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Classification of finite minimal non-metacyclic groups

MARIO CURZIO

Dedicated to the memory of Carlo Miranda

Let (MC) be the class of *metacyclic* groups $(G\in(\mathrm{MC})$ if and only if it has a cyclic factor group G/H with H cyclic). A group G is said to be *minimal non-metacyclic* (*minimal non-*(MC)) if and only if $G\notin(\mathrm{MC})$ and $H\in(\mathrm{MC})$ for every subgroup $H<G$.

N. Blackburn [1] (Theorem 3.2) determined all finite minimal non-(MC) p-groups (p prime). In the present paper we construct all other finite minimal non-(MC) groups (see Theorems 1.2, 2.7, 2.8 and 2.10). They are generally monomial (see 2.12), and for the set $\pi(G)$ of all prime divisors of the order of G we have $|\pi(G)|=2, 3$ (see 2.11). Moreover, every $G_p\in\mathrm{Syl}_p(G)$ $(p\neq\min\pi(G))$ is either cyclic, or of order p^2 and exponent p. Finally, the metacyclic p-group G_p is rather general for $p=\min\pi(G)$.

All groups we shall deal with are finite.

Notation and terminology are the usual ones in group theory (see for instance [3], [6] and [7]). We just point out that $G/\mathscr{L}(G)$ will denote the largest nilpotent factor group of G, Q_8 the quaternion group.

1. A minimal non-(MC) group is either supersolvable or minimal non-supersolvable. In this section we shall determine the structure of non-supersolvable and minimal non-(MC) groups.

1.1. *Let G be a non-supersolvable and minimal non-*(MC) *group, and G_p its normal Sylow subgroup*[1]. *Then:*

(1) *if $\Phi(G)=1$, then G is minimal non-abelian and its order is p^2q (q prime);*

(2) *$G=G_pG_q$ with G_q cyclic and $\Phi(G_q)\lhd G$;*

(3) *if $p>2$, then $|G_p|=p^2$ and $\exp G_p=p$;*

(4) *if $p=2$, then either $G_2\cong Q_8$ or $|G_2|=4$ and $\exp G_2=2$.*

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[1]) A minimal non-supersolvable group has a unique normal Sylow subgroup (see [2], Hilfssatz C).

Proof. (1) Suppose $\Phi(G)=1$. G_p is the only minimal normal subgroup of G (see [2], Satz 1a); it is elementary abelian, not cyclic and metacyclic, hence $|G_p|=p^2$. In contrary to our claim, assume there is a $G_{p'}$ of composite order; then there exists a subgroup $H=G_pM\lhd G$ with $1<M<G_{p'}$. If H' is cyclic and non-trivial, there is an $N\leqq H'$, minimal normal in G, so $N=G_p$, which is a contradiction, as G_p is not cyclic. If $H'=1$, then $H=G_p\times M$, and G has a minimal normal subgroup $N\leqq M$, hence $N\neq G_p$, again a contradiction.

(2) By (1) we have $G=G_pG_q$ and $|G/\Phi(G)|=p^2q$. The Sylow q-subgroup S of $\Phi(G)$ is maximal in some G_q and normal in G. If $x\in G_q-S$, then $G=\langle G_p, S, x\rangle=$ $=G_p\langle x\rangle$ and G_q is cyclic; thus $\Phi(G_q)=S\lhd G$.

(3) Suppose $p>2$, hence (see [2], Satz 1f) $\exp G_p=p$. Since G_p is metacyclic of order greater than p, it follows that $|G_p|=p^2$.

(4) Suppose $p=2$, hence (see [2], Satz 1f) $\exp G_2\leqq 4$. If G_2 is abelian, then by (2) G is minimal non-abelian, so $\exp G_2=2$, whence $|G_2|=4$. Suppose now G_2 is not abelian, hence $\exp G_2=4$; G being metacyclic, it follows that either $|G_2|=8$ or $|G_2|=16$; whether the latter case occurs or G_2 is dihedral (of order 8), $|\mathrm{Aut}(G_2)|$ is a power of 2, thus $G=[G_2]G_q$, whence the contradiction $G=G_2\times G_q$. This proves that $G_2\cong Q_8$.

Theorem 1.2. *A non-supersolvable group G is minimal non-*(MC) *if, and only if, one of the following holds:*

(a) *G is minimal non-abelian of order p^2q^n ($p\neq q$ primes, $G_p\lhd G$);*

(b) *$G=\langle Q_8, x\rangle$, where $|x|=3^n$ and x induces on Q_8 an automorphism of order* 3.

Proof. Let G be a minimal non-(MC) group. By 1.1, $G=[G_p]G_q$; if G_p is abelian, then (a) holds. If G_p is not abelian, by 1.1 we get $G_p=G_2\cong Q_8$, and (b) holds.

2. Minimal non-(MC) p-groups were classified by Blackburn [1]. In this section we construct all other supersolvable minimal non-(MC) groups (see Theorems 2.7, 2.8 and 2.10).

2.1. *Let $G=MN$ be a metacyclic p-group ($p>2$) with $M\neq 1$, $N\neq 1$ subgroups such that $M\cap N=1$. Then both M and N are cyclic.*

Proof. G is modular (see [8], Proposition 1.8), so $\Omega_1=\{x\in G \mid x^p=1\}$ is a metacyclic p-group of exponent p; then $|\Omega_1|=p^2$, whence the assertion follows.

2.2. *Let $G=[A]B$ with A cyclic of odd order, B nilpotent, $|\pi(G)|>1$, and suppose each $H<G$ is metacyclic. Then*

(1) *$B_p\in \mathrm{Syl}(B)$ is cyclic for any $p\in\pi(A)\cap\pi(B)$;*

(2) *G is metacyclic if $|\pi(B)|>1$;*

(3) *G is metacyclic if $|\pi(A)|>2$.*

Proof. (1) follows from 2.1.

(2) Suppose $|\pi(B)|>1$ and let $p\in\pi(B)-\pi(A)$. The subgroup $K=AB_p<G$ is metacyclic, so there exists a cyclic subgroup $C\lhd K$ with cyclic factor K/C. The Sylow p-subgroup C_p of C is normal in G and B_p/C_p is cyclic. By (1) we have $B=(\bigtimes_{p\nmid|A|} B_p)\times T$, with T cyclic. The normal subgroup $H=A\times(\bigtimes_{p\nmid|A|} C_p)$ is cyclic, as is $G/H\cong T\times(\bigtimes_{p\nmid|A|} B_p/C_p)$.

(3) Assume $|\pi(A)|>2$. After (2) we may suppose B is a p-group. If p divides $|A|$, then B is cyclic (see (1)) and G is metacyclic. So let p be relatively prime to $|A|$. Since $|\pi(A)|>2$, we get that $A=R\times S\times T$, with R, S and T non-trivial Hall subgroups. $K=(R\times S)B<G$ is metacyclic, so there exists $N\lhd G$ cyclic with K/N cyclic. $N_p\in\mathrm{Syl}\,(N)$ is contained in $C_B(R)\cap C_B(S)$; moreover, B/N_p is a cyclic p-group, hence $C_B(R)$ and $C_B(S)$ are comparable. Arguing as before, we see that $C_B(R)$, $C_B(S)$ and $C_B(T)$ are pairwise comparable; assuming $C_B(R)$ is the smallest one, N_p centralizes $R\times S\times T=A$. The normal subgroup $H=A\times N_p$ is cyclic, as is $G/H\cong B/N_p$.

2.3. *Suppose G has a modular subgroup $G_p\in\mathrm{Syl}\,(G)$ with $p>2$. Then $G_p\cap\cap Z(G)\cap\mathscr{L}(G)=1$.*

Proof. See HUPPERT [5], 3.2. Satz.

Lemma 2.4. *Let G be a supersolvable group such that for each $p\neq\min\pi(G)$, $G_p\in\mathrm{Syl}\,(G)$ is modular. Then $G=[\mathscr{L}(G)]M$, with M a system normalizer.*

Proof. Let $q=\min\pi(G)$. Then $(G_q)'$ is normal in G (see [4], Satz 4) and $G/(G_q)'$ has abelian Sylow q-subgroups. Now apply the following theorem of HUPPERT [5] (3.3. Satz): For a solvable group G with each G_p modular and each G_2 abelian, $G=[\mathscr{L}(G)]M$, with M a system normalizer. Thus we have $G/(G_q)'=[A/(G_q)']M/(G_q)'$ with $A/(G_q)'=\mathscr{L}(G/(G_q)')$ and $M/(G_q)'$ a system normalizer. $A/(G_q)'\leqq(G/(G_q)')'$ is nilpotent and its order is relatively prime to q. As G is supersolvable, M is nilpotent and $A=B\times(G_q)'$ with B nilpotent and $|B|$ relatively prime to q. We have $G=AM=(B\times(G_q)')M=[B]M$, and $\mathscr{L}(G)\leqq B$; on the other hand, $G/(\mathscr{L}(G)\times\times(G_q)')$ is nilpotent, hence $\mathscr{L}(G)\times(G_q)'\geqq A=B\times(G_q)'$. From this we get $B=\mathscr{L}(G)$. In a similar way the assertion about M can also be proved.

2.5. *Let G be a non-primary, supersolvable and minimal non-(MC) group. Then either $\mathscr{L}(G)=G_p$ ($|G_p|\leqq p^2$ and $\exp G_p=p$), or $\mathscr{L}(G)=G_p\times G_q$ ($|G_p|=p$ and $|G_q|=q$).*

Proof. A metacyclic p-group of odd order is modular (see [8], Proposition 1.8) hence $G=[\mathscr{L}(G)]M$ (by Lemma 2.4) and $\mathscr{L}(G)\leqq G'$ is nilpotent of odd order. Suppose there is a non-abelian $\mathscr{L}_p\in\mathrm{Syl}\,(\mathscr{L}(G))$. If $\mathscr{L}_p<G_p$, then $G_p=\mathscr{L}_pM_p<$

$<G$ is metacyclic, hence $\mathcal{L}_p$ is cyclic (see 2.1), a contradiction. If $\mathcal{L}_p=G_p$, there exists a non-trivial factor G/H which is a p-group (see [5], 1.5. Satz, saying that for a group G with $G_p\in\mathrm{Syl}\,(G)$ of odd order, metacyclic and not abelian, there is a non-trivial factor group G/H which is a p-group); this is again a contradiction, as $G_p= =\mathcal{L}_p\leqq\mathcal{L}(G)$. Thus $\mathcal{L}(G)$ is abelian, and we can consider the following cases:

(i) There exists $\mathcal{L}_p\in\mathrm{Syl}\,(\mathcal{L}(G))$ having a socle of order p^2. Arguing by contradiction, let $S<\mathcal{L}(G)$. If $K=SM$, $\mathcal{L}(K)\leqq K'$ is cyclic, and K splits on $\mathcal{L}(K)$. On the other hand, $\mathcal{L}(K)\leqq S$ hence either $K=S\times M$, or $K=[N_1](N_2\times M)$ with $|N_i|=p$. In both cases, G has a non-trivial central subgroup which is contained in $G_p\cap Z(G)\cap\mathcal{L}(G)$, contradicting 2.3. Then we have $|\mathcal{L}(G)|=p^2$ and $\exp\mathcal{L}(G)=p$. Assume $\mathcal{L}(G)<G_p$; then $[\mathcal{L}(G)]M_p$ should be metacyclic, hence $\mathcal{L}(G)$ cyclic. which contradicts the hypothesis. Thus $\mathcal{L}(G)=G_p$.

(ii) $\mathcal{L}(G)$ is cyclic. We have $|\pi(\mathcal{L}(G))|\leqq2$ and $|\pi(M)|=1$ (see 2.2), hence either $\mathcal{L}(G)=G_p$, or $\mathcal{L}(G)=G_p\times G_q$. In both cases, let $P\leqq G_p$ be of order p. $K=[G_p]C_M(P)$ splits on $\mathcal{L}(K)\leqq G_p$, and either $\mathcal{L}(K)=G_p$ or $\mathcal{L}(K)=1$, since G_p is cyclic. In the first case $1\neq P\leqq K_p\cap\mathcal{L}(K)\cap Z(K)$, which contradicts 2.3. Hence $K=G_p\times C_M(P)$ and $C_M(P)=C_M(G_p)$.

Suppose now $\mathcal{L}(G)=G_p$ and, by contradiction, $\mathcal{L}(G)>P$. The subgroup $PM<G$ is metacyclic, so there exists a cyclic subgroup $X\leqq C_M(P)$, $X\lhd G$ with M/X cyclic; but $C_M(P)=C_M(G_p)$ and G is metacyclic, a contradiction.

Suppose finally $\mathcal{L}(G)=G_p\times G_q$ and consider $P\leqq G_p$ and $Q\leqq G_q$ of order p and q respectively; as before, $C_M(P)=C_M(G_p)$ and $C_M(Q)=C_M(G_q)$. If $(P\times Q)M$ were metacyclic, there should exist a cyclic subgroup $X\lhd M$, with cyclic factor M/X, such that $X\leqq C_M(P\times Q)=C_M(G_p)\cap C_M(G_q)$; then G should be metacyclic.

This proves that $P\times Q=\mathcal{L}(G)$.

2.6. *Let G be a supersolvable minimal non-*(MC) *group and suppose* $|\pi(G)|=3$. *Then:*

(1) *either* $G=(G_p\times G_q)G_r$ *with* $\mathcal{L}(G)=G_p\times G_q$ *and* $|\mathcal{L}(G)|=pq$, *or* $G= =G_p(G_q\times G_r)$ *with* $\mathcal{L}(G)=G_p$ ($|G_p|=p^2$ *and* $\exp G_p=p$);

(2) *if* $\mathcal{L}(G)=G_p\times G_q$, *then* $G_r=M_1M_2$ ($M_i\lhd G$, M_i *cyclic,* $M_1<C_G(G_p)$ *and* $M_2<C_G(G_q)$);

(3) *if* $\mathcal{L}(G)=G_p\times G_q$, *then* $C_{G_r}(G_p)$ *and* $C_{G_r}(G_q)$ *are maximal subgroup of* G_r.

Proof. (1) By 2.5, either $\mathcal{L}(G)=G_p\times G_q$ ($|\mathcal{L}(G)|=pq$), or $\mathcal{L}(G)=G_p$ ($|G_p|\leqq p^2$ and $\exp G_p=p$). If $|\mathcal{L}(G)|=p$, from 2.2 we would have $|\pi(G)|=2$, a contradiction; now (1) readily follows.

(2) Let $\mathcal{L}(G)=G_p\times G_q$. G_pG_r and G_qG_r are metacyclic; then there are cyclic subgroups $M_i\lhd G_r$, with cyclic factor groups G_r/M_i, such that $M_1<C_G(G_p)$ and $M_2<C_G(G_q)$. Arguing by contradiction, suppose $M_1M_2<G_r$. $(G_p\times G_q)M_1M_2< <G$ is metacyclic; then we can find a cyclic subgroup $X\lhd M_1M_2$ with M_1M_2/X

cyclic and $X<C_G(G_p\times G_q)$; M_1M_2/X is primary, so that one can suppose $M_2X\geqq\geqq M_1X\geqq M_1$, thus $M_1<C_G(G_p\times G_q)$ and G is metacyclic.

(3) After (2), assuming $\mathscr{L}(G)=G_p\times G_q$, one has $G_r=M_1C_{G_r}(G_q)$. Denoting by N_1 the maximal subgroup of M_1, suppose $N_1\not\leqq C_{G_r}(G_q)$. Then $N_1C_{G_r}(G_q)<G_r$, so that $(G_p\times G_q)N_1C_{G_r}(G_q)$ is metacyclic; hence there is a cyclic subgroup $X\leqq\leqq C_{G_r}(G_p\times G_q)$, normal in $N_1C_{G_r}(G_q)$ with primary cyclic quotient. Thus either $C_{G_r}(G_q)=XC_{G_r}(G_q)\geqq XN_1\geqq N_1$, or $XN_1\geqq XC_{G_r}(G_q)=C_{G_r}(G_q)\geqq M_2$. In the first case the contradiction is clear. In the second case we get $M_2\leqq C_{G_r}(G_p\times G_q)$ and G is metacyclic, again a contradiction.

Theorem 2.7. *Let G be a supersolvable group with $|\pi(G)|=3$. Then G is a minimal non-*(MC) *group if and only if it has one of the following structures:*

(a) $G=[G_p\times G_q]G_r$, *where* $|G_pG_q|=pq$, $G_r=M_1M_2$ ($M_i\lhd G_r$, M_i *cyclic*), $C_{G_r}(G_p)\geqq M_1$ *and* $C_{G_r}(G_q)\geqq M_2$ *are maximal subgroups of* G_r;

(b) $G=G_p(G_q\times G_r)$, *where* $G_q\times G_r$ *is cyclic,* $G_p=N_1\times N_2$ ($N_i\lhd G$ *and* $|N_i|=p$), $G_q<C_G(N_1)$, $\Phi(G_q\times G_r)<C_G(G_p)$, N_1G_r *and* N_2G_q *are non-abelian.*

Proof. Assume G is a minimal non-(MC) group. Then either $\mathscr{L}(G)=G_p\times G_q$ and $|\mathscr{L}(G)|=pq$, or $G=G_p(G_q\times G_r)$ with $\mathscr{L}(G)=G_p$ of order p^2 and exponent p (see 2.7 (1)). In the first case, (a) holds (see 2.6 (2) and (3)). Let us look at the second possibility. We have $G_p=N_1\times N_2$ with N_i minimal normal in G. $(N_1\times N_2)G_q<G$ has a cyclic commutator subgroup, so G_q centralizes only one of the N_i's. Indeed, were $G_q<C_G(N_1\times N_2)$, $G=G_q\times G_pG_r$ would be metacyclic since G_q and G_pG_r are metacyclic and of coprime orders. Suppose G_p centralizes N_1. We cannot have $G_r<\!<C_G(N_1)$ for this implies $G=N_2(G_q\times G_r)\times N_1$, which contradicts the meaning of $\mathscr{L}(G)=N_1\times N_2$. Thus $G_r<C_G(N_2)$. Neither G_q nor G_r centralizes $G_p=N_1\times N_2$, hence $x\notin C_G(N_2)$ and $y\notin C_G(N_1)$ for suitable $x\in G_q$ and $y\in G_r$. $\langle G_p; x, y\rangle$ has a non-cyclic commutator subgroup; hence it coincides with G; so $G_q=\langle x\rangle$ and $G_r=\langle y\rangle$. Denoting by M the maximal subgroup of $\langle y\rangle$, $(N_1\times N_2)(\langle x\rangle\times M)$ has a cyclic commutator subgroup, thus $M<C_G(G_p)$; similarly, the maximal subgroup of $\langle x\rangle$ centralizes G_p, so G is like in (b).

Vice versa, if (b) holds, G is clearly minimal non-(MC). Assume (a) holds. G is not metacyclic, since, modulo $G_{G_r}(G_p)\cap C_{G_r}(G_q)$, G_r is not cyclic. Suppose now $M<G$ is a maximal subgroup. If $(G:M)=q$, $M=G_pG_r$ is metacyclic as $G_p\times M_1$ and $M/(G_p\times M_1)\cong G_r/M_1$ are cyclic; similarly M turns out to be metacyclic when $(G:M)=p$. Finally, suppose $(G:M)=r$, so that $M=(G_p\times G_q)X$ with X maximal in G_r. We can assume $M_1\not\leqq X$, since $G_r=M_1M_2$. Then $M_1\cap X\leqq C_{G_r}(G_p)$ is the maximal subgroup of M_1 and we also get $M_1\cap X\leqq C_{G_r}(G_q)$, since $G_r=M_1C_{G_r}(G_q)$ with M_1 cyclic and $C_{G_r}(G_q)$ maximal in G_r, implying that the maximal subgroup of M_1 is contained in $C_{G_r}(G_q)$. Hence it follows that $H=(G_p\times G_q)\times(X\cap M_1)$ is cyclic, as is $M/H\cong X(M_1\cap X)\cong G_r/M_1$.

Theorem 2.8. *Let G be a supersolvable group with $|\pi(G)|=2$ and G_p not cyclic* ($p=\max\pi(G)$). *Then G is minimal non-*(MC) *if and only if it has the following structure: $G=(N_1\times N_2)G_q$, where is G_q cyclic, $N_i\lhd G$ and $|N_i|=p$, $\Phi(G_q)<C_G(N_1)$, N_1G_q and N_2G_q are non-abelian.*

Proof. A group with the above structure is clearly minimal non-(MC).

Vice versa, suppose G is minimal non-(MC). By 2.5 we have $\mathscr{L}(G)=G_p=N_1\times N_2$ ($N_i\lhd G$ and $|N_i|=p$), $G=G_pG_q$. If G_q centralizes N_1, then $G=N_1\times N_2G_q$, which contradicts the meaning of $\mathscr{L}(G)=N_1\times N_2$; similarly, $G_q\not\leqq C_G(N_2)$.

Let M be a maximal subgroup of G_q; the commutator subgroup of $(N_1\times N_2)M\lessdot G$ is cyclic, so it centralizes at least one of the N_i's. If G_q were not cyclic, there should be at least three maximal subgroups in G_q, hence two maximal subgroups of G should centralize the same N_i (for instance N_1); hence we get the contradiction $G_q<C_G(N_1)$.

Definition 2.9. Let G_p be a group of prime order $p>2$, G_q a q-group (q prime), metacyclic with a subgroup $C\lhd G_q$ such that G_q/C is a cyclic and $|G_q/C|$ divides $p-1$. Moreover, suppose there is no cyclic quotient G_q/X with X cyclic and $X\leqq C$, while for every maximal subgroup $M<G_q$ there exists a cyclic factor M/X_M with X_M cyclic and $X_M\leqq C\cap M$. Under these hypotheses, there exists an homomorphism $\alpha:G_q\to\operatorname{Aut}G_p$ such that $\operatorname{Ker}\alpha=C$. We shall call the semidirect product $G=[G_p]G_q$ (determined by α) a *group of type G_α*.

An easy example of such a group can be obtained in the following way. Let us denote by G_2 the dihedral group of order 8 and by G_p a group of prime order $p>2$. Then for any maximal non-cyclic subgroup $C<G_2$, the hypotheses of Definition 2.9 hold, hence the semidirect product $G=[G_p]G_2$ determined by the homomorphism $\alpha:G_2\to\operatorname{Aut}G_p$ with kernel C is of type G_α.

Remark. Let $G_q\not\cong Q_8$ be a metacyclic non-abelian q-group (q prime). With standard calculations (omitted here for the sake of brevity) we can prove the existence of a subgroup $C\lhd G_q$ such that: G_q/C is cyclic and there is no cyclic quotient G_q/X with X cyclic and $X\leqq C$, while for every maximal subgroup $M<G_q$ there is a cyclic factor M/X_M with X_M cyclic and contained in $C\cap M$. From this it follows that in Definition 2.9 the q-Sylow subgroups of G can be almost arbitrary.

We thank Mercede Maj for this remark.

Theorem 2.10. *Let G be a supersolvable group with $|\pi(G)|=2$ and G_p cyclic* ($p=\max\pi(G)$). *Then G is a minimal non-*(MC) *group if and only if it is of type G_α.*

Proof. Let G be minimal non-(MC). By 2.5, $\mathscr{L}(G)=G_p$ and $|G_p|=p$; $G=G_pG_q$ is of type G_α (see Definition 2.9), where $C=C_{G_q}(G_p)$.

The converse statement is trivial.

2.11. *Let G be a supersolvable and minimal non-*(MC) *group. Then* $|\pi(G)| \leqq 3$.

Proof. If $\mathscr{L}(G)$ is cyclic, the statement follows from 2.2 and 2.4. Assume $\mathscr{L}(G)$ is not cyclic; then (see 2.5) $G=[\mathscr{L}(G)]M$ and $\mathscr{L}(G)=G_p=N_1 \times N_2$ ($N_i \lhd G$ and $|N_i|=p$). Arguing by contradiction, suppose $M=A\times B\times C$ with A, B and C non-trivial Hall subgroups. The commutator subgroup of $(N_1\times N_2)(A\times B)$ is cyclic; hence we can assume $A\times B \leqq C_G(N_1)$. Similarly, either $A\times C < C_G(N_2)$ or $A\times C < C_G(N_1)$, whence either $G=A\times G_p(B\times C)$, or $C=N_1\times N_2 M$; in the first case G is metacyclic, since A and $G_p(B\times C)$ are metacyclic of coprime orders; in the second case we get a contradiction to the meaning of $\mathscr{L}(G)=N_1\times N_2$.

By 2.11, Theorems 1.2, 2.7, 2.8 and 2.10 characterize the non-primary and minimal non-metacyclic groups; thus the theory of group extentions allows us to give an effective construction of these groups. Furthermore:

2.12. *Let G be a minimal non-*(MC) *group, without any normal* $G_2 \in \mathrm{Syl}(G)$ *isomorphic to* Q_8. *Then any irreducible representation of G over an algebraically closed field K such that* $\mathrm{ch}\, K \nmid |G|$ *is monomial.*

Proof. G is either supersolvable or metabelian (see Theorem 1.2), hence the assertion is an immediate consequence of the following well-known result by Huppert [6] (V. 18.4. Satz): Every solvable group G having a supersolvable quotient G/H such that H has abelian Sylow subgroups is monomial.

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ISTITUTO MATEMATICO DELL'UNIVERSITÀ
VIA MEZZOCANNONE, 8
80 134 NAPOLI, ITALY