee s11p_1 = s11p_1\vee s01p_0,$$

and, for any s, $p_0, p_1\in P$ with $\breve{s}=1$, define

$$sl = s00p_0\vee s10p_1 = s10p_1\vee s00p_0.$$

Now let

$$\bar{Q} = \{sr\mid \breve{s}=0\}\cup\{sl\mid \breve{s}=1\},$$
$$\bar{A}_i = \bar{Q}\cup\{s\mid \breve{s}=i,\ |s| \text{ is even},\ x\in\{e,l,r\}\},\quad i=0,1,$$
$$\bar{B}_i = \bar{Q}\cup\{s\mid \breve{s}=i,\ |s| \text{ is odd},\ x\in\{e,l,r\}\},\quad i=0,1.$$

(iv) For $a, b\in P$, define $a\leqq b$ if and only if either $a=b$ or there exist a positive integer n, and elements $a_0, a_1, \dots, a_{n-1}, a_n, c_0, c_1, \dots, c_{n-1}\in P$, such that $a=a_0$, $a_n=b$ and the relations $a_i\vee c_i=a_{i+1}$, $i=0,1,\dots,n-1$ hold by (i), (ii), or (iii).

This relation is a partial ordering on P. If $a \leqq b$, define $a \vee b = b \vee a = b$ and $a \wedge b = b \wedge a = a$.

A part of P together with all non-trivial joins (there are only trivial meets) is illustrated in the Figure.

Finally, let $L = F(P)$, the free lattice over P, let $Q = [\bar{Q}]$, $A_i = [\bar{A}_i]$, and $B_i = [\bar{B}_i]$, $i = 0, 1$, in L, and let $G = \bar{B}_0 \cup \bar{B}_1$.

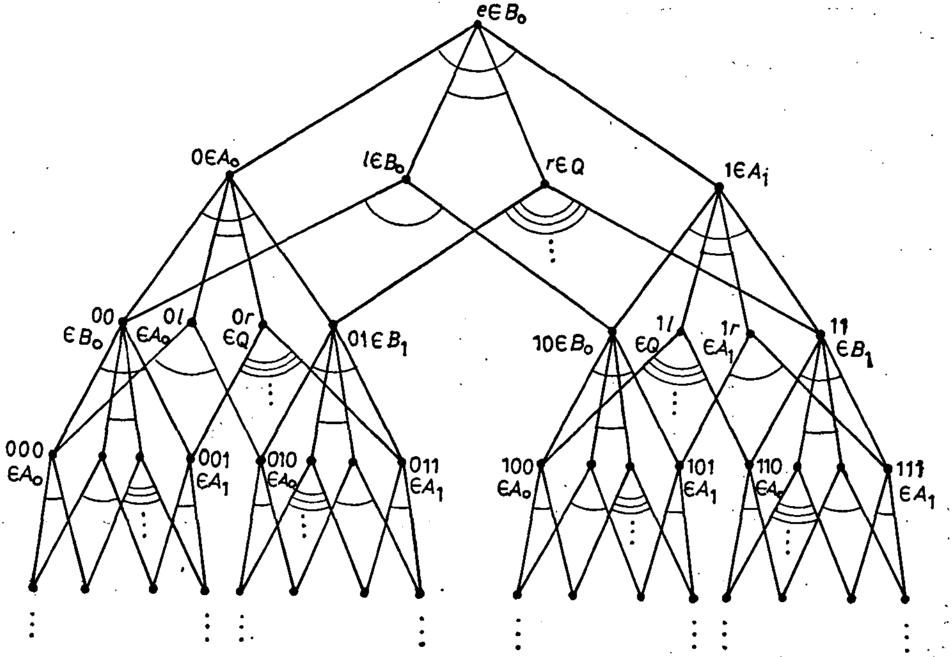

Fig. 1.

3. *P* is a partial lattice

This statement is of primary importance in the proof of Theorem 1 (see the proof of Lemma 6). In this section we shall give a proof. The following lemma will be used to prove that P is a weak partial lattice.

Lemma 1. *For any $a, b, c \in P$, if $a \leqq c$, $b \leqq c$, and $a \vee b$ is defined, then $a \vee b \leqq c$.*

The proof of this lemma proceeds via checking all the possible cases (i), (ii), (iii), and (iv) of how $b \vee c$ is defined and establishing the assertion in these separate cases. We omit the details.

Lemma 2. *P is a weak partial lattice.*

Proof. The following four statements and their duals are to be proved.

(a) for any $a\in P$, $a\vee a$ is defined and $a\vee a=a$;

(b) for any $a, b\in P$, if $a\vee b$ is defined, then $b\vee a$ is defined and $a\vee b=b\vee a$;

(c) for any $a, b\in P$, if $a\vee b$, $(a\vee b)\vee c$, $b\vee c$ are defined, then $a\vee(b\vee c)$ is defined and $(a\vee b)\vee c=a\vee(b\vee c)$;

(d) for an $a, b\in P$, if $a\wedge b$ is defined, then $a\vee(a\wedge b)$ is defined and $a\vee(a\wedge b)=a$.

Of these only (c) is non-trivial. We consider the following five cases.

First case: $a\vee b=b$. Then $a\leqq b\leqq b\vee c$, thus the right hand side in (c) exists and equals $b\vee c(=(a\vee b)\vee c)$.

Second case: $b<a\vee b$, $c\|b$, and $c\|a\vee b$. Observe that, under these conditions, the joins $b\vee c$ and $(a\vee b)\vee c$ can only be defined if, for suitable elements $p_0, p_1, p_2, s\in P$, one of the following four subcases holds:

$$\breve{s}=0,\quad c=s01p_0,\quad b=s11p_1,\quad a\vee b=s11p_2;$$

$$\breve{s}=0,\quad c=s11p_0,\quad b=s01p_1,\quad a\vee b=s01p_2;$$

$$\breve{s}=1,\quad c=s00p_0,\quad b=s10p_1,\quad a\vee b=s10p_2;$$

$$\breve{s}=1,\quad c=s10p_0,\quad b=s00p_1,\quad a\vee b=s00p_2.$$

In the first two subcases $a\vee(b\vee c)$ exists and equals sr, which is also the value of $(a\vee b)\vee c$. The last two subcases are similar, only the common value of the two sides is s.

Third case: $b<a\vee b$, $b\leqq c$, and $c\|a\vee b$. This case is impossible, for $(a\vee b)\vee c$ is defined and two incomparable elements whose join is defined cannot have a common lower bound (check the definitions (i), (ii), and (iii)).

Fourth case: $b<a\vee b$, $b\|c$, $c\leqq a\vee b$. Applying Lemma 1, we have that $b\leqq b\vee c\leqq a\vee b$. If the join $a\vee b$ was defined in (i) or (ii), then $b\vee c=b$, or $b\vee c=a\vee b$. But $b\vee c=b$ contradicts $b\|c$, thus $b\vee c=a\vee b$. Then $a\vee(b\vee c)$ is defined and $a\vee(b\vee c)=a\vee(a\vee b)=a\vee b=(a\vee b)\vee c$. If $a\vee b$ was defined in (iv), then $a\geqq b$, thus $a=a\vee b$. Hence $a\vee(b\vee c)=a=(a\vee b)\vee c$. Finally, if $a\vee b$ was defined in (iii), then we again have to consider four subcases as in the second case; we check only one of these:

$$a=s01p_0,\quad b=s11p_1,\quad \breve{s}=0.$$

Then $a\vee b=sr$, whence $s11p_1\leqq b\vee c\leqq sr$. Thus either $b\vee c=sr=a\vee b$, which can be handled similarly as the cases (i) or (ii), or there is a factorization $p_1=p_2p_3$ such that $b\vee c=s11p_2$ ($q_1=e$ is allowed, too). But then, (iii) applies again, whence $a\vee(b\vee c)=sr=a\vee b=(a\vee b)\vee c$.

Fifth case: $b<a\vee b$ and c is comparable with both b and $a\vee b$. Then the sub-

cases $c \leqq b$ and $a \vee b \leqq c$ are trivial and $b \leqq c \leqq a \vee b$ can be handled similarly as the fourth case.

These five cases exhaust all possibilities.

To finish the proof of the statement formulated in the heading of this section, we have to prove the following lemma (and its dual, but the latter is obvious).

Lemma 3. *If* $a, b, c \in P$ *and* $(a] \vee (b] = (c]$ *in the ideal lattice of* P, *then* $a \vee b = c$ *in* P.

Proof (by R. W. Quackenbush). Suppose that $(a] \vee (b] = (c]$ and $a \vee b$ is *not* defined. Let $a * b = s$ be the largest common initial segment of a and b. Then $(a] \vee (b] \subseteq (s]$, so $c \leqq s$. Now $a, b \in \{sr, sl, s0p, s1q\}$ for some p, q.

Case 1. $a = sl$. Then $b = s0p$ or $s1q$ since $sl \vee sr = s$.

1.1: $b = s0p$. Since $s \succ sl$, $c = s$.

Claim. $(s0] \vee (sl] = (s0] \cup (sl]$.

Proof. Let $d \leqq s0$ and $e \leqq sl$. Thus $d = s0p$; we assume that $d \vee e$ is defined. Thus $e \neq sl$; so $e = s00q$ or $s10q$. If $e = s00q$ then $d \vee e \leqq s0$. Thus let $e = s10q$. Then $d \vee e = sl$. This contradicts $(a] \vee (b] = (s]$ since $a \leqq s0$ and $b = sl$.

1.2: $b = s1q$. Similar to 1.1 using $(s1] \vee (sl] = (s1] \cup (sl]$.

Case 2: $a = sr$. By symmetry with Case 1.

Case 3: $a = s0p$, $b = s1q$. By symmetry, this is the last case.

3.1: $a = s0$. Thus $q \neq \emptyset$. We compute $(s0] \vee (s1q]$. Let $d \leqq s0$ and $e \leqq s1q$ and let us assume that $d \vee e$ is defined.

3.11: $q = 1q'$. The only possibilities are:

$$d \vee e = s01 \vee s11 = sr, \quad d \vee e = s01p' \vee s11q' = sr.$$

Thus either $sr \in (s0] \vee (s1q]$ and so $(s0] \vee (s1q] = (s0] \vee (sr] = (s0] \cup (sr]$ or $(s0] \vee (s1q] = (s0] \cup (s1q]$.

3.12: $q = 0q'$. Then similarly to 1.11, $(s0] \vee (s1q] = (s0] \cup (s1q]$ or $(s0] \cup (sl]$.

3.2: $b = s1$. So $p = \emptyset$. By symmetry with 1.1.

3.3: $a = s00p'$, $b = s11q'$. If $d \leqq a$ and $e \leqq b$, then $d = s00p''$, $e = s11q''$ and $d \vee e$ is not defined. Thus $(a] \vee (b] = (a] \cup (b]$.

3.4: $a = s01p'$, $b = s10q'$. Similar to 3.3.

3.5: $a = s00p'$, $b = s10q'$. Let $d \leqq a$ and $e \leqq b$. Since $a \vee b$ is not defined we must have $p' \neq \emptyset$ or $q' \neq \emptyset$ and we must have $\breve{s} = 0$. But then $d \vee e$ is not defined, since $d = s00p''$, $e = s10q''$. Hence $(a] \vee (b] = (a] \cup (b]$.

3.6: $a = s01p'$, $b = s11q'$. Similar to 3.5.

Now the above results, together with Funayama's characterization of partial lattices (see G. Grätzer [3]), gurantee that P is a partial lattice.

4. Proof of Theorem 1

We shall need a description of the free lattice generated by a partial lattice. The description we use is due to R. A. Dean [2] (see also H. Lakser [5]). Let $\langle X;\ \wedge, \vee\rangle$ (or briefly X) be a partial lattice, and let $F(X)$ denote the free lattice generated by X. Denote by $FP(X)$ the algebra of polynomial symbols in the two binary operation symbols $\wedge$ and $\vee$ generated by the set X. Then $F(X)$ is the image of $FP(X)$ under a homomorphism $\varrho\colon FP(X)\to F(X)$ with $x\varrho=x$ for $x\in X$. For each $p\in FP(X)$, we define an ideal p_X and a dual ideal p^X of X as follows.

$$p_X = \{x\in X | x \leqq p\varrho \text{ in } F(X)\},\quad p^X = \{x\in X | p\varrho \leqq x \text{ in } F(X)\}.$$

Now the description of $F(X)$ is found in the following three propositions. Actually, we need here only Propositions 2 and 3; Proposition 1 will be used in Section 5.

Proposition 1. *If $p, q\in FP(X)$, then $p\varrho\leqq q\varrho$ iff it follows by applying the following five rules.*

(W_C) $p^X\cap q_X \neq \emptyset$;

$({}_\vee W)$ $p = p_0\vee p_1$, $p_0\varrho \leqq q\varrho$ *and* $p_1\varrho \leqq q\varrho$;

$({}_\wedge W)$ $p = p_0\wedge p_1$, $p_0\varrho \leqq q\varrho$ *or* $p_1\varrho \leqq q\varrho$;

$(W_\vee)$ $q = q_0\vee q_1$, $p\varrho \leqq q_0\varrho$ *or* $p\varrho \leqq q_1\varrho$;

$(W_\wedge)$ $q = q_0\wedge q_1$, $p\varrho \leqq q_0\varrho$ *and* $p\varrho \leqq q_1\varrho$.

If $p\in P(X)$, then p_X and p^X can be calculated as follows.

Proposition 2. *For $p\in X$, $p_X=(p]$ (in $\langle X;\ \wedge, \vee\rangle$) and $p^X=[p)$. For $p=p_0\vee p_1$,*

$$p_X = (p_0)_X\vee(p_1)_X,\quad p^X = (p_0)^X\wedge(p_1)^X,$$

and, for $p=p_0\wedge p_1$,

$$p_X = (p_0)_X\wedge(p_1)_X,\quad p^X = (p_0)^X\vee(p_1)^X$$

where the $\vee$ and $\wedge$ on the right hand sides are to be formed in the lattice of all ideals (respectively, dual ideals) of $\langle X;\ \wedge, \vee\rangle$.

By a *binary tree* we mean a finite poset T with greatest element such that every element of T is either minimal or has exactly two lower covers. Now the join and meet of a set of ideals of $\langle X;\ \wedge, \vee\rangle$ can be formed as follows. The operations on the dual ideals are analogous.

Proposition 3. *Let $I_j, j\in J$ be ideals of $\langle X;\ \wedge, \vee\rangle$. Then $x\in\bigvee(I_j|j\in J)$ iff there is a binary tree T and there exist elements $x_t\in X$, $t\in T$ such that*

(1) $x=x_{\sup T}$;

(2) *if t is a minimal element in T, then $x_t \in I_j$ for some $j \in J$;*
(3) *if u and v are different lower covers of t, then $x_u \vee x_v$ is defined in $\langle X; \wedge, \vee \rangle$, and $x_t \leqq x_u \vee x_v$.*

$\wedge (I_j | j \in J)$ *is the intersection of* $\{I_j | j \in J\}$.

The proof of Theorem 1 will be completed by the following three lemmas.

Lemma 4. *L is freely generated by $\bar{A}_0 \cup \bar{A}_1$ as well as by $\bar{B}_0 \cup \bar{B}_1$.*

Proof. It is enough to show that all the elements of P can be expressed by elements of $\bar{A}_0 \cup \bar{A}_1$, and these expressions obey all the relations (i) to (iv) in $F(\bar{A}_0 \cup \bar{A}_1)$, that is, (i) to (iv) can be derived from the relations valid in $\bar{A}_0 \cup \bar{A}_1$. (The statement concerning $\bar{B}_0 \cup \bar{B}_1$ can be proved analogously.) In fact, let $s \in P$, $s \notin A_0 \cup A_1$. Then an expression of s by elements of $\bar{A}_0 \cup \bar{A}_1$ is

$$s = s0 \vee s1 \quad \text{if} \quad \breve{s} \in \{0, 1\}, \tag{4}$$

$$s = s'000 \vee s'001 \vee s'100 \vee s'101 \quad \text{if} \quad s = s'l, \tag{5}$$

$$s = s'010 \vee s'011 \vee s'110 \vee s'111 \quad \text{if} \quad s = s'r. \tag{6}$$

It is straightforward to check the relations (i) to (iv). Let us consider only one example: $s = s0 \vee s1$, $s \in \bar{A}_0 \cup \bar{A}_1$. In fact, applying (ii) within $\bar{A}_0 \cup \bar{A}_1$ and (4) we have

$$s = sl \vee sr = (s00 \vee s10) \vee (s01 \vee s11) = (s00 \vee s01) \vee (s10 \vee s11) = s0 \vee s1.$$

Lemma 5. *$L = A_0 *_Q A_1 = B_0 *_Q B_1$.*

Proof. Let $\bar{A}_0 \cup_{\bar{Q}} \bar{A}_1$ be the weakest partial lattice defined on the set $\bar{A}_0 \cup \bar{A}_1$ having $\bar{A}_0$ and $\bar{A}_1$ as sublattices. The same proof as that of Lemma 4 yields that $L = F(\bar{A}_0 \cup_{\bar{Q}} \bar{A}_1)$, for every join defined in $\bar{A}_0 \cup \bar{A}_1$ is defined either within A_0 or within A_1. Now $Q \subseteqq A_0$ and $Q \subseteqq A_1$, thus we can form the union $A_0 \dot{\cup} A_1$ subject to the condition $A_0 \cap A_1 = Q$. Let $A_0 \cup_Q A_1$ be the weakest partial lattice on $A_0 \dot{\cup} A_1$ extending the operations defined in A_0 or A_1. Since $A_0 \cup_Q A_1$ contains a copy of $\bar{A}_0 \cup_{\bar{Q}} \bar{A}_1$, there is a homomorphism φ of $L = F(\bar{A}_0 \cup_{\bar{Q}} \bar{A}_1)$ onto $F(A_0 \cup_Q A_1)$. Since L contains copies of A_0 and A_1 with $Q \subseteqq A_0$, $Q \subseteqq A_1$, there is a homomorphism ψ of $F(A_0 \cup_{\bar{Q}} A_1)$ into L. $\varphi\psi$ is the identity on $A_0 \cup_Q A_1$, hence it is the identity on L. Thus φ is one-to-one. Summarizing

$$L = F(\bar{A}_0 \cup_{\bar{Q}} \bar{A}_1) \cong F(A_0 \cup_Q A_1) = A_0 *_Q A_1.$$

This isomorphism is the identity on A_0 and on A_1, therefore L is the Q-free product of its sublattices A_0 and A_1. Analogously, $L = B_0 *_Q B_1$, completing the proof.

By Lemma 4, L has the free generating set $G = \bar{B}_0 \cup \bar{B}_1$, and, by Lemma 5, it has the Q-free decomposition $L = A_0 *_Q A_1$. Thus the following lemma proves Theorem 1.

Lemma 6. *$G \cap A_i \subseteq Q$, that is, $[G \cap A_i]$ is a proper part of A_i.*

Proof. By symmetry it is enough to show that $\bar{B}_0 \cap A_0 \subseteq Q$. Let us assume that an element $b_0 \in \bar{B}_0$ can be expressed by elements of $\bar{A}_0$, that is, $b_0 = p(a_0, \ldots, a_n)$, where p is a polynomial and $a_0, \ldots, a_n \in \bar{A}_0$. Then, by Proposition 2,

$$[b_0) = p^\delta([a_0), \ldots, [a_n))$$

holds in the lattice of all dual ideals of P, where p^δ is the polynomial dual to p. This lattice is distributive and, by distributivity, p^δ can be rearranged in such a way that all the joins precede all the meets in it:

$$[b_0) = \bigvee(\bigwedge([a_j)|j \in J_i)|i \in I) \tag{7}$$

with $J_i \subseteq \{0, 1, \ldots, n\}$, for all $i \in I$, while, by the distributive inequality,

$$b_0 \leqq \bigwedge\bigvee(a_j|j \in J_i|i \in I) \tag{8}$$

holds in L. Since $[b_0)$ is a principal dual ideal, from (7) we obtain that there exists i in I such that

$$[b_0) = \bigwedge([a_j)|j \in J_i).$$

By Proposition 2, we have

$$b_0 \geqq \bigvee(a_j|j \in J_i).$$

This, together with (8) yields

$$b_0 = \bigvee(a_j|j \in J_i).$$

Again, by Proposition 2, we have

$$(b_0] = \bigvee((a_j]|j \in J_i).$$

Now we show that this is impossible unless $b_0 \in Q$. We carry out the proof for $b_0 = e$; for other choices of b_0 there is no essential difference in the proof.

We show that $e \notin \bigvee(a_j]$ if a_j runs over all elements of $\bar{A}_0$. Consider a binary tree $\bar{T}$ and a set $X = \{x_t|t \in T\}$ with the properties (1) to (3), with $I_j = (a_j]$. There are only two joins with the value e, namely $e = 0 \vee 1$ and $e = l \vee r$. Thus X contains 0 and 1 or l and r. Of these $1 \nleqq a_j$ (respectively, $l \nleqq a_j$) for all j, therefore, by (2), there is a $t \in T$ (t not minimal), such that $1 = x_t$ (respectively, $l = x_t$). Thus (3) can be applied: 10 (and 11) or $1r$ (and $1l$) (respectively, 10 (and 00)) are contained in X. (2) does not apply for 10 and $1r$, thus we can proceed by (3): $101 \in X$ or $10l \in X$. Now, by induction, we obtain that $101\ldots01 \in X$ or $101\ldots0l \in X$, which contradicts the fact that X is finite. This contradiction completes the proof.

5. Some remarks

First of all, we prove the statement already announced in the introduction that the example is no counterexample for the common refinement property. It is worth mentioning that it is exactly the characterization theorem of the existence of common refinements in [4] that will be used to prove this assertion.

Theorem 2. *The two Q-free products* $L=A_0 *_Q A_1 = B_0 *_0 B_1$ *have a common refinement.*

We need the following lemma.

Lemma 7. *Let* $b_0, \ldots, b_m \in B_0$ *and let p be a polynomial in* $b_0, \ldots, b_m$. *Then for any* $x \in \bar{A}_0$ *satisfying* $x \leqq p(b_0, \ldots, b_m)$ *in* $F(P)$, *there exists an element* $c \in A_0 \cap B_0$ *with* $x \leqq c \leqq p(b_0, \ldots, b_m)$.

Before proceeding to the proof, we present another lemma, which will be used in the proof of Lemma 7.

Lemma 8. *Let* $b_0, \ldots, b_m \in B_0$ *and let* $p = p_0 \vee p_1$ *be a polynomial in* $b_0, \ldots, b_m$. *Assume that, for any* $x \in \bar{A}_0$ *satisfying* $x \leqq p_i(b_0, \ldots, b_m)$ *for* $i=0$ *or* $i=1$, *there exists an element* $c \in A_0 \cup B_0$ *such that* $x \leqq c \leqq p_i$. *Let, furthermore, T be a binary tree and let* x_t, $t \in T$, *be elements of P satisfying the condition* $x_{\sup T} \in \bar{A}_0$ *as well as the conditions (2) and (3) of Section 4 with* $(p_j)_P$, $j=0, 1$, *and P in the place of* I_j, $j \in J$, *and* $\langle X; \wedge, \vee \rangle$, *respectively. Then there exists an element* $c \in A_0 \cap B_0$ *such that* $x_{\sup T'} \leqq c \leqq p(b_0, \ldots, b_m)$.

Proof of Lemma 8. We proceed by an induction. Set $b = p(b_0, \ldots, b_m)$. If $T = \{t\}$ is a singleton, then, by (2), $x_t \leqq p_i(b_0, \ldots, b_m)$, for $i=0$ or $i=1$. By one of our assumptions $x_t \leqq c \leqq p_i(t_0, \ldots, b_m)$ for a suitable $c \in A_0 \cap B_0$, whence $x_t \leqq c \leqq p(b_0, \ldots, b_m) = b$. Assume that T consists of more than one element and the statement is valid for any proper binary subtree of T. Let u and v denote the different maximal elements of $T - \{\sup T\}$. Now $x_{\sup T} \leqq x_u \vee x_v \leqq b$. If $x_u \vee x_v \in \bar{A}_1$ (respectively, $x_u \vee x_v \in \bar{B}_1$), then, by Lemma 5, there exists an element $q \in Q$ such that $x_u \vee x_v \leqq q \leqq x_{\sup T}$ (respectively, $x_{\sup T} \leqq q \leqq b$), proving the statement of the lemma. If $x_u \vee x_v \in \bar{B}_0$, then we may assume that there exists no $y \in \bar{A}_1 \cup \bar{B}_1$ with $x_{\sup T} \leqq y \leqq x_u \vee x_v$, else we could find an element $q \in Q$ with $x_{\sup T} \leqq q \leqq x_u \vee x_v$ similarly as above. Thus it follows that the interval $[x_{\sup T}, x_u \vee x_v]$ contains a prime interval $[y_0, y_1]$ of P with $y_0 \in \bar{A}_0, y_1 \in \bar{B}_0$. Then, using the notation of Section 2, $y_0 = y_1 0$. Let $c = y_1 0 \vee y_1 r$. Obviously, $c \in A_0$. Compute:

$$c = y_1 0 \vee y_1 r = y_1 00 \vee y_1 01 \vee y_1 r = y_1 00 \vee y_1 r.$$

Now $y_1 r \in Q$ and $y_1 00 \in B_0$, hence $c \in B_0$, which again proves the lemma. We may assume that $x_u \vee x_v \in \bar{A}_0$. We may also assume that $x_u \vee x_v \neq x_u, x_v$. Thus, by the definition of P, either $x_u, x_v \in \bar{A}_0$ or $x_u \in \bar{B}_0, x_v \in \bar{B}_1$. In the former case we can apply the induction hypothesis for the subtrees $(u], (v] \subseteq T$, whence there exist elements $c_0, c_1 \in A_0 \cap B_0$ with $x_u \leqq c_0 \leqq b$, $x_v \leqq c_1 \leqq b$. Thus $x_u \vee x_v \leqq c_0 \vee c_1 \leqq b$ and $c_0 \vee c_1 \in \in A_0 \cap B_0$. In the latter case, using again the notations introduced in Section 2, $x_u = x_u 0 \vee x_u 1$, $x_v = x_v 0 \vee x_v 1$, and $x_u 1 \vee x_v 1 \in Q$. Now, replacing the element x_u by $x_u 0$ and x_v by $x_v 0$, we may apply the induction hypothesis for the subtrees $(u]$ and $(v]$. Hence we obtain that there exist elements $c_0, c_1 \in A_0 \cap B_0$, with $x_u 0 \leqq c_0 \leqq b$, $x_v 0 \leqq \leqq c_1 \leqq b$. Therefore

$$x_u \vee x_v = (x_u 0 \vee x_u 1) \vee (x_v 0 \vee x_v 1) \leqq c_0 \vee c_1 \vee (x_u 1 \vee x_v 1) \in A_0 \cap B_0,$$

completing the proof of Lemma 8.

Proof of Lemma 7. We again use an induction. Set $b = p(b_0, \ldots, b_m)$. If p is a projection, that is $b \in \bar{B}_0$, then we may assume that there exists no $y \in \bar{A}_1 \cup \bar{B}_1$ with $x \leqq y \leqq b$. In fact, for example the existence of such an $y \in A_1$ would imply the existence of a $q \in Q$ with $x \leqq q \leqq y \leqq b$, proving the lemma. Thus the interval $[x, b]$ contains a prime interval $[y_0, y_1]$ with $y_0 \in \bar{A}_0$, $y_1 \in \bar{B}_0$, and we can proceed similarly as in the proof of Lemma 8. Consider the case $p = p_0 \wedge p_1$. By the induction hypothesis, there are elements $c_0, c_1 \in A_0 \cap B_0$ with $x \leqq c_i \leqq p_i(b_0, \ldots, b_m)$. Hence $x \leqq c_0 \wedge c_1 \leqq \leqq p(b_0, \ldots, b_m)$. Thus we may assume that $p = p_0 \vee p_1$, and the polynomials p_i have the property described in the lemma. By Proposition 2, we have $x \in (p_0)_p \vee (p_1)_p$. By Proposition 3, there exists a binary tree T and elements $x_t \in P$, $t \in T$, satisfying conditions (1) to (3) of Section 4, with $(p_j)_p$, $j=0, 1$, and $\bar{P}$ in the place of I_j, $j \in J$, and $\langle X; \wedge, \vee \rangle$ respectively. Now an application of (1) and Lemma 8 completes the proof.

Proof of Theorem 2. By the main theorem of GRÄTZER, HUHN [4] and by symmetry, it suffices to prove that, for any $a \in A_0$ and $b \in B_0$ with $a \leqq b$ in L, there is an element $c \in A_0 \cap B_0$ with $a \leqq c$ and $c \leqq b$. Let $a = p'(a_0, \ldots, a_n)$, $b = p(b_0, \ldots \ldots, b_m)$, $a_0, \ldots, a_n \in \bar{A}_0$, $b_0, \ldots, b_m \in \bar{B}_0$, $p, p' \in FP(P)$. We apply an induction following the description in Proposition 1. Assume $a \leqq b$ by $({}_\vee W)$, that is $p' = p'_0 \vee p'_1$ and $p'_i(a_0, \ldots, a_n) \leqq p(b_0, \ldots, b_m)$, $i=0, 1$. Then, by the induction hypothesis there are elements $c_0, c_1 \in A_0 \cap B_0$, with $p'_i(a_0, \ldots, a_n) \leqq c_i \leqq p(b_0, \ldots, b_m)$. Hence

$$p'(a_0, \ldots, a_n) = p'_0(a_0, \ldots, a_n) \vee p'_1(a_0, \ldots, a_n) \leqq c_0 \vee c_1 \leqq p(b_0, \ldots, b_m),$$

as claimed. The proof is similar if $a \leqq b$ by $({}_\wedge W)$, $W_\vee)$, or $(W_\wedge)$. Thus we may assume that $a \leqq b$ follows from (W_C), that is, there is an element $x \in P$ with $a \leqq x \leqq b$. If $x \in \bar{A}_1$ (respectively, $x \in \bar{B}_1$), then, by Lemma 5, there exists an element $q \in Q$ with $a \leqq q \leqq x$ (respectively, $x \leqq q \leqq b$), and we can choose $c = q$. If $x \in \bar{A}_0$, then, by Lemma 7, $[x, b] \cap (A_0 \cap B_0) \neq \emptyset$. If $x \in \bar{B}_0$, then the dual of Lemma 7 yields that $[a, x] \cap$

$\bigcap(A_0 \cap B_0) \neq \emptyset$. (The dual of Lemma 7 could be proved similarly as Lemma 7 but the proof is much easier, for the operations on the dual ideals of P are the set operations.) This completes the proof.

We conclude this paper by mentioning an open problem. There is an obvious similarity between our main theorem and M. E. ADAMS' theorem [1] that a generating set of a free product (without amalgamation) need not contain generating sets of the components. This gives rise to the following question.

Problem. Need a free generating set of a free product always contain free generating sets of the components?

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