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On self-injectivity and strong regularity

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A generalization of quasi-injectivity, called I-injectivity, is introduced and various properties are derived. Semi-prime left q-rings (studied in [10]) are characterized in terms of I-injectivity. Left non-singular left I-injective rings are proved to be left continuous regular. Fully left idempotent rings whose essential left ideals are two sided (which effectively generalize semi-prime left q-rings and strongly regular rings) are studied. Characteristic properties of strongly regular rings are given. Certain rings having von Neumann regular centre are considered.

Introduction

Throughout, A represents an associative ring with identity and A-modules are unitary. J, Z, Y will denote respectively the Jacobson radical, the left singular ideal and the right singular ideal of A. As usual, a left (right) ideal of A is called reduced iff it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal. A is called a left V-ring iff every simple left A-module is injective (cf. [5]). Recall that (1) A is ELT (resp. MELT) iff every essential (resp. maximal essential, if it exists) left ideal of A is an ideal; (2) A is a left CM-ring iff for any maximal essential left ideal M of A (if it exists), every complement left subideal is an ideal of M (cf. [21]). ELT (MELT) rings generalize left q-rings [10], left duo rings while left CM-rings generalize left PCI rings [5, p. 140], left uniform rings and left duo rings.

It is well known that A is von Neumann regular iff every left (right) A-module is flat. A theorem of I. Kaplansky asserts that a commutative ring is regular iff it is a V-ring [5, Corollary 19.53]. For completeness, recall that a left A-module M is p-injective iff for any principal left ideal P of A, any left A-homomorphism $g\colon P\to M$, there exists $y\in M$ such that $g(b)=by$ for all $b\in P$. Then A is regular iff every left (right) A-module is p-injective. If I is a p-injective left ideal of A, then A/I is a flat

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left A-module [20, Remark 1]. Consequently, a finitely generated p-injective left ideal is a direct summand of ${}_AA$. For several years, von Neumann regular rings (introduced in [18]), self-injective rings, V-rings and associated rings have been studied by many authors (cf. [1] to [17]).

Rings whose left ideals are quasi-injective, called left q-rings, are studied in [10], where they are characterized as ELT left self-injective rings. We now introduce the following generalization of quasi-injectivity.

Definition. A left A-module M is called I-injective if, for all left submodules N, P which are isomorphic, any left A-homomorphism of N into P extends to an endomorphism of ${}_AM$.

(If Q, R are non-isomorphic quasi-injective non-injective left A-modules such that $Q \cap R = 0$ and their injective hulls are isomorphic, then $Q \oplus R$ is I-injective but not quasi-injective (cf. [7, p. 53, ex. 1].)

If every simple left A-module is p-injective, then A is fully left idempotent (cf. [14, Proposition 6]). Since any simple left A-module is I-injective, we see that I-injectivity does not even imply p-injectivity. The converse is not true either (cf. Remark 3 below).

1. I-injectivity

Our first result characterizes semi-prime left q-rings in terms of I-injectivity. A is called left I-injective iff ${}_AA$ is I-injective.

Theorem 1. *The following conditions are equivalent:*

(1) *A is an* ELT *left and right self-injective regular, left and right V-ring of bounded index;*

(2) *A is a semi-prime left q-ring;*

(3) *A is a semi-prime* ELT *left I-injective ring;*

(4) *A is a* MELT *left T-injective ring whose simple right modules are flat;*

(5) *A is an* ELT *left non-singular left I-injective ring.*

Proof. By [10, Theorem 2.3], (1) implies (2) while (2) implies (3) and (4). Since a semi-prime ELT ring is left non-singular, (3) implies (5).

If A is a MELT ring whose simple right modules are flat, then any simple left A-module is either injective or projective which implies that A is ELT (because any proper essential left ideal is an intersection of maximal left ideals). Consequently, (4) implies (5).

Assume (5). Let I be an essential left ideal of A, $g\colon I \to A$ a non-zero left A-homomorphism. For any $b \in I$, let K be a complement left ideal such that $L = l(b) \oplus K$ is an essential left ideal. If $f\colon Kb \to K$ is the map given by $f(kb) = k$ for all $k \in K$,

f is an isomorphism and by hypothesis, f extends to an endomorphism h of ${}_AA$. If $h(1)=d$, then $k=f(kb)=h(kb)=kbh(1)=kbd$ for all $k\in K$, which implies $L\subseteq l(b-bdb)$, whence $b-bdb\in Z=0$. Now $g(b)=g(bdb)=bg(db)\in I$ (because A is ELT), which shows that g is an endomorphism of ${}_AI$ and by hypothesis, g extends to an endomorphism of ${}_AA$. This proves that A is left self-injective and then (5) implies (1) by [21, Lemma 1.1].

The next corollary improves [10, Theorem 2.13].

Corollary 1.1. *A is simple Artinian iff A is a prime* ELT *left I-injective ring.*

Corollary 1.2. *The following conditions are equivalent:*

(1) *A is a direct sum of a semi-simple Artinian ring and a left and right self-injective strongly regular ring;*

(2) *A is a semi-prime* ELT *left I-injective ring.*

(Apply [10, Theorem 2.19] to Theorem 1.)

Since a prime ELT fully idempotent ring is primitive fully left idempotent, therefore [8, Theorem 6.10] and Corollary 1.1 imply

Corollary 1.3. *Suppose that A is an* ELT *fully idempotent ring such that any primitive factor ring is left I-injective. Then A is a unit-regular left and right V-ring.*

(Following [6], A is called fully idempotent (resp. fully left idempotent) iff every ideal (resp. left ideal) of A is idempotent.)

It is well-known that if A is left self-injective, then $Z=J$ (cf. for example [5, p. 78]). This is generalized in our first remark.

Remark 1. (a) If A is left I-injective, then $Z=J$ and every left or right A-module is divisible; (b) A left I-injective left Noetherian ring is left Artinian.

The following question is due to the referee: when do the rings of Remark 1 (b) coincide with quasi-Frobeniusean rings?

Theorem 2. *The following conditions are equivalent:*

(1) *A is left and right self-injective strongly regular;*

(2) *A is left non-singular left I-injective such that every maximal left ideal is an ideal;*

(3) *A is left non-singular left I-injective such that every maximal right ideal is an ideal;*

(4) *A is a reduced left I-injective ring.*

Proof. (1) implies (2) and (3) evidently.

If $J=0$ and every maximal left (resp. right) ideal of A is an ideal, then A is reduced. Consequently, either of (2) or (3) implies (4) by Remark 1 (a).

Assume (4). Since $Z=0$, the proof of Theorem 1 shows that A is von Neumann regular. Since A is reduced, A is strongly regular and hence (4) implies (1) by Theorem 1.

Corollary 2.1. *The following conditions are equivalent:*

(1) *A is either semi-simple Artinian or left and right self-injective strongly regular;*

(2) *A is a left non-singular left* CM *left I-injective ring.*

Quasi-injective left A-modules are I-injective. The proof of [7, Theorem 2.16] yields the following analogue of a well-known theorem of C. Faith—Y. Utumi concerning quasi-injective modules.

Theorem 3. *Let M be an I-injective left A-module, $E=\mathrm{End}\,({}_AM)$, $J(E)=$ the Jacobson radical of E. Then $E/J(E)$ is von Neumann regular and $J(E)=\{f\in E\,|\ker f$ is essential in ${}_AM\}$.*

Recall that A is a left QI-ring iff each quasi-injective left A-module is injective [5]. Left QI-rings are left Noetherian left V-rings [5, p. 114]. ELT left QI-rings are then semi-simple Artinian by [21, Theorem 1.11].

The next proposition shows that, in general, a direct sum of I-injective left A-modules need not be I-injective.

Proposition 4. *The following conditions are equivalent:*

(1) *Each direct sum of I-injective left A-modules is I-injective;*

(2) *A is a left QI-ring and each I-injective left A-module is injective.*

Proof. Assume (1). Let M be an I-injective left A-module, $\hat{M}$ the injective hull of ${}_AM$. If $S={}_AM\oplus{}_A\hat{M}$, $j\colon M\to\hat{M}$ and $t\colon \hat{M}\to S$ are the inclusion maps, $u\colon M\to S$ the natural injection, $p\colon S\to M$ the natural projection, $i\colon M\to M$ the identity map, then i extends to an endomorphism h of ${}_AS$, since ${}_AS$ is I-injective. Hence $htj(m)=$ $=ui(m)$ for all $m\in M$, which implies that $htj=ui$ and hence $phtj=pui=i$. Thus $g=pht\colon \hat{M}\to M$ such that $gj=$the identity map on M which implies that ${}_AM$ is a direct summand of ${}_A\hat{M}$, whence $M=\hat{M}$ is injective. Since any quasi-injective left A-module is I-injective, therefore A is a left QI-ring and hence (1) implies (2).

(2) implies (1) by [5, Theorem 20.1].

It is well-known that A is left hereditary iff the sum of any two injective left A-modules is injective. The next corollary then follows.

Corollary 4.1. *If the sum of any two I-injective left A-modules is I-injective, then A is a left Noetherian, left hereditary, left V-ring.*

Since any direct sum of p-injective left A-modules is p-injective, then the proof of Proposition 4 yields

Remark 2. Suppose that every p-injective left A-module is I-injective. Then A is a left Noetherian ring whose p-injective left modules are injective.

Applying [5, Theorem 24.5] to Remark 2, we get

Remark 3. If A is a left p-injective ring whose p-injective left modules are I-injective, then A is quasi-Frobeniusean.

We now proceed to prove that a left non-singular left I-injective ring is left continuous regular. Recall that A is left continuous (in the sense of UTUMI [17, p. 158]) iff every left ideal of A which is isomorphic to a complement left ideal is a direct summand of ${}_AA$.

Lemma 5. *Let M be an I-injective left A-module. K a complement left submodule of M. Then*

(1) *If N is a left submodule of M containing K, then any left A-homomorphism f of N into K extends to one of M into K;*

(2) *${}_AK$ is a direct summand of ${}_AM$.*

Proof. (1) The set of left submodules P of M containing N such that f extends to a left A-homomorphism of P into K has a maximal member U by Zorn's Lemma. Let $h\colon U\to K$ be the extension of f to U. If $j\colon K\to U$ is the inclusion map, then by hypothesis, jh extends to an endomorphism t of ${}_AM$. If $t(M)\nsubseteq K$, and D is a left submodule of M which is maximal with respect to $K\cap D=0$, then $(t(M)+K)\cap \cap D\neq 0$. If $0\neq d\in (t(M)+K)\cap D$, $d=t(m)+k$, $m\in M$, $k\in K$, then $t(m)=d-k\in \in D\oplus K$, $t(m)\notin K$ and therefore $m\notin U$. If $E=\{b\in M \mid t(b)\in D\oplus K\}$, then E strictly contains U. If p is the natural projection of $D\oplus K$ onto K, then $pt\colon E\to K$ extends f to E, which contradicts the maximality of U. This proves that t maps M into K and for any $n\in N$, $t(n)=jh(n)=h(n)=f(n)$.

(2) If C is a complement left ideal of A such that $K\oplus C$ is an essential left ideal, $p\colon K\oplus C\to K$ the natural projection, then by (1), p extends to a left A-homomorphism $g\colon M\to K$. Since $K\cap \ker g=0$, then for any $m\in M$, $m=g(m)+(m-g(m))$, where $g(m)\in K$, $(m-g(m))\in \ker g$, which proves that $M=K\oplus \ker g$.

If A is left I-injective, then A/Z is von Neumann regular (cf. the proof of Theorem 1). Consequently, Lemma 5(2) yields

Proposition 6. *If A is left non-singular, left I-injective, then A is left continuous regular.*

Corollary 6.1. *A left I-injective, left or right V-ring is left continuous regular.*

Corollary 6.2. *A left I-injective ring whose I-injective left modules are p-injective is left continuous regular.*

Applying [6, Theorem 16] to Proposition 6, we get

Corollary 6.3. *A semi-prime left I-injective ring which satisfies a polynomial identity is a left continuous regular, left and right V-ring.*

[16, Theorem 3] and a theorem of K. GOODEARL [5], Corollary 19.67] yield

Corollary 6.4. *A is primitive left self-injective regular iff A is prime left non-singular left I-injective.*

If M is a left A-module, N a left submodule of M, the usual closure of N in M is $Cl_M(N)=\{y\in M \mid Ly\subseteq N$ for some essential left ideal L of $A\}$. $Z(M)=Cl_M(0)$ is the singular submodule of M.

Proposition 7. *If A is left non-singular, then any quotient module Q of an I-injective left A-module contains its singular submodule Z(Q) as a direct summand.*

Proof. Let M be an I-injective left A-module, Q a quotient module of M, $f\colon M\to Q$ the canonical projection. Since $Z=0$, $Cl_M(\ker f)$ is a complement left submodule of $_AM$ and therefore $f^{-1}(Z(Q))=Cl_M(\ker f)$ is a direct summand of $_AM$ by Lemma 5(2). If $M=f^{-1}(Z(Q))\oplus N$, then $Q=f(M)=Z(Q)\oplus f(N)$.

2. Strongly regular rings

We now turn to characterizations of strongly regular rings.

Lemma 8. *The following conditions are equivalent:*

(1) *A is a division ring;*

(2) *A is a prime ring containing a non-zero reduced p-injective right ideal.*

Proof. Obviously (1) implies (2).

Assume (2). Let I be a non-zero reduced p-injective right ideal of A, $0\neq b\in I$, $i\colon bA\to I$ the inclusion map. Then there exists $c\in I$ such that $b=i(b)=cb$ and since I is reduced, $l(b)\subseteq r(b)$ which implies $A(1-c)\subseteq l(b)\subseteq r(b)$, whence $AbA(1-c)=0$. Since A is prime, therefore $1=c\in I$ which implies $A=I$ is a right p-injective integral domain. For any $0\neq c\in A$, if $f\colon cA\to A$ is the map $f(ca)=a$ for all $a\in A$, then there exists $d\in A$ such that $1=f(c)=dc$ which proves that (2) implies (1).

Lemma 9. *Let A be an* ELT *fully left idempotent ring. Then*

(1) *Any non-zero-divisor of A is invertible. Consequently, every left or right A-module is divisible;*

(2) *Any reduced principal left ideal is a direct summand of $_AA$;*

(3) *Any reduced principal right ideal is a direct summand of $_AA$.*

Proof. (1) Let c be a non-zero-divisor of A. If $Ac\neq A$, let M be a maximal left ideal containing Ac. If $M=l(e)$, where $e=e^2\in A$, then $ce=0$ implies $e=0$, whence

$M=A$, which is impossible. Therefore M is an essential left ideal and hence an ideal of A. Since A is fully left idempontent, $c=dc$ for some $d\in AcA\subseteq M$ and then $1=d\in M$, again contradicting $M\neq A$. This proves that c is left invertible and since c is a non-zero-divisor, c is invertible in A. For any left A-module M, $M=cbM\subseteq cM\subseteq M$, where $cb=bc=1$, which yields $M=cM$. Similarly, any right A-module is divisible.

(2) Let $a\in A$ be such that Aa is reduced. Suppose that $Aa+l(a)\neq A$. If M is a maximal left ideal containing $Aa+l(a)$, and if $M=l(e)$, $e=e^2\in A$, then $e\in r(a)\subseteq l(a)$ (because Aa is reduced) which implies $e=e^2=0$, contradicting $M\neq A$. Thus M is a maximal essential left ideal which is therefore an ideal of A. Since A is fully left idempotent, therefore A/M_A is flat [13, Lemma 2.3] which implies that $u\in Mu$ for all $u\in M$. In particular, $a=da$ for some $d\in M$ which yields $1-d\in l(a)\subseteq M$, whence $1\in M$, again a contradiction. This proves that $Aa+l(a)=A$ and therefore $a=ca^2$ for some $c\in A$ and since Aa is reduced, $(a-aca)^2=0$ implies $a=aca$, whence Aa is a direct summand of ${}_AA$.

(3) Let $b\in A$ be such that bA is reduced and K a complement left ideal such that $L=Ab\oplus K$ is an essential left ideal. Then A/L_A is flat which implies $b=db$ for some $d\in L$, whence $b=bd$ (since bA is reduced). If $d=cb+k$, $c\in A$, $k\in K$, then $b-bcb=bk\in Ab\cap K=0$ which proves that bA is a direct summand of A_A.

Corollary 9.1. *If A is an* ELT *left V-ring, then* (a) *any non-zero-divisor is invertible;* (b) *any reduced principal left or right ideal is generated by an idempotent.*

Corollary 9.2. *If A is a prime* ELT *left idempotent ring, then A is either a division ring or a primitive ring with non-zero socle such that every non-zero left or right ideal contains a non-zero nilpotent element.*

Remark 4. If A is ELT fully left idempotent, then $J=Z=Y=0$.

Remark 5. [2, Corollary 6] holds for the following classes of rings: (1) ELT fully left idempotent rings; (2) Fully right idempotent rings whose essential right ideals are ideals; (3) Right I-injective rings.

Note that (a) rings whose essential left ideals are idempotent need not be semiprime (cf. for example, V. S. Ramamurthi and K. M. Rangaswamy, *Math. Scand.*, **31** (1972), 69—77); (b) reduced V-rings need not be regular (even when they are prime) [6, p. 109, Example 2].

Theorem 10. *The following conditions are equivalent:*

(1) *A is strongly regular;*

(2) *A is reduced such that any prime factor ring is left I-injective;*

(3) *A is regular such that every non-zero factor ring contains a non-zero reduced right ideal;*

(4) *A is left V-ring such that every non-zero factor ring contains a non-zero reduced p-injective right ideal;*

(5) *A is right V-ring such that every non-zero factor ring contains a non-zero reduced p-injective right ideal;*

(6) *Every non-zero factor ring of A is semi-prime containing a non-zero reduced p-injective right ideal;*

(7) *A is a reduced ring such that every non-zero factor ring contains a non-zero p-injective right ideal;*

(8) *A is an* ELT *reduced fully idempotent ring;*

(9) *A is a reduced* MELT *ring whose essential left ideals are idempotent;*

(10) *A is a reduced* MELT *ring whose essential right ideals are idempotent;*

(11) *A is an* ELT *fully idempotent ring whose proper prime ideals are completely prime.*

Proof. It is easy to see that (1) implies (2) through (5).

Assume (2). Let P be a proper prime ideal such that A/P is an integral domain. Then A/P is a division ring by Theorem 2 and (2) implies (6) by [8, Theorem 1.21].

Any one of (3), (4) or (5) implies (6).

Assume (6). Then A is a fully idempotent ring such that any non-zero prime factor ring is a division ring by Lemma 8. A is therefore strongly regular by [8. Corollary 1.18 and Theorem 3.2]. Thus (6) implies (7).

(7) implies (8) by [8, Theorem 1.21] and Lemma 8.

It is clear that (8) implies (9).

Assume (9). Let B be a prime factor ring of A, $0\neq b\in B$, $T=BbB$. Let K be a complement left subideal of T such that $L=Bb\oplus K$ is an essential left subideal of T. Since ${}_BT$ is essential in ${}_BB$, then so is ${}_BL$, whence $L=L^2$ (because every essential left ideal of B is idempotent). Now $b\in L^2$ implies $b=\sum_{i=1}^{n}(b_ib+k_i)(d_ib+c_i)$, where b_i, $d_i\in B$, k_i, $c_i\in K$, whence $b-\sum_{i=1}^{n}(b_ib+k_i)d_ib=\sum_{i=1}^{n}(b_ib+k_i)c_i\in Bb\cap K=0$ and therefore $b=\sum_{i=1}^{n}(b_ib+k_i)d_ib\in Tb=(Bb)^2$ which proves that B is fully left idempotent. If, further, B is an integral domain, then B is a division ring by Lemma 9(2) (because a MELT fully left idempotent ring is ELT). Thus (9) implies (10) by [8, Theorem 1.21].

Similarly, (10) implies (11).

Assume (11). If B is a non-zero prime factor ring of A, then B is an ELT fully idempotent domain which implies that B is a division ring. Consequently, (11) implies (1) by [8, Theorem 3.2].

Applying [16, Theorem 3] to Theorem 10(2), we get

Corollary 10.1. *If A is a left continuous regular ring such that any proper non-zero factor ring contains a non-zero reduced right ideal, then A is either left self-injective or right continuous strongly regular.*

Then Theorem 2 and Proposition 6 yield

Corollary 10.2. *If A is left non-singular left I-injective such that any proper non-zero factor ring contains a non-zero reduced right ideal, then A is left self-injective regular.*

We now consider rings having von Neumann regular centre. The centre of A will always be denoted by C. Rings whose simple left modules are either p-injective or flat need not be semi-prime (the converse is not true either).

Proposition 11. *Let A belong to any one of the following classes of rings:* (1) *A is semi-prime such that every essential left ideal is idempotent* (2) *A is such that each factor ring B satisfies one of the following conditions:* (a) *B is semi-prime;* (b) *The intersection of the Jacobson radical, the left singular ideal and the right singular ideal of B is zero;* (c) *Every simple left B-module is either p-injective or flat;* (3) *A is semi-prime such that for any non-zero element a of A, there exists a positive integer n such that Aa^n is a non-zero left annihilator. Then C, the centre of A, is von Neumann regular.*

Proof. (1) Let $c \in C$. If K is a complement left ideal such that $L = Ac \oplus K$ is an essential left ideal of A, then $c \in L^2 = L$ and since $AcK \subseteq Ac \cap K = 0$, $(KAc)^2 = 0$ implies $KAc = 0$ (A being semi-prime), whence $c \in (Ac)^2 + K^2$ which yields $c \in (Ac)^2$. Thus $c = cdc$ for some $d \in A$ and it follows from the proof of [18, Theorem 3] that $c = cvc$ for some $v \in C$.

(2) Suppose that $c \in C$ such that $c^2 = 0$.

(a) If A is semi-prime, then $(Ac)^2 = Ac^2 = 0$ implies $c = 0$.

(b) Let $J \cap Z \cap Y = 0$. If K is a complement right ideal of A such that $R = r(c) \oplus K$ is an essential right ideal, then $Kc \subseteq Ac = cA \subseteq r(c)$ implies $ck = Kc \subseteq \subseteq r(c) \cap K = 0$, whence $K \subseteq r(c)$ and therefore $K = 0$, implying that $c \in Y$. Similarly, $c \in Z$. Also, for any $a \in A$, $(1+ac)(1-ac) = 1$ which proves that $c \in J$. Thus $c \in J \cap \cap Z \cap Y = 0$.

(c) Suppose that every simple left A-module is either p-injective or flat. If $c \neq 0$, M a maximal left ideal containing $l(c)$, then ${}_AA/M$ is either p-injective or flat. If ${}_AA/M$ is flat, the proof of Lemma 9(2) shows that we shall end with a contradiction. If ${}_AA/M$ is p-injective, the map $Ac \to A/M$ given by $ac \to a + M$ for all $a \in A$ leads again to a contradiction. Thus $c^2 = 0$ implies $c = 0$ in (2) which proves that C (and hence the centre of any factor ring) is reduced. In particular, for any $u \in C$, $u + Au^2$ is a nilpotent element of the centre of A/Au^2 which implies $u \in Au^2$, whence $u = uvu$ for some $v \in C$.

(3) Since A is semi-prime, C is reduced (cf. (2)). If $0 \neq c \in C$, Ac^n is a non-zero left annihilator for some positive integer n. For any $b \in r(Ac^n)$, $(Acb)^n \subseteq Ac^n b = 0$ implies $b \in r(Ac)$ and hence $r(Ac^n) = r(Ac)$. Now $c \in l(r(Ac)) = l(r(Ac^n)) = Ac^n$. If $n > 1$, then $c = cac^{n-1}$, $a \in A$, which proves that Ac is a direct summand of ${}_AA$. Thus, whether $n = 1$ or $n > 1$, Ac is always a left annihilator for any non-zero $c \in C$. In particular, Ac^2 is a left annihilator and the preceding argument yields $c \in Ac^2$, whence $c = cvc$ for some $v \in C$.

Applying [1, Theorem 3] to Proposition 11, we get

Corollary 11.1. *Suppose that for each maximal ideal M of C, A/AM is regular. Then A is regular iff A satisfies any one of conditions* (1), (2), (3) *of Proposition* 11.

The proof of Proposition 11(2) and Corollary 11.1 yield

Proporisiton 12. *Suppose that A is semi-prime such that the centre C is not a field. Then A is regular iff for each non-zero ideal T of C, A/AT is regular.*

For any left A-module M, any left submodule N, write $K_M(N) = \{y \in M \mid cy \in N$ for some non-zero-divisor c of $A\}$. In general, $K_M(N) \neq Cl_M(N)$. If A has a classical left quotient ring, then $K_M(N)$ is a left submodule of M. Note that A has a classical left quotient ring iff A satisfies the left Ore condition (cf. for example [7, p. 101]). By [7, Theorem 3.34], the two "closures" $K_M(N)$ and $Cl_M(N)$ coincide over semi-prime left Goldie rings. To simplify the notation, write $K_A(I) = K(I)$ and $Cl_A(I) = Cl(I)$ for any left ideal I of A. If A is either left p-injective or a ring whose simple left modules are flat, then $K_M(N) = N$ for all left A-modules M and submodules N. Note that A is semi-simple Artinian iff $Cl_M(N) = N$ for all left A-modules M and submodules N.

Proposition 13. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*

(2) *A is an* ELT *left hereditary left I-injective ring;*

(3) *A is an* ELT *fully left idempotent ring such that $K(I) = Cl(I)$ for any left ideal I of A;*

(4) *A is a left I-injective ring such that $K(I)$ is a complement left ideal for any left ideal I;*

(5) *A is semi-prime left I-injective satisfying the maximum condition on left annihilators;*

(6) *The direct sum of a projective and an I-injective left A-modules is I-injective.*

Proof. Obviously, (1) implies (2) through (6).

Since a well-known result of B. Osofsky asserts that a left self-injective left hereditary ring is semi-simple Artinian, (2) implies (1) by Theorem 1.

Assume (3). By Lemma 9(1), A is its own classical left quotient ring. Since a semi-prime ELT ring is left non-singular, then $K(I)=Cl(I)$ is a complement left ideal for any left ideal I. In particular, if L is an essential left ideal, $K(L)=A$ which implies that L contains a non-zero-divisor c. By Lemma 9(1), c is invertible in A which yields $L=A$. This proves that (3) implies (1).

Similarly, (4) implies (1) by Remark 1(a).

(5) implies (1) by Proposition 6.

Assume (6). If P is a projective left A-module, H the injective hull of ${}_AP$, then $P\oplus H$ is a left I-injective A-module and the proof of Proposition 4 shows that ${}_AP$ is injective. Therefore every injective left A-module is projective by [5, Theorem 24.20] and from Proposition 4, every I-injective left A-module is injective which implies that every simple left A-module is projective. Thus (6) implies (1).

Remark 6. The following conditions are equivalent for a left CM-ring A: (1) is semi-prime left Goldie; (2) For any left A-module M and every left submodule N, $K_M(N)=Cl_M(N)$; (3) Every essential left ideal of A contains a non-zero-divisor.

We add a last remark on rings whose essential left ideals are idempotent.

Remark 7. Suppose that every essential left ideal of A is idempotent. If A is either ELT or left CM, then the centre of A is von Neumann regular.

The referee has kindly drawn my attention to the following papers:

(1) V. E. Govorov, Semi-injective modules, *Algebra i Logika,* **2** (1963), 21—49.

(2) A. A. Tuganbaev, Quasi-injective and weakly injective modules, *Bull. Moscow Univ. Math. Mech.,* Series № 2 (1977).

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