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On questions of hereditariness of radicals

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Introduction

All rings considered are associative. We shall use the following notation: $\mathscr{R}$ is a radical class, $\mathscr{S}\mathscr{R}$ the corresponding semisimple class; $\lhd$ indicates an ideal; ann (A) is the two-sided annihilator of a ring A; $\mathscr{B}$ is the lower Baer radical; $L(\ )=$ =lower radical class, for instance, $\mathscr{B}=L$(zero-rings).

A radical class $\mathscr{R}$ is said to be a *hereditary class* if $\mathscr{R}$ satisfies:

$$B\lhd A, \quad A\in\mathscr{R} \Rightarrow B\in\mathscr{R}.$$

In [1] a weak version of hereditariness was introduced, which arose in connection with the finite closure property of radicals under subdirect sums. If a radical class $\mathscr{R}$ is closed under finite subdirect sums, then $\mathscr{R}$ has the property:

$$I\lhd A, \quad A\in\mathscr{R}, \quad I\subseteqq \text{ann}\, A \Rightarrow I\in\mathscr{R}.$$

Such a radical is said to be *hereditary for annihilator ideals* ([1], Proposition 1.7). Although this condition is not sufficient for the finite closure property of $\mathscr{R}$, very little is needed to make $\mathscr{R}$ hereditary. Hereditary radical classes are closed under finite subdirect sums. We investigate these questions in §2.

In [3] a new characterization was found for the *maximal hereditary subradical* $h_{\mathscr{R}}$ of a radical $\mathscr{R}$, in fact

$$h_{\mathscr{R}} = \bar{\bar{\mathscr{R}}} = \{A| \quad \text{any ideal of} \quad A \quad \text{is in} \quad \bar{\mathscr{R}}\}$$

where $\bar{\mathscr{R}}=\{A|$ any ideal of A is in $\mathscr{R}\}$. We use this result to sharpen Proposition 1.6 of [1], where $h_{\mathscr{R}}$ was given as an intersection of an infinite number of radical classes. We show that the chain, used in [1], stops at the second step. We also show that, for any radical $\mathscr{R}$ containing $\mathscr{B}$ or being subidempotent,

$$h_{\mathscr{R}} = \bar{\mathscr{R}} = \{A| \quad \text{any ideal of} \quad A \quad \text{is in} \quad \mathscr{R}\},$$

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i.e. $\bar{\mathcal{R}}$ is hereditary. Here a radical class $\mathcal{R}$ is called *subidempotent* if any ring A in $\mathcal{R}$ is idempotent.

Our terminology for radical theory is the usual one. Both a radical and a radical class are denoted by $\mathcal{R}$. A ring A is in the radical class $\mathcal{R}$ or A is an $\mathcal{R}$-ring if $A=\mathcal{R}(A)$, where $\mathcal{R}(A)$ is the radical of the ring A. The semisimple class $\mathcal{S}\mathcal{R}$ of the radical $\mathcal{R}$ consists of all rings A, such that $\mathcal{R}(A)=0$, i.e.

$$\mathcal{S}\mathcal{R} = \{A \mid A \text{ has no non-zero ideal in } \mathcal{R}\}.$$

A class $\mathbf{M}$ is said to be *closed under finite subdirect sums* if $A_1, \ldots, A_n \in \mathbf{M}$ implies that $A_1 + \ldots + A_n \in \mathbf{M}$ (subdirect sum) for any finite number n of rings $A_1, \ldots, A_n$. In order to show closure under finite subdirect sums one needs only consider $n=2$.

I would like to thank Dr. R. Wiegandt for his criticism and valuable remarks in preparing this paper. Originally I tried to do something with quasi-radicals, but he remarked that an order-preserving quasi-radical is complete, which, together with idempotency, makes it a radical (cf. [2]).

1. In our first result we deal with sums of ideals (cf. Problem 12 in [4]).

Theorem 1. *Let A be a ring with ideals B, C and $B \cap C \in \mathcal{R}$ for some radical $\mathcal{R}$. Then $\mathcal{R}(B+C)=\mathcal{R}(B)+\mathcal{R}(C)$.*

Proof. The inclusion $\mathcal{R}(B)+\mathcal{R}(C) \subseteq \mathcal{R}(B+C)$ is clear. Obviously, we have the direct decomposition

$$B+C/B\cap C = B/B\cap C \oplus C/B\cap C.$$

By the assumption $B\cap C \subseteq \mathcal{R}(B+C)$, therefore the above direct decomposition yields

$$\mathcal{R}(B+C)/B\cap C = K/B\cap C \oplus L/B\cap C$$

for ideals K resp. L in B resp. C. Clearly $K/B\cap C$ is an $\mathcal{R}$-ring and contained in $\mathcal{R}(B/B\cap C)=\mathcal{R}(B)/B\cap C$. Similarly

$$L/B\cap C \subseteq \mathcal{R}(C/B\cap C) = \mathcal{R}(C)/B\cap C.$$

Hence

$$\mathcal{R}(B+C)/B\cap C \subseteq \mathcal{R}(B)/B\cap C \oplus \mathcal{R}(C)/B\cap C$$

giving

$$\mathcal{R}(B+C) \subseteq \mathcal{R}(B)+\mathcal{R}(C).$$

In addition we have

Theorem 2. *For any ring A with arbitrary ideals B, C and I, J and for any radical $\mathcal{R}$ the following two statements are equivalent:*

(i) A/B, $A/C \in \mathcal{R}$, $\mathcal{R}(B)=\mathcal{R}(C)$ *implies* $A/(B\cap C) \in \mathcal{R}$,

(ii) A/I, $A/J \in \mathcal{R}$, $\mathcal{R}(I)=\mathcal{R}(J)=0$ *implies* $A/(I\cap J) \in \mathcal{R}$.

Proof. (i)⇒(ii) is trivial.

Let A/B, $A/C\in\mathscr{R}$ with $\mathscr{R}(B)=\mathscr{R}(C)$. Then

$$\frac{A/\mathscr{R}(B)}{B/\mathscr{R}(B)}\cong A/B\in\mathscr{R},\quad \frac{A/\mathscr{R}(B)}{C/\mathscr{R}(B)}\cong A/C\in\mathscr{R}$$

with

$$\mathscr{R}\left(\frac{B}{\mathscr{R}(B)}\right)=\mathscr{R}\left(\frac{C}{\mathscr{R}(B)}\right)=0\quad (\mathscr{R}(B)=\mathscr{R}(C)).$$

Hence

$$\frac{A/\mathscr{R}(B)}{B/\mathscr{R}(B)\cap C/\mathscr{R}(B)}=\frac{A/\mathscr{R}(B)}{(B\cap C)/\mathscr{R}(B)}\cong\frac{A}{B\cap C}\in\mathscr{R}.$$

In order to show that a radical class $\mathscr{R}$ is closed under finite subdirect sums we might use the following reduction:

Theorem 3. *If for any ring A and arbitrary ideals I, J in A with $I\cap J=0$ the condition $A/I, A/J\in\mathscr{R}$ implies that $A/(I\cap J)\cong A\in\mathscr{R}$, then $\mathscr{R}$ is closed under finite subdirect sums.*

Proof. The symbol $\oplus$ will mean "direct sum". Let I, J be ideals of A such that $I\cap J=0$, $A/I\in\mathscr{R}$ and $A/J\in\mathscr{R}$. By $I\cap J=0$ we have

(1) $$\big(I\oplus\mathscr{R}(J)\big)\cap\big(\mathscr{R}(I)\oplus J\big)=\mathscr{R}(I)\oplus\mathscr{R}(J).$$

and also

(2) $$\big(I\oplus\mathscr{R}(J)\big)/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)\cong I/\mathscr{R}(I)\in\mathscr{S}\mathscr{R}$$

(3) $$\big(\mathscr{R}(I)\oplus J\big)/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)\cong J/\mathscr{R}(J)\in\mathscr{S}\mathscr{R}.$$

In (2) and (3) the left hand sides are ideals of $A/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)$ and by (1) these ideals have zero intersection. Since

$$\frac{A/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)}{\big(I\oplus\mathscr{R}(J)\big)/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)}\cong A/\big(I\oplus\mathscr{R}(J)\big)\in\mathscr{R}$$

and

$$\frac{A/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)}{\big(\mathscr{R}(I)\oplus J\big)/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)}\cong A/\big(\mathscr{R}(I)\oplus J\big)\in\mathscr{R}$$

the imposed condition is applicable yielding

$$A/\big(\mathscr{R}(I)\oplus\mathscr{R}(J)\big)\in\mathscr{R};$$

and so the extension property of $\mathscr{R}$ implies $A\in\mathscr{R}$.

Lemma 4a. *Let A be a ring with ideals I, J such that $I\cap J=0$, $A/I\in\mathscr{R}$* and *$A/J\in\mathscr{R}$. If $\mathscr{R}$ is hereditary for annihilator ideals, then* ann $A\in\mathscr{R}$. *Moreover, if, in addition, $I, J\in\mathscr{S}\mathscr{R}$, then* $I\cap\operatorname{ann} A=J\cap\operatorname{ann} A=0$.

Proof. $\operatorname{ann} A/\operatorname{ann} A\cap I \cong (\operatorname{ann} A+I)/I$ is an annihilator ideal of $A/I\in\mathcal{R}$, so $\operatorname{ann} A/\operatorname{ann} A\cap I\in\mathcal{R}$. Also

$$\operatorname{ann} A\cap I \cong \frac{(\operatorname{ann} A\cap I)+J}{J} \subseteq \frac{J+\operatorname{ann} A}{J}\in\mathcal{R},$$

since $(J+\operatorname{ann} A)/J\subseteq \operatorname{ann} A/J$, $A/J\in\mathcal{R}$. Again, since $((\operatorname{ann} A\cap I)+J)/J$ is an annihilator ideal of $(J+\operatorname{ann} A)/J$,

$$\frac{(\operatorname{ann} A\cap I)+J}{J} \cong \operatorname{ann} A\cap I\in\mathcal{R}.$$

The extension property of $\mathcal{R}$ implies $\operatorname{ann} A\in\mathcal{R}$. Now assume that $I, J\in\mathcal{S}\mathcal{R}$. Then $\operatorname{ann} A\cap I$ is an ideal in $I\in\mathcal{S}\mathcal{R}$ implies $\operatorname{ann} A\cap I=0$. Similarly $\operatorname{ann} A\cap J=0$.

From the proof of Lemma 4a we see that $I\in\mathcal{S}\mathcal{R}$ implies that $\operatorname{ann} A\cap I=0$. Clearly $\operatorname{ann} A\subseteq I^*=\{a\in A|aI=Ia=(0)\}$, as $I\subseteq A$. We can say more if we assume that $\mathcal{B}\subseteq\mathcal{R}$.

Lemma 4b. *Let $\mathcal{B}\subseteq\mathcal{R}$; A is a ring with ideals I, J such that $I\in\mathcal{S}\mathcal{R}$, $I\cap J=0$, $A/I\in\mathcal{R}$ and $A/J\in\mathcal{R}$. Also $\mathcal{R}$ is hereditary for annihilator ideals. Then any ideal K in A such that $K\cap I=0$ is contained in I^* and I^* is maximal with respect to $I^*\cap I=0$. Moreover $A/I^*\in\mathcal{R}$, whereas* $\operatorname{ann}(A/I^*)=0$.

Proof. $(I^*\cap I)^2\subseteq I^*\cdot I=0$, so $(I^*\cap I)\lhd I\in\mathcal{S}\mathcal{R}$ gives $I^*\cap I=0$. As $J\cap I=0$, by Zorn's lemma there exists an ideal M, maximal relative to $M\cap I=0$. Since $MI=IM=0$, $M\subseteq I^*$ and the maximality of M ensures $M=I^*$.

Let K be any ideal in A such that $K\cap I=0$. If K is not contained in I^*, then $(K+I^*)\cap I\neq 0$. Now let x, y be arbitrary elements in $(K+I^*)\cap I$, then $x=k+a$ $(k\in K, a\in I^*)$, $y\in I$. Hence $xy=(k+a)y=ky+ay=0$, as $K\cap I=I^*\cap I=0$. So $[(K+I^*)\cap I]^2=0$. But $[(K+I^*)\cap I]\lhd I\in\mathcal{S}\mathcal{R}$ and $\mathcal{S}\mathcal{R}\subseteq\mathcal{S}\mathcal{B}$ implies $(K+I^*)\cap I$ is a semiprime ring, consequently $(K+I^*)\cap I=0$. This contradicts $(K+I^*)\cap I\neq 0$, so $K\subseteq I^*$. In particular, $J\subseteq I^*$ and $A/J\in\mathcal{R}$ implies $A/I^*\in\mathcal{R}$. The ideal $I\cong(I+I^*)/I^*$ is essential in A/I^*: if $B/I^*\neq 0$ is an ideal of A/I^*, then $B\cap I\nsubseteq I^*$, otherwise $B\cap I\subseteq I^*\cap I=0$ implies $B\cap I=0$ which is impossible by the maximality of I^*. Hence

$$0\neq((B\cap I)+I^*)/I^*\subseteq B/I^*\cap(I+I^*)/I^*.$$

As $\mathcal{R}$ is hereditary for annihilator ideals and $(I+I^*)/I^*\cap\operatorname{ann} A/I^*\subseteq\operatorname{ann} A/I^*$, it follows that $(I+I^*)/I^*\cap\operatorname{ann} A/I^*\in\mathcal{R}$. On the other hand $I\cong(I+I^*)/I^*\in\mathcal{S}\mathcal{R}$, so $(I+I^*)/I^*\cap\operatorname{ann} A/I^*\in\mathcal{S}\mathcal{R}$ yielding $(I+I^*)/I^*\cap\operatorname{ann} A/I^*=0$. The essential property of $(I+I^*)/I^*$ in A/I^* implies $\operatorname{ann} A/I^*=0$.

For any ring A and any ideal I in A we define $[I:A]:=\{x\in A|xA\subseteq I,\ Ax\subseteq I\}$.

Theorem 5. *Let $\mathcal{R}$ be an arbitrary radical class. $\mathcal{R}$ is closed under finite subdirect sums if and only if*

(i) *Whenever I and J are ideals in a ring A with $I\cap J=0$, then $A/[I: A], A/[J: A]\in\mathcal{R}$ implies $A/[I: A]\cap[J: A]\in\mathcal{R}$.*

(ii) *$\mathcal{R}$ is hereditary for annihilator ideals.*

Proof. Suppose that (i) and (ii) are satisfied. Let I, J be ideals in A with $I\cap J=0$ and suppose A/I, $A/J\in\mathcal{R}$. Since $I\subseteqq[I: A]$ and $J\subseteqq[J: A]$ it follows that $A/[I: A]$, $A/[J: A]\in\mathcal{R}$. It can easily be seen that $I\cap J=0$ implies $\operatorname{ann} A=[I: A]\cap[J: A]$. Hence (i) implies that $A/\operatorname{ann} A\in\mathcal{R}$. From Lemma 4a we get, using (ii), that $\operatorname{ann} A\in\mathcal{R}$. The extension property of $\mathcal{R}$ implies $A\in\mathcal{R}$.

The converse is clear by Proposition 1.7 [1].

Note that $\operatorname{ann}(A/I)=[I: A]/I$, so we may replace (i) by

$$\frac{A/I}{\operatorname{ann}(A/I)},\quad \frac{A/J}{\operatorname{ann}(A/J)}\in\mathcal{R}\quad \text{implies}\quad \frac{A}{\operatorname{ann} A}\in\mathcal{R}.$$

Corollary 6. *Let $\mathcal{R}$ be a radical class such that $\mathcal{B}\subseteqq\mathcal{R}$. Then $\mathcal{R}$ is closed under finite subdirect sums if and only if*

$$\frac{A}{I^*},\quad \frac{A}{[I: A]}\in\mathcal{R}\quad \textit{implies}\quad \frac{A}{I^*\cap[I: A]}\in\mathcal{R}$$

for any ideal I in any ring A.

Proof. Obviously $\mathcal{B}\subseteqq\mathcal{R}$ implies that $\mathcal{R}$ is hereditary for annihilator ideals. Let A be a ring with ideals I, J such that $I\cap J=0$; A/I, $A/J\in\mathcal{R}$. We have to show that $A\in\mathcal{R}$. If $I\notin\mathcal{S}\mathcal{R}$, then $I/\mathcal{R}(I)$, $(J+\mathcal{R}(I))/\mathcal{R}(I)$ are ideals in $A/\mathcal{R}(I)$ and $I\cap(J+\mathcal{R}(I))=\mathcal{R}(I)+(I\cap J)=\mathcal{R}(I)$. So $A/\mathcal{R}(I)$ is a ring with ideals $I/\mathcal{R}(I)$, $(J+\mathcal{R}(I))/\mathcal{R}(I)$ having zero-intersection; also A/I, $A/(J+\mathcal{R}(I))\in\mathcal{R}$, as $A/J\in\mathcal{R}$. Now $I/\mathcal{R}(I)\in\mathcal{S}\mathcal{R}$. If we can show that $A/\mathcal{R}(I)\in\mathcal{R}$, we are done by the extension property.

Hence, without loss of generality, we may assume: $I\triangleleft A, J\triangleleft A$; A/I, $A/J\in\mathcal{R}$ and $I\in\mathcal{S}\mathcal{R}$.

Now apply Lemma 4b. Then $J\subseteqq I^*$ and $I\subseteqq[I: A]$ imply A/I^*, $A/[I: A]\in\mathcal{R}$. Hence $A/(I^*\cap[I: A])\in\mathcal{R}$. By Lemma 4b we know that $\operatorname{ann}(A/I^*)=0$, i.e. $[I^*: A]=I^*$. From $I\cap I^*=0$, as $I\in\mathcal{S}\mathcal{R}$, it follows that $\operatorname{ann} A=[I: A]\cap[I^*: A]=I^*\cap[I^*: A]$. Hence $A/\operatorname{ann} A\in\mathcal{R}$. Then Lemma 4a implies that $\operatorname{ann} A\in\mathcal{R}$ and consequently $A\in\mathcal{R}$. So the condition is sufficient. The converse is obvious.

The above proof of Corollary 6 suggests the next result which is a further reduction for the question of finite subdirect closure for radicals (cf. Theorem 3).

Theorem 7. *If for any ring A and arbitrary ideals I, J in A with $I\cap J=0$, $I, J\in\mathcal{S}\mathcal{R}$ the condition $A/J, A/I\in\mathcal{R}$ implies that $A\in\mathcal{R}$, then $\mathcal{R}$ is closed under finite subdirect sums.*

Proof. Let A be a ring with ideals I, J such that $I\cap J=0$; A/I, $A/J\in\mathcal{R}$. By Theorem 3 we have to show that $A\in\mathcal{R}$. Now the ring $A/(\mathcal{R}(I)\oplus\mathcal{R}(J))$ has ideals $(I\oplus\mathcal{R}(J))/(\mathcal{R}(I)\oplus\mathcal{R}(J))$, $(\mathcal{R}(I)\oplus J)/(\mathcal{R}(I)\oplus\mathcal{R}(J))$ with zero intersection and both ideals are in $\mathcal{S}\mathcal{R}$ (see the proof of Theorem 3). Hence $A/(\mathcal{R}(I)\oplus\mathcal{R}(J))\in\mathcal{R}$ and $A\in\mathcal{R}$.

Theorem 8. *Let $\mathcal{R}$ be a radical class. Then $\mathcal{R}$ is hereditary for annihilator ideals if and only if AI, $IA\in\mathcal{R}$ imply $I\in\mathcal{R}$ for any ring $A\in\mathcal{R}$ and any ideal I in A.*

Proof. Let $I\lhd A$ with $A\in\mathcal{R}$ and $I\subseteq \operatorname{ann} A$. Then $AI=IA=0\in\mathcal{R}$ implies $I\in\mathcal{R}$. Conversely, let $I\lhd A$ with $A\in\mathcal{R}$ such that AI, $IA\in\mathcal{R}$. Now

$$\frac{I}{AI+IA}\lhd\frac{A}{AI+IA}$$

and clearly

$$\frac{I}{AI+IA}\subseteq \operatorname{ann}\left(\frac{A}{AI+IA}\right),$$

so

$$\frac{A}{AI+IA}\in\mathcal{R} \quad\text{implies}\quad \frac{I}{AI+IA}\in\mathcal{R}.$$

Also

$$\frac{AI+IA}{AI}\cong\frac{IA}{AI\cap IA}\in\mathcal{R},$$

as $IA\in\mathcal{R}$. Hence

$$\left(\frac{I}{AI}\right)\Big/\left(\frac{AI+IA}{AI}\right)\cong\frac{I}{AI+IA}\in\mathcal{R}$$

implies $I/AI\in\mathcal{R}$. But $AI\in\mathcal{R}$, so $I\in\mathcal{R}$.

2. In a number of cases we get that $\mathcal{R}$ is hereditary for annihilator ideals implies that $\mathcal{R}$ is hereditary. We need some kind of extra condition, otherwise the condition of hereditariness for annihilator ideals would be sufficient for closure under finite subdirect sums. In [1] a counter-example is given.

Theorem 9. *Let $\mathcal{R}$ be a radical class which is hereditary for annihilator ideals. Then $\mathcal{R}$ is hereditary if and only if $I\lhd A\in\mathcal{R}$ implies AI, $IA\in\mathcal{R}$.*

Proof. From the above proof in Theorem 8 we infer that $I\lhd A\in\mathcal{R}$ together with $AI, IA\in\mathcal{R}$ implies $I\in\mathcal{R}$. Hence $\mathcal{R}$ is hereditary. The converse is trivial.

Theorem 10. *Let $\mathcal{R}$ be a radical class which is hereditary for annihilator ideals. Then $\mathcal{R}$ is hereditary if and only if $I \lhd A \in \mathcal{R}$, $I \subseteq A^2$ implies $I \in \mathcal{R}$.*

Proof. Again let $I \lhd A \in \mathcal{R}$. Now $AI \subseteq A^2$, $IA \subseteq A^2$ with both AI and IA ideals in $\mathcal{R}$ imply AI, $IA \in \mathcal{R}$. As $\mathcal{R}$ is hereditary for annihilator ideals, it follows that $I \in \mathcal{R}$ (Theorem 8), so $\mathcal{R}$ is hereditary. The converse is trivial.

Another condition which ensures hereditariness of $\mathcal{R}$ is contained in the following

Theorem 11. *A radical class $\mathcal{R}$ is hereditary if and only if $I \lhd A \in \mathcal{R}$ implies $I \in \mathcal{R}$ whenever $I^2=(0)$ or $I \subseteq A^2$.*

Proof. This is a direct consequence of Theorem 10, since the condition

$$I \lhd A \in \mathcal{R}, \quad I^2 = (0) \Rightarrow I \in \mathcal{R}$$

yields also

$$I \lhd A \in \mathcal{R}, \quad AI = 0 = IA \Rightarrow I \in \mathcal{R}$$

so that $\mathcal{R}$ is hereditary for annihilator ideals.

Corollary 12. *Let $\mathcal{R}$ be a radical class which contains $\mathcal{B}$. Then $\mathcal{R}$ is hereditary if and only if*

$$I \lhd A \in \mathcal{R}, \quad I \subseteq A^2 \Rightarrow I \in \mathcal{R}.$$

Proof. Let $I \lhd A \in \mathcal{R}$. Now $I/I^2 \in \mathcal{B} \subseteq \mathcal{R}$. But $I^2 \subseteq A^2$, so $I^2 \in \mathcal{R}$, hence $I \in \mathcal{R}$ and $\mathcal{R}$ is hereditary.

We might remark that Corollary 12 is an easy consequence of Theorem 10, since any radical class $\mathcal{R}$ which contains $\mathcal{B}$ is hereditary for annihilator ideals (see the proof of Corollary 6).

The proof of Corollary 12 also indicates the next result:

Corollary 13. *Let $\mathcal{R}$ be a radical class which contains $\mathcal{B}$. Then $\mathcal{R}$ is hereditary if and only if*

$$I \lhd A \in \mathcal{R} \Rightarrow I^2 \in \mathcal{R}.$$

Proof. See Corollary 12.

Theorem 14. *A radical class $\mathcal{R}$ is hereditary if and only if $\mathcal{R}$ is hereditary for annihilator ideals and*

$$\mathcal{R}(A)\big(I \cap \mathcal{R}(A)\big) \subseteq \mathcal{R}(I), \quad \big(I \cap \mathcal{R}(A)\big)\mathcal{R}(A) \subseteq \mathcal{R}(I)$$

for any ideal I in any ring A.

Proof. Obviously if $\mathcal{R}$ is hereditary, then using $I \cap \mathcal{R}(A) = \mathcal{R}(I)$ for any ideal I in any ring A, we get the conditions.

Conversely, let I be an ideal in a ring A. Then $(I\cap\mathcal{R}(A))/\mathcal{R}(I)\lhd\mathcal{R}(A)/\mathcal{R}(I)$ and the second condition implies that $(I\cap\mathcal{R}(A))/\mathcal{R}(I)\subseteqq\operatorname{ann}\mathcal{R}(A)/\mathcal{R}(I)$. Hence, since $\mathcal{R}(A)/\mathcal{R}(I)\in\mathcal{R}$, the first condition gives $(I\cap\mathcal{R}(A))/\mathcal{R}(I)\in\mathcal{R}$. This says $I\cap\mathcal{R}(A)\in\mathcal{R}$ or $I\cap\mathcal{R}(A)\subseteqq\mathcal{R}(I)$. Always $\mathcal{R}(I)\subseteqq I\cap\mathcal{R}(A)$, whence $I\cap\mathcal{R}(A)=\mathcal{R}(I)$ and $\mathcal{R}$ is hereditary.

Corollary 15. *A radical class $\mathcal{R}$ is hereditary if and only if $\mathcal{R}$ is hereditary for annihilator ideals and*

$$I\lhd A\in\mathcal{R},\quad AI+IA\subseteqq\mathcal{R}(I)\Rightarrow I\in\mathcal{R}$$

for any ring $A\in\mathcal{R}$ and any ideal I in A.

Proof. The necessity being trivial, let $I\lhd A\in\mathcal{R}$. Then $\mathcal{R}(A)(I\cap\mathcal{R}(A))=A(I\cap A)\subseteqq AI\subseteqq\mathcal{R}(I)$ and $(I\cap\mathcal{R}(A))\mathcal{R}(A)=(I\cap A)A\subseteqq IA\subseteqq\mathcal{R}(I)$, if $AI+IA\subseteqq\mathcal{R}(I)$ is assumed. Now apply Theorem 14.

It might be noted that Theorem 9 follows directly from Corollary 15. For, if $I\lhd A\in\mathcal{R}$, then AI, $IA\in\mathcal{R}$ implies AI, $IA\subseteqq\mathcal{R}(I)$, so $AI+IA\subseteqq\mathcal{R}(I)$. Corollary 15 gives $I\in\mathcal{R}$ or $\mathcal{R}$ is hereditary.

We conclude this section with a more general result.

Theorem 16. *Let $\mathcal{R}$ and S resp. be radicals such that S-semi-simple rings are $\mathcal{R}$-radical. Then $\mathcal{R}$ is hereditary if and only if*

$$I\lhd A\in\mathcal{R},\quad I\subseteqq\mathrm{S}(A)\Rightarrow I\in\mathcal{R}$$

for any ring $A\in\mathcal{R}$ and any ideal I in A.

Proof. Suppose the condition be satisfied and assume that $I\lhd A\in\mathcal{R}$. As $I/\mathrm{S}(I)$ is S-semi-simple, we have $I/\mathrm{S}(I)\in\mathcal{R}$. Now $\mathrm{S}(I)\lhd A\in\mathcal{R}$ and $\mathrm{S}(I)\subseteqq\mathrm{S}(A)$, so $\mathrm{S}(I)\in\mathcal{R}\Rightarrow I\in\mathcal{R}$. Then $\mathcal{R}$ is hereditary. The converse is obvious.

Example. Let $\mathcal{R}$ be the class of idempotent rings, i.e. the rings A with $A^2=A$. Let S be the upper radical determined by the Boolean rings. A ring A is called a Boolean ring if $a^2=a$ for every element $a\in A$. Since Boolean rings form a special class of rings, S is a special radical and the S-semi-simple rings are subdirect sums of Boolean rings, so they are again Boolean rings. Any Boolean ring is idempotent, hence any S-semi-simple ring is $\mathcal{R}$-radical. It is known that $\mathcal{R}$ is not hereditary. If we take the subradical class $\mathcal{R}'$ (of $\mathcal{R}$) of the hereditarily idempotent rings, we get a hereditary radical $\mathcal{R}'$. Again any S-semi-simple ring is $\mathcal{R}'$-radical, as any Boolean ring is hereditarily idempotent. (If A is a Boolean ring and $I\lhd A$, then I is again a Boolean ring and idempotent).

3. It is known that for any radical $\mathcal{R}$ there exists a unique maximal hereditary radical $h_{\mathcal{R}}$, contained in $\mathcal{R}$. In [3] it is shown that $h_{\mathcal{R}}=\overline{\overline{\mathcal{R}}}$, where $\overline{\mathcal{R}}=\{A|$any ideal of A is in $\mathcal{R}\}$. It can easily be proved that $\bar{\mathcal{R}}$ is a radical and $\mathcal{R}$ is hereditary if and only if $\mathcal{R}=\bar{\mathcal{R}}$. Let $(\mathcal{S}\mathcal{R})_k$ be the essential closure of the semisimple class $\mathcal{S}\mathcal{R}$ of the radical $\mathcal{R}$. A ring $A\in(\mathcal{S}\mathcal{R})_k$ if A has an essential ideal $B\in\mathcal{S}\mathcal{R}$.

Lemma 17. *For any radical $\mathcal{R}$, $\bar{\mathcal{R}}=\mathcal{U}(\mathcal{S}\mathcal{R})_k$ (upper radical).*

Proof. Let $A\in\bar{\mathcal{R}}$ and suppose that $A\notin\mathcal{U}(\mathcal{S}\mathcal{R})_k$. Then there exists a non-zero homomorphic image $A/I\in(\mathcal{S}\mathcal{R})_k$ and A/I has an essential ideal $B/I\in\mathcal{S}\mathcal{R}$. But $A\in\bar{\mathcal{R}}$, so $A/I\in\bar{\mathcal{R}}$. By definition of $\bar{\mathcal{R}}$, it follows that $B/I\in\mathcal{R}$, which implies $B/I\in\mathcal{R}\cap\mathcal{S}\mathcal{R}=0$. Since this is impossible for an essential ideal, we get that $A\in\mathcal{U}(\mathcal{S}\mathcal{R})_k$.

Conversely, assume that $A\in\mathcal{U}(\mathcal{S}\mathcal{R})_k$. If $A\notin\bar{\mathcal{R}}$, A has a non-zero ideal I, $I\notin\mathcal{R}$. Then $0\neq I/\mathcal{R}(I)$ is an ideal in $A/\mathcal{R}(I)$ and $I/\mathcal{R}(I)\in\mathcal{S}\mathcal{R}$. Now there exists a homomorphic image A/J of $A/\mathcal{R}(I)$ containing an isomorphic copy of $I/\mathcal{R}(I)$, such that this copy is an essential ideal in A/J. But $A\in\mathcal{U}(\mathcal{S}\mathcal{R})_k$ implies that $A/J\in\mathcal{U}(\mathcal{S}\mathcal{R})_k$, hence $A/J\in\mathcal{U}(\mathcal{S}\mathcal{R})_k\cap(\mathcal{S}\mathcal{R})_k=0$ or $A=J$. Contradiction, so $A\in\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}=\mathcal{U}(\mathcal{S}\mathcal{R})_k$.

For our next result we use the notation of [1]. $\mathcal{R}$ is a radical class.

$$\mathcal{G}^0_{\mathcal{R}}:=\{(S,A)|\quad S\lhd A \quad\text{and}\quad S\in\mathcal{S}\mathcal{R}\},$$

$$\bar{\mathcal{G}}^0_{\mathcal{R}}:=\{A|\quad\text{every}\quad 0\neq A/I \quad\text{has no nonzero ideals in}\quad \mathcal{S}\mathcal{R}\}.$$

$\bar{\mathcal{G}}^0_{\mathcal{R}}$ is a radical class [1].

$$\mathcal{G}^1_{\mathcal{R}}:=\{(S,A)|\quad S\lhd A \quad\text{and}\quad S\in\mathcal{S}(\bar{\mathcal{G}}^0_{\mathcal{R}})\},$$

$$\bar{\mathcal{G}}^1_{\mathcal{R}}:=\{A|\quad\text{every}\quad 0\neq A/I \quad\text{has no nonzero ideals in}\quad \mathcal{S}(\bar{\mathcal{G}}^0_{\mathcal{R}})\};$$

$\bar{\mathcal{G}}^1_{\mathcal{R}}$ is a radical class [1].

Continuing in this way, one gets a chain of radical classes:

$$\mathcal{R}\supseteqq\bar{\mathcal{G}}^0_{\mathcal{R}}\supseteqq\ldots\supseteqq\bar{\mathcal{G}}^n_{\mathcal{R}}\supseteqq\ldots.$$

In [1] it was shown that $\bigcap_n\bar{\mathcal{G}}^n_{\mathcal{R}}$ is the unique maximal hereditary radical subclass of $\mathcal{R}$. An improvement of this result is given in the next theorem.

Theorem 18. *For any radical class $\mathcal{R}$ we have: $\bar{\mathcal{G}}^1_R$ is the unique maximal hereditary radical subclass of $\mathcal{R}$.*

Proof. We show that, with the above notation, $\bar{\mathcal{G}}^0_{\mathcal{R}}=\bar{\mathcal{R}}$. Let $A\in\bar{\mathcal{G}}^0_{\mathcal{R}}$. Since for any $I\lhd A$ we have $\mathcal{R}(I)\lhd A$ and $I/\mathcal{R}(I)\in\mathcal{S}\mathcal{R}$, the assumption $A\in\bar{\mathcal{G}}^0_{\mathcal{R}}$ yields $I/\mathcal{R}(I)=0$. Thus $A\in\bar{\mathcal{R}}$.

Conversely, let $A\in\bar{\mathcal{R}}$ and take any $0\neq A/I$. If A/I has a nonzero ideal $B(I)\in\in\mathcal{S}\mathcal{R}$, then $A/I\in\bar{\mathcal{R}}$ yields that $B/I\in\bar{\mathcal{R}}\cap\mathcal{S}\mathcal{R}=0$, which is a contradiction. Hence $0\neq A/I$ has no nonzero ideals in $\mathcal{S}\mathcal{R}$, i.e. $A\in\bar{\mathcal{G}}^0_{\mathcal{R}}$. Using Lemma 17 we have established: $\bar{\mathcal{R}}=\mathcal{U}(\mathcal{S}\mathcal{R})_k=\bar{\mathcal{G}}^0_{\mathcal{R}}$. Apply now Lemma 17 again to the radical $\bar{\mathcal{G}}^0_{\mathcal{R}}$: $\bar{\bar{\mathcal{R}}}=\mathcal{U}(\mathcal{S}\bar{\mathcal{R}})_k=\bar{\mathcal{G}}^0_{\bar{\mathcal{R}}}$. From $\bar{\mathcal{R}}=\bar{\mathcal{G}}^0_{\mathcal{R}}$ and the definitions of $\mathcal{G}^0_{\mathcal{R}}$ and $\mathcal{G}^1_{\mathcal{R}}$ resp. we infer that $\mathcal{G}^0_{\bar{\mathcal{R}}}=\mathcal{G}^1_{\mathcal{R}}$. Hence we get: $\bar{\mathcal{G}}^0_{\bar{\mathcal{R}}}=\bar{\mathcal{G}}^1_{\mathcal{R}}$ or $\bar{\mathcal{G}}^1_{\mathcal{R}}=\bar{\bar{\mathcal{R}}}$, which is the unique maximal hereditary subradical of $\mathcal{R}$.

Note that the above chain now reads:

$$\mathcal{R}\supseteqq\bar{\mathcal{R}}\supseteqq\bar{\bar{\mathcal{R}}}=\bar{\bar{\mathcal{R}}}=\dots$$

since $\bigcap_n\bar{\mathcal{G}}^n_{\mathcal{R}}=\bar{\mathcal{G}}^1_{\mathcal{R}}=\bar{\bar{\mathcal{R}}}$.

An example in [1] shows that, in general, $\bar{\mathcal{G}}^0_{\mathcal{R}}=\bar{\mathcal{R}}$ need not be hereditary. In fact, $\bar{\mathcal{G}}^0_{\mathcal{R}}$ is hereditary if and only if $\bar{\mathcal{G}}^0_{\mathcal{R}}=\bar{\mathcal{G}}^1_{\mathcal{R}}$ or, in our notation, $\bar{\mathcal{R}}$ is hereditary if and only if $\bar{\mathcal{R}}=\bar{\bar{\mathcal{R}}}$.

Theorem 19. *If a radical class $\mathcal{R}$ is hereditary for annihilator ideals, then $\bar{\mathcal{R}}$ is hereditary.*

Proof. Let A be a zero-ring and suppose that $A\in\mathcal{R}$. Then any ideal I of A is in $\mathcal{R}$, so $A\in\bar{\mathcal{R}}$. Therefore any zero-ring in $\mathcal{R}$ is in $\bar{\mathcal{R}}$, which implies $\bar{\mathcal{R}}=\bar{\bar{\mathcal{R}}}$ ([3], Proposition 1 and Corollary 1).

The next result is well-known. For a radical class $\mathcal{R}$ the following are equivalent.

a) $\mathcal{R}$ contains all zero-rings;

b) $\mathcal{R}$ contains all nilpotent rings;

c) $\mathcal{B}\subseteqq\mathcal{R}$.

The above proof of Theorem 19 indicates that any radical class $\mathcal{R}$ containing all zero-rings satisfies: $\bar{\mathcal{R}}$ is hereditary. So we get

Corollary 20. *Let $\mathcal{R}$ be a radical with $\mathcal{B}\subseteqq\mathcal{R}$. Then $\bar{\mathcal{R}}$ is the maximal hereditary subradical of $\mathcal{R}$.*

Proof. Obviously $\mathcal{B}\subseteqq\mathcal{R}$ implies that $\mathcal{R}$ is hereditary for annihilator ideals, so Corollary 20 is a direct consequence of Theorem 19.

Remark. We will see that the condition of Theorem 19 for hereditariness of $\bar{\mathcal{R}}$ is not necessary (after Theorem 24).

The counterpart is formed by the radicals $\mathcal{R}$ containing no nonzero zero-rings.

Lemma 21. *For a radical class $\mathcal{R}$ the following are equivalent:*

a) *$\mathcal{R}$ contains no nonzero zero-rings;*

b) $\mathcal{R}$ *contains no nonzero nilpotent rings;*

c) $\mathcal{R}$ *is subidempotent i.e. any ring A in* $\mathcal{R}$ *is idempotent.*

Proof. Since the proof is straightforward, we omit it.

In order to study radicals $\mathcal{R}$ with the above property, we introduce

$$\mathcal{G}_{\mathcal{R}} := \{(S, A)| \quad S \in \mathcal{S}\mathcal{R} \quad \text{and} \quad S \subseteqq \operatorname{ann} A\},$$

where S is a subring of A. This implies $S \lhd A$.

$\bar{\mathcal{G}}_{\mathcal{R}} := \{A| \quad \text{every} \quad 0 \neq A/I \quad \text{has no nonzero ideals in} \quad \operatorname{ann}(A/I) \quad \text{and in} \quad \mathcal{S}\mathcal{R}\}$. Then $\bar{\mathcal{G}}_{\mathcal{R}}$ is a radical class and $\mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ is the maximal radical subclass of $\mathcal{R}$ which is hereditary for annihilator ideals ([1], Proposition 1.8).

Define

$$\mathcal{E}_6 := \{A| \quad \text{every} \quad 0 \neq A/I \quad \text{has} \quad \operatorname{ann}(A/I) = 0\}.$$

Then $\mathcal{E}_6$ is a radical class (cf. [4]). It is clear that for any radical $\mathcal{R}$ one has: $\mathcal{E}_6 \subseteqq \bar{\mathcal{G}}_{\mathcal{R}}$. The next lemma shows that equality holds for subidempotent radicals $\mathcal{R}$.

Lemma 22. *Let* $\mathcal{R}$ *be a subidempotent radical. Then* $\bar{\mathcal{G}}_{\mathcal{R}} = \mathcal{E}_6$.

Proof. We only need to prove that $\bar{\mathcal{G}}_{\mathcal{R}} \subseteqq \mathcal{E}_6$. Let $A \in \bar{\mathcal{G}}_{\mathcal{R}}$ and take any $0 \neq \neq A/I = \bar{A}$. Then $\operatorname{ann} \bar{A}/\mathcal{R}(\operatorname{ann} \bar{A}) \subseteqq \operatorname{ann}(\bar{A}/\mathcal{R}(\operatorname{ann} \bar{A}))$. Since $A \in \bar{\mathcal{G}}_{\mathcal{R}}$, it follows that $\operatorname{ann} \bar{A}/\mathcal{R}(\operatorname{ann} \bar{A}) = 0$, so $\operatorname{ann} \bar{A} \in \mathcal{R}$. But $(\operatorname{ann} \bar{A})^2 = 0$, so $\operatorname{ann} \bar{A} = 0$, as $\mathcal{R}$ is subidempotent. Hence $A \in \mathcal{E}_6$.

In general one can show that

$$\bar{\mathcal{G}}_{\mathcal{R}} = \{A| \quad \text{any} \quad 0 \neq A/I \quad \text{has the property:} \ J/I \lhd A/I, \ J/I \subseteqq \operatorname{ann}(A/I) \Rightarrow J/I \in \mathcal{R}\}.$$

From the definitions of $\mathcal{G}_{\mathcal{R}}$ and $\mathcal{G}^0_{\mathcal{R}}$ resp. we get immediately: $\mathcal{G}_{\mathcal{R}} \subseteqq \mathcal{G}^0_{\mathcal{R}}$ yielding $\bar{\mathcal{G}}^0_{\mathcal{R}} \subseteqq \bar{\mathcal{G}}_{\mathcal{R}}$ for any radical $\mathcal{R}$. Always $\bar{\mathcal{G}}^0_{\mathcal{R}} \subseteqq \mathcal{R}$, hence $\bar{\mathcal{R}} = \bar{\mathcal{G}}^0_{\mathcal{R}} \subseteqq \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ for any radical $\mathcal{R}$.

In the following theorem we will give a sufficient condition in order that $\bar{\mathcal{R}} = \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$.

Theorem 23. *Let* $\mathcal{R}$ *be a radical class such that* $A \in \mathcal{R}$ *implies* $AS, SA \in \mathcal{R}(S)$ *for any ring A and any ideal S in A. Then* $\bar{\mathcal{R}} = \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ *and* $\bar{\mathcal{R}}$ *is the unique maximal radical subclass of* $\mathcal{R}$ *which is hereditary for annihilator ideals.*

Proof. We have to show that $A \in \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ implies $A \in \bar{\mathcal{R}}$. Assume $A \in \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ and let $S \lhd A$. Then $S/\mathcal{R}(S) \lhd A/\mathcal{R}(S) \in \bar{\mathcal{G}}_{\mathcal{R}}$, as $A \in \bar{\mathcal{G}}_{\mathcal{R}}$. Also $S/\mathcal{R}(S) \subseteqq \subseteqq \operatorname{ann}(A/\mathcal{R}(S))$, as AS, $SA \in \mathcal{R}(S)$. Hence $S/\mathcal{R}(S) \in \mathcal{R}$, as $A \in \bar{\mathcal{G}}_{\mathcal{R}}$ (see the above characterization of $\bar{\mathcal{G}}_{\mathcal{R}}$). Therefore $S = \mathcal{R}(S)$ or $S \in \mathcal{R}$. It follows that $A \in \bar{\mathcal{R}}$. By Proposition 1.8 [1] $\bar{\mathcal{R}} = \mathcal{R} \cap \bar{\mathcal{G}}_{\mathcal{R}}$ has the required property of maximality.

We have seen that radicals $\mathcal{R}$ with $\mathcal{B}\subseteq\mathcal{R}$ have the property that $\bar{\mathcal{R}}$ is the maximal hereditary subradical of $\mathcal{R}$. Our final result contains another class of radicals $\mathcal{R}$ for which this phenomenon occurs.

Theorem 24. *Let $\mathcal{R}$ be a subidempotent radical. Then $\mathcal{R}$ is hereditary for annihilator ideals if and only if $\mathcal{R}\subseteq\mathcal{E}_6$.*

For any subidempotent radical $\mathcal{R}$ we have that $\bar{\mathcal{R}}$ is the maximal hereditary subradical of $\mathcal{R}$. $\bar{\mathcal{R}}$ is a hereditarily idempotent radical.

Proof. From [1], Proposition 1.8 it follows that $\mathcal{R}$ is hereditary for annihilator ideals if and only if $\mathcal{R}\subseteq\bar{\mathcal{G}}_{\mathcal{R}}$ for any radical $\mathcal{R}$. So for a subidempotent radical we get the first result immediately from Lemma 22. Now let $\mathcal{R}$ be an arbitrary subidempotent radical. Take any ring $A\in\mathcal{R}$. If $A^2=0$, then $A=0$, so any zero-$\mathcal{R}$-ring is in $\bar{\mathcal{R}}$, hence $\bar{\mathcal{R}}$ is hereditary ([3], Proposition 1) and $\bar{\mathcal{R}}$ is a hereditarily idempotent radical.

Remark. As not every subidempotent radical $\mathcal{R}$ is contained in $\mathcal{E}_6$, it follows that a subidempotent radical $\mathcal{R}$ need not be hereditary for annihilator ideals. This shows that the sufficient condition in Theorem 19 is not necessary.

In the light of the previous results we examine the Examples 1.4 and 1.5 in [1]. Consider the ring R whose additive group is $Q+Q$ (direct sum) and whose multiplication is given by

$$(a, b)(c, d) = (ac, ad+bc).$$

The homomorphic images of R are 0, Q and R, while the ideals of R are 0, $I\,(\cong Q^0)$ and R (Q^0 is the zero-ring on Q).

Let $\mathcal{D}$ be the (radical) class of rings with divisible additive groups. Then both R and I are in $\mathcal{D}$. Since I is the only non-trivial ideal in R, we get that $R\in\bar{\mathcal{D}}$. However, $I\notin\bar{\mathcal{D}}$, as $I(\cong Q^0)$ has non-zero reduced ideals. So $\bar{\mathcal{D}}$ is not hereditary. By Theorem 19 we get that $\mathcal{D}$ is not hereditary for annihilator ideals. Note that $\mathcal{B}$ is not contained in $\mathcal{D}$, since $Z^0\notin\mathcal{D}$, $Z^0\in\mathcal{B}$ (Z^0 is the zero-ring on Z). In addition, $\mathcal{D}$ is not subidempotent, since $I\in\mathcal{D}$, but $I^2=0$. This is in accordance with Corollary 20 and Theorem 24, since any radical $\mathcal{R}$ containing $\mathcal{B}$ or being subidempotent has a hereditary subradical $\bar{\mathcal{R}}$.

We also consider the lower radical class $L(\{R\})$, determined by R. Now R is a non-simple ring with identity $(1, 0)$. Since I is the only non-trivial ideal of R and $R/I\cong Q$, Q not isomorphic to R, we see that R satisfies the conditions (i) and (ii) of Theorem 3.5 in [1]. Hence $L(\{R\})$ is not closed under finite subdirect sums.

On the other hand, R is idempotent and $R/I\cong Q$ is idempotent, so that $R\in\mathcal{E}_6$. Therefore $L(\{R\})\subseteq\mathcal{E}_6$. Also $L(\{R\})$ is a subidempotent radical, as any radical contained in $\mathcal{E}_6$ is subidempotent. Hence $L(\{R\})$ is hereditary for annihilator ideals (Theorem 24).

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