On the a.e. convergence of multiple orthogonal series. II (Unrestricted convergence of the rectangular partial sums)

F. MÓRICZ and K. TANDORI

1. Preliminaries and notations

Let Z^d_+ be the set of all d-tuples $k=(k_1, \ldots, k_d)$ with positive integral coordinates. In case $d=1$, Z^1_+ is the set of the positive integers, which is well-ordered. For $d\geqq 2$, Z^d_+ is only partially ordered by agreeing that for $k=(k_1, \ldots, k_d)$ and $n=(n_1, \ldots, n_d)$ we write $k\leqq n$ iff $k_j\leqq n_j$ for each $j(=1, 2, \ldots, d)$. Further, sometimes we write 1 for the d-tuple $(1, \ldots, 1)$.

Let $\varphi=\{\varphi_k(x)\colon k\in Z^d_+\}$ be an orthonormal system (in abbreviation: ONS) on the unit interval $I=(0, 1)$. Since we are interested in the questions of almost everywhere (in abbreviation: a.e.) convergence behaviour, in this paper we do not make any distinction among open, half-closed, and closed intervals. Consider the d-multiple orthogonal series

(1) $$\sum_{k\in Z^d_+} a_k\varphi_k(x) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} a_{k_1, \ldots, k_d}\varphi_{k_1, \ldots, k_d}(x),$$

where $\mathfrak{a}=\{a_k\colon k\in Z^d_+\}$ is a d-multiple sequence of real numbers (coefficients), for which

(2) $$\sum_{k\in Z^d_+} a_k^2 < \infty.$$

By the well-known Riesz—Fischer theorem, there exists a function $f(x)\in L^2(I)$ such that the rectangular partial sums

$$s_n(x) = \sum_{k\leqq n} a_k\varphi_k(x) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} a_{k_1, \ldots, k_d}\varphi_{k_1, \ldots, k_d}(x)$$

Received January 10, 1983.

of series (1) converge to $f(x)$ in L^2-metric:

$$\int_0^1 [s_n(x)-f(x)]^2\,dx \to 0 \quad \text{as} \quad \min_{1\leqq j\leqq d} n_j \to \infty.$$

It is a fundamental fact that condition (2) itself does not ensure the pointwise convergence of $s_n(x)$ to $f(x)$ (see [2] for $d=1$ and [5] for $d\geqq 2$). Our goal is to give a necessary and sufficient condition in order to ensure the a.e. convergence of the rectangular partial sums $s_n(x)$ of series (1) for every ONS φ on I. The case $d=1$ was elaborated by the second author in [6] and [7]. Some of the results for $d\geqq 2$ were announced by the first author in [4].

In this paper we do not suppose any restriction on the ratios n_j/n_i, $1\leqq i,j\leqq d$, that is, we are concerned ourselves with the a.e. unrestricted convergence of the rectangular partial sums $s_n(x)$ of series (1).

Given a d-multiple sequence $\mathfrak{a}=\{a_k\colon k\in Z_+^d\}$, let us introduce the following quantity:

$$\|\mathfrak{a}\| = \sup_{\varphi}\left\{\int_0^1 \Big(\sup_{m,n\in Z_+^d:\, m\leqq n}\Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2 dx\right\}^{1/2},$$

where the first supremum is extended over all ONS φ on I. Here and in the sequel

$$\sum_{m\leqq k\leqq n} a_k\varphi_k(x) = \sum_{k_1=m_1}^{n_1}\cdots\sum_{k_d=m_d}^{n_d} a_{k_1,\ldots,k_d}\varphi_{k_1,\ldots,k_d}(x).$$

Given an arbitrary subset Q of Z_+^d, we consider another d-multiple sequence $\mathfrak{a}(Q)=\{a_k(Q)\colon k\in Z_+^d\}$ defined as follows

$$a_k(Q) = \begin{cases} a_k & \text{for } k\in Q,\\ 0 & \text{for } k\in Z_+^d\setminus Q.\end{cases}$$

In particular, we write

$$Q_N=\{k\in Z_+^d\colon k_j\leqq N \text{ for each } j\} \quad (N=1,2,\ldots).$$

In this case we may write

$$\|\mathfrak{a}(Q_N)\| = \sup_{\varphi}\Big\{\int_0^1 \Big(\max_{m,n\in Q_N:\, m\leqq n}\Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2 dx\Big\}^{1/2}. \tag{3}$$

It is clear that $\|\mathfrak{a}(Q_N)\|\leqq\|\mathfrak{a}\|$ for every $N=1,2,\ldots$. On the other hand, by Beppo Levi's theorem, it follows that

$$\lim_{N\to\infty}\|\mathfrak{a}(Q_N)\| = \|\mathfrak{a}\|. \tag{4}$$

Denote by

$$\mathfrak{M} = \{\mathfrak{a}\colon \|\mathfrak{a}\|<\infty\}.$$

It will turn out that $\mathfrak{M}$ is the very class of those d-multiple sequences $\mathfrak{a}=\{a_k: k\in Z_+^d\}$, for which series (1) converges a.e. for every ONS φ on I.

Remark 1. Let us observe that

$$\sum_{m\leqq k\leqq n} a_k\varphi_k(x) = \sum_{\delta_1=0}^{1} \cdots \sum_{\delta_d=0}^{1} (-1)^{\delta_1+\ldots+\delta_d} s_{\delta_1(m_1-1)+(1-\delta_1)n_1,\ldots,\delta_d(m_d-1)+(1-\delta_d)n_d}(x)$$

with the agreement of taking $s_{k_1,\ldots,k_d}(x)=0$ if $k_j=0$ for at least one j. Thus, introducing another quantity:

$$\|\mathfrak{a}\|_* = \sup_{\varphi}\Big\{\int_0^1 \big(\sup_{n\in Z_+^d} \big|\sum_{1\leqq k\leqq n} a_k\varphi_k(x)\big|\big)^2\,dx\Big\}^{1/2}$$

for every d-multiple sequence $\mathfrak{a}$ we have

$$\|\mathfrak{a}\|_* \leqq \|\mathfrak{a}\| \leqq 2^d\|\mathfrak{a}\|_*.$$

This means that the corresponding classes $\mathfrak{M}$ and $\mathfrak{M}_*=\{\mathfrak{a}: \|\mathfrak{a}\|_*<\infty\}$ coincide. However, the use of $\|\mathfrak{a}\|$ is more convenient for our purposes.

Remark 2. The definition of $\|\mathfrak{a}\|$ and the theorems below remain valid if the interval I of orthogonality is replaced by any finite, nonatomic, positive measure space $(X, \mathcal{F}, \nu)$, in particular $X=I^d$. In addition, the treatment can be extended, with some simple modifications, to the case when we consider ONS φ of complex-valued functions and d-multiple sequences $\mathfrak{a}$ of complex numbers.

2. Auxiliary results

We begin with

Lemma 1. *For every positive integer N we have*

$$\Big\{\sum_{k\in Q_N} a_k^2\Big\}^{1/2} \leqq \|\mathfrak{a}(Q_N)\| \leqq \sum_{k\in Q_N} |a_k|. \tag{5}$$

Proof. It immediately follows from the following inequalities:

$$\Big|\sum_{k\in Q_N} a_k\varphi_k(x)\Big| \leqq \max_{m,n\in Q_N: m\leqq n}\Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big| \leqq \sum_{k\in Q_N} |a_k\varphi_k(x)|.$$

Theorem 1. *The mapping $\|\cdot\|$: $\mathfrak{a}(\in\mathfrak{M})\to\|\mathfrak{a}\|$ is a norm, and $\mathfrak{M}$ is a Banach space with respect to the usual vector operations and the norm $\|\cdot\|$.*

Proof. Obviously $\|\mathfrak{a}\|\geqq 0$. By (4) and (5),

$$\Big\{\sum_{k\in Z_+^d} a_k^2\Big\}^{1/2} \leqq \|\mathfrak{a}\| \leqq \sum_{k\in Z_+^d} |a_k|. \tag{5'}$$

Hence it follows that $\|\mathfrak{a}\|=0$ if and only if $a_k=0$ for each $k\in Z_+^d$.

It is also clear that $\|\alpha\mathfrak{a}\|=|\alpha|\|\mathfrak{a}\|$ for every real number α and sequence $\mathfrak{a}$.

Now let two sequences $\mathfrak{a}=\{a_k: k\in Z_+^d\}$ and $\mathfrak{b}=\{b_k: k\in Z_+^d\}$ be given. Then for every positive integer N

$$\max\Big|\sum_{m\leqq k\leqq n}(a_k+b_k)\varphi_k(x)\Big|\leqq\max\Big|\sum_{m\leqq k\leqq n}a_k\varphi_k(x)\Big|+\max\Big|\sum_{m\leqq k\leqq n}b_k\varphi_k(x)\Big|,$$

where all the three maxima are taken under the conditions $m,n\in Q_N$ and $m\leqq n$. Applying the Bunjakovskiĭ—Schwartz inequality and definition (3), we get that

$$\|(\mathfrak{a}+\mathfrak{b})(Q_N)\|\leqq\|\mathfrak{a}(Q_N)\|+\|\mathfrak{b}(Q_N)\|.$$

Hence, via (4),

$$\|\mathfrak{a}+\mathfrak{b}\|\leqq\|\mathfrak{a}\|+\|\mathfrak{b}\|.$$

Thus we have shown that $\mathfrak{M}$ is a linear space. Now we prove the completeness with respect to the norm $\|\cdot\|$. To this effect, let $\mathfrak{a}^{(p)}=\{a_k^{(p)}: k\in Z_+^d\}$ $(p=1,2,\dots)$ be an ordinary sequence of elements from $\mathfrak{M}$ satisfying the Cauchy convergence criterion:

$$\|\mathfrak{a}^{(p)}-\mathfrak{a}^{(q)}\|\to 0\quad\text{as}\quad p,q\to\infty.$$

By (5′),

$$\sum_{k\in Z_+^d}(a_k^{(p)}-a_k^{(q)})^2\to 0\quad\text{as}\quad p,q\to\infty.$$

So there exists an $\mathfrak{a}=\{a_k: k\in Z_+^d\}$ such that

$$a_k^{(p)}\to a_k\quad\text{as}\quad p\to\infty\quad\text{for each}\quad k\in Z_+^d.$$

Let an $\varepsilon>0$ be given. By assumption there exists a positive integer $p_0=p_0(\varepsilon)$ such that

$$\|\mathfrak{a}^{(p)}-\mathfrak{a}^{(q)}\|\leqq\varepsilon\quad\text{whenever}\quad p,q\geqq p_0.$$

Given a positive integer N, a fortiori

$$\|\mathfrak{a}^{(p)}(Q_N)-\mathfrak{a}^{(q)}(Q_N)\|\leqq\varepsilon\quad\text{whenever}\quad p,q\geqq p_0.$$

By (5) and the triangle inequality,

$$\|\mathfrak{a}^{(p)}(Q_N)-\mathfrak{a}(Q_N)\|\leqq\|\mathfrak{a}^{(p)}(Q_N)-\mathfrak{a}^{(q)}(Q_N)\|+\|\mathfrak{a}^{(q)}(Q_N)-\mathfrak{a}(Q_N)\|\leqq$$
$$\leqq\varepsilon+\sum_{k\in Q_N}|a_k^{(q)}-a_k|.$$

Letting q tend to infinity, hence

$$\|\mathfrak{a}^{(p)}(Q_N)-\mathfrak{a}(Q_N)\|\leqq\varepsilon\quad\text{whenever}\quad p\geqq p_0.$$

This holds true for each $N=1,2,\dots$. Thus, by (4)

$$\|\mathfrak{a}^{(p)}-\mathfrak{a}\|\leqq\varepsilon\quad\text{whenever}\quad p\geqq p_0,$$

in particular, $\mathfrak{a}\in\mathfrak{M}$. Being $\varepsilon>0$ arbitrary,

$$\|\mathfrak{a}^{(p)}-\mathfrak{a}\|\to 0\quad\text{as}\quad p\to\infty.$$

Remark 3. By (5′), if $\mathfrak{a}\in\mathfrak{M}$, then condition (2) is necessarily satisfied.

Theorem 2. *If* $\mathfrak{a}=\{a_k\colon k\in Z_+^d\}$ *and* $\mathfrak{b}=\{b_k\colon k\in Z_+^d\}$ *are such that*

$$|a_k| \leqq |b_k| \quad \textit{for every} \quad k\in Z_+^d, \tag{6}$$

then $\|\mathfrak{a}\|\leqq\|\mathfrak{b}\|$.

This immediately yields

Corollary 1. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be such that* (6) *is satisfied. If* $\mathfrak{b}\in\mathfrak{M}$, *then* $\mathfrak{a}\in\mathfrak{M}$; *and consequently, if* $\mathfrak{a}\notin\mathfrak{M}$, *then* $\mathfrak{b}\notin\mathfrak{M}$.

Proof of Theorem 2. By (4), it is enough to prove that for every positive integer N

$$\|\mathfrak{a}(Q_N)\| \leqq \|\mathfrak{b}(Q_N)\|. \tag{7}$$

By (6), if $b_k=0$ for every $k\in Q_N$, then also $a_k=0$ for every $k\in Q_N$. Thus, (7) is trivially satisfied:

$$\|\mathfrak{a}(Q_N)\| = \|\mathfrak{b}(Q_N)\| = 0.$$

Now assume that the set

$$R_N = \{k\in Q_N\colon \quad b_k \neq 0\}$$

is non-empty. If $k\in Q_N\setminus R_N$, then $b_k=0$ and $a_k=0$. For a given $\varepsilon>0$, let us choose an ONS $\{\varphi_k(x)\colon k\in Q_N\}$ in such a way that

$$\|\mathfrak{a}(Q_N)\|^2-\varepsilon \leqq \int_0^1 \Big(\max_{m,n\in Q_N:\, m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2\,dx. \tag{8}$$

We define for $k\in R_N$

$$\bar\varphi_k(x) = \begin{cases} \sqrt{3}\, a_k b_k^{-1}\varphi_k(3x) & \text{for} \quad x\in(0,\, 1/3),\\ \sqrt{3}\,(1-a_k^2 b_k^{-2})^{1/2}\varphi_k(3x-1) & \text{for} \quad x\in(1/3,\, 2/3),\\ 0 & \text{for} \quad x\in(2/3,\, 1);\end{cases}$$

and for $k\in Q_N\setminus R_N$

$$\bar\varphi_k(x) = \begin{cases} 0 & \text{for} \quad x\in(0,\, 2/3),\\ \sqrt{3}\,\varphi_k(3x-2) & \text{for} \quad x\in(2/3,\, 1).\end{cases}$$

It is easy to check that $\{\bar\varphi_k(x)\colon k\in Q_N\}$ is also an ONS on I. Further, (8) implies that

$$\|\mathfrak{b}(Q_N)\|^2 \geqq \int_0^1 \Big(\max\Big|\sum_{m\leqq k\leqq n} b_k\bar\varphi_k(x)\Big|\Big)^2 dx \geqq 3\int_0^{1/3} \Big(\max\Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(3x)\Big|\Big)^2 dx =$$

$$= \int_0^1 \Big(\max\Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2 dx \geqq \mathfrak{a}(Q_N)\|^2-\varepsilon,$$

where all the three maxima are taken under the conditions $m, n \in Q_N$ and $m \leqq n$. Being $\varepsilon > 0$ arbitrary, hence the wanted inequality (7) follows.

In the sequel we shall need the following

Lemma 2. *Let $\mathfrak{a}(Q_N)=\{a_k: k\in Q_N\}$ be given, where N is a positive integer. Then there exist an ONS $\psi=\{\psi_k(x): k\in Q_N\}$ of step functions on I and a simple subset E of I having the following properties:*

$$\text{(9)} \qquad \operatorname{mes} E \geqq C_1$$

and

$$\text{(10)} \qquad \max_{m,n\in Q_N: m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k \psi_k(x)\Big| \geqq \|\mathfrak{a}(Q_N)\| \quad \text{for every} \quad x\in E,$$

where C_1 is a positive constant.

A set E is said to be simple if it is the union of finitely many disjoint intervals and $\operatorname{mes} E$ stands for the sum of the lengths of these intervals (i.e. the Lebesgue measure of E). In the following, by $C_2, C_3, \ldots$ we shall denote positive constants, sometimes depending on d.

Proof. If $\|\mathfrak{a}(Q_N)\|=0$, then statements (9) and (10) are satisfied for $E=(0, 1)$, $C_1=1$, and arbitrary ONS ψ of step functions.

From now on we assume that $\|\mathfrak{a}(Q_N)\|>0$. Without loss of generality, we may also assume that $\|\mathfrak{a}(Q_N)\|=1$. By definition, there exists on ONS φ on I, for which

$$\text{(11)} \qquad \int_0^1 \Big(\max_{m,n\in Q_N: m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2 dx \geqq \frac{1}{2}.$$

Let $\varepsilon>0$ be arbitrary, and let $\chi_k(x)$, $k\in Q_N$, be step functions on I such that

$$\int_0^1 [\varphi_k(x)-\chi_k(x)]^2\, dx \leqq \varepsilon \quad (k\in Q_N).$$

We set

$$\alpha_{k,m} = \int_0^1 \chi_k(x)\chi_m(x)\, dx$$

and

$$\eta_k = \sum_{m\in Q_N: m\neq k} |\alpha_{k,m}| \quad (k, m\in Q_N).$$

It is not hard to see that if $\varepsilon>0$ is small enough, then we have

$$\text{(12)} \qquad \int_0^1 \Big(\max_{m,n\in Q_N: m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k\chi_k(x)\Big|\Big)^2 dx \geqq \frac{1}{4}$$

and

$$\int_0^1 \Big(\max_{m,n\in Q_N:\, m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k \Big(1-\frac{1}{\sqrt{\alpha_{k,k}+\eta_k}}\Big)\chi_k(x)\Big|\Big)^2 dx \leqq \frac{1}{8}. \tag{13}$$

We shall define an ONS $\{\bar\chi_k(x):\ k\in Q_N\}$ of step functions on the interval $(0, 2)$ in the following way. We divide the interval $(1, 2)$ into $N^d(N^d-1)$ subintervals $I_{k,m}$ of equal length, where $k, m\in Q_N$ and $k\neq m$. Then, for $k\in Q_N$, we set

$$\bar\chi_k(x) = \begin{cases} \chi_k(x) & \text{for } x\in(0, 1),\\ \left\{\dfrac{|\alpha_{k,m}|}{2 \operatorname{mes} I_{k,m}}\right\}^{1/2} & \text{for } x\in I_{k,m},\\ -\left\{\dfrac{|\alpha_{k,m}|}{2 \operatorname{mes} I_{k,m}}\right\}^{1/2} \operatorname{sign}\alpha_{k,m} & \text{for } x\in I_{m,k},\\ 0 & \text{otherwise,}\end{cases}$$

where in the second and third lines m runs over Q_N except k. Taking into account that

$$\int_0^2 \bar\chi_k^2(x)\, dx = \alpha_{k,k}+\eta_k,$$

it is obvious that the step functions

$$\bar\psi_k(x) = \frac{\bar\chi_k(x)}{\sqrt{\alpha_{k,k}+\eta_k}} \quad (k\in Q_N)$$

constitute an ONS on the interval $(0, 2)$. Furthermore, by (12) and (13)

$$\int_0^2 \Big(\max_{m,n\in Q_N:\, m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k\bar\psi_k(x)\Big|\Big)^2 dx \geqq \frac{1}{8}. \tag{14}$$

Now we set

$$F(x) = \max_{m,n\in Q_N:\, m\leqq n} \Big|\sum_{m\leqq k\leqq n} a_k\bar\psi_k(x)\Big|.$$

Since $F(x)$ is a step function, we can divide the interval $(0, 2)$ into disjoint subintervals $J_1, J_2, \ldots, J_\varrho$ such that it is constant on each J_r; denote by w_r this constant value $(r=1, 2, \ldots, \varrho)$. Then (14) can be rewritten into the following form:

$$S = \sum_{r=1}^{\varrho} w_r^2 \operatorname{mes} J_r \geqq \frac{1}{8}.$$

Taking ε sufficiently small, we may assume that $S\leqq 2$. We set

$$u_0 = 0,\quad u_r = \frac{1}{2}\sum_{s=1}^{r} w_s^2 \operatorname{mes} J_s \quad (r = 1, 2, \ldots, \varrho),$$

and, for $k\in Q_N$,

$$\psi_k(x)=\begin{cases}\dfrac{\sqrt{2}}{w_{r+1}}\bar{\psi}_k\left(\dfrac{2}{w_{r+1}^2}(x-u_r)+\dfrac{1}{2}\sum_{s=1}^{r}\operatorname{mes} J_s\right) & \text{for } x\in(u_r, u_{r+1}),\\ & r=0,1,\dots,\varrho-1, \quad \text{provided } w_r\neq 0;\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to verify that these functions $\psi_k(x)$, $k\in Q_N$, the simple set $E=\bigcup_{r=0}^{\varrho-1}(u_r, u_{r+1})$ with $C_1=1/8$ satisfy all requirements of Lemma 2.

Theorem 3. *Let* $\mathfrak{a}=\{a_k\colon k\in Z_+^d\}$ *be given. If* Q' *and* $Q''\subseteqq Z_+^d$ *are such that*

$$Q'\cap Q''=\emptyset \quad \text{and} \quad Q'\cup Q''=Z_+^d,$$

then

$$\|\mathfrak{a}(Q')\|^2+\|\mathfrak{a}(Q'')\|^2\leqq\|\mathfrak{a}\|^2.$$

Proof. Given an $\varepsilon>0$, there exist two ONS $\{\varphi_k'(x)\colon k\in Z_+^d\}$ and $\{\varphi_k''(x)\colon k\in Z_+^d\}$ such that

$$(15)\qquad \begin{aligned}&\int_0^1\Big(\sup_{m,n\in Z_+^d:\, m\leqq n}\Big|\sum_{m\leqq k\leqq n:\, k\in Q'}a_k\varphi_k'(x)\Big|\Big)^2dx\geqq\|\mathfrak{a}(Q')\|^2-\varepsilon,\\ &\int_0^1\Big(\sup_{m,n\in Z_+^d:\, m\leqq n}\Big|\sum_{m\leqq k\leqq n:\, k\in Q''}a_k\varphi_k''(x)\Big|\Big)^2dx\geqq\|\mathfrak{a}(Q'')\|-\varepsilon.\end{aligned}$$

We define for $k\in Q'$

$$\varphi_k(x)=\begin{cases}\sqrt{2}\,\varphi_k'(2x) & \text{for } x\in(0,1/2),\\ 0 & \text{for } x\in(1/2,1);\end{cases}$$

and for $k\in Q''$

$$\varphi_k(x)=\begin{cases}0 & \text{for } x\in(0,1/2),\\ \sqrt{2}\,\varphi_k''(2x-1) & \text{for } x\in(1/2,1).\end{cases}$$

It is clear that $\{\varphi_k(x)\colon k\in Z_+^d\}$ is an ONS on I. Furthermore, by (15)

$$\|\mathfrak{a}\|^2\geqq\int_0^1\Big(\sup\Big|\sum_{m\leqq k\leqq n}a_k\varphi_k(x)\Big|\Big)^2dx=2\int_0^{1/2}\Big(\sup\Big|\sum_{m\leqq k\leqq n:\, k\in Q'}a_k\varphi_k''(2x)\Big|\Big)^2dx+$$

$$+2\int_{1/2}^1\Big(\sup\Big|\sum_{m\leqq k\leqq n:\, k\in Q''}a_k\varphi_k''(2x-1)\Big|\Big)^2dx=$$

$$=\int_0^1\Big(\sup\Big|\sum_{m\leqq k\leqq n:\, k\in Q'}a_k\varphi_k'(x)\Big|\Big)^2dx+\int_0^1\Big(\sup\Big|\sum_{m\leqq k\leqq n:\, k\in Q''}a_k\varphi_k''(x)\Big|\Big)^2dx\geqq$$

$$\geqq\|\mathfrak{a}(Q')\|^2+\|\mathfrak{a}(Q'')\|^2-2\varepsilon,$$

where all the five suprema are taken over all $m, n \in Z_+^d$ such that $m \leqq n$. Being $\varepsilon > 0$ arbitrary, the proof is complete.

Corollary 2. *If* $\mathfrak{a} \in \mathfrak{M}$, *then*

$$\lim_{N\to\infty} \|\mathfrak{a}(Z_+^d \setminus Q_N)\| = 0.$$

Proof. Given $\varepsilon > 0$, by (4) there exists a positive integer N_0 such that

$$\|\mathfrak{a}(Q_N)\|^2 \geqq \|\mathfrak{a}\|^2 - \varepsilon \quad \text{whenever} \quad N \geqq N_0.$$

On the other hand, in virtue of Theorem 3

$$\|\mathfrak{a}(Q_N)\|^2 + \|\mathfrak{a}(Z_+^d \setminus Q_N)\|^2 \leqq \|\mathfrak{a}\|^2 < \infty.$$

Combining the two estimates above, we find that

$$\|\mathfrak{a}(Z_+^d \setminus Q_N)\|^2 \leqq \varepsilon \quad \text{whenever} \quad N \geqq N_0.$$

Corollary 3. $\mathfrak{M}$ *is separable.*

Proof. On the one hand, by Corollary 2,

$$\|\mathfrak{a} - \mathfrak{a}(Q_N)\| = \|\mathfrak{a}(Z_+^d \setminus Q_N)\| \leqq \varepsilon$$

if N is large enough. On the other hand, we can choose $\mathfrak{b}(Q_N) = \{b_k : k \in Q_N\}$ in such a way that all b_k, $k \in Q_N$, are rational numbers and by (5)

$$\|\mathfrak{a}(Q_N) - \mathfrak{b}(Q_N)\| \leqq \sum_{k \in Q_N} |a_k - b_k| \leqq \varepsilon.$$

Since the class $\bigcup_{N=1}^{\infty} \{\mathfrak{b}(Q_N)$: all b_k are rational numbers for $k \in Q_N\}$ is countable, the proof is complete.

Theorem 4. *If* $\mathfrak{a} \in \mathfrak{M}$, *then there exists a d-multiple sequence* $\lambda = \{\lambda_k : k \in Z_+^d\}$ *of positive numbers such that*

$$\lambda_k \to \infty \quad as \quad \max_{1 \leqq j \leqq d} k_j \to \infty \quad and \quad \lambda\mathfrak{a} \in \mathfrak{M}. \tag{16}$$

If $\mathfrak{a} \notin \mathfrak{M}$, *then there exists a d-multiple sequence* $\mu = \{\mu_k : k \in Z_+^d\}$ *of positive numbers such that*

$$\mu_k \to 0 \quad as \quad \max_{1 \leqq j \leqq d} k_j \to \infty \quad and \quad \mu\mathfrak{a} \notin \mathfrak{M}. \tag{17}$$

Proof. If $\mathfrak{a} \in \mathfrak{M}$, then by Corollary 2 there exists a sequence $(0=)N_0 < N_1 < \ldots$ $\ldots < N_p < \ldots$ of integers for which

$$\|\mathfrak{a}(Q_{N_p} \setminus Q_{N_{p-1}})\| \leqq p^{-3} \quad (p = 2, 3, \ldots).$$

We set

$$\lambda_k = p \quad \text{for} \quad k \in Q_{N_p} \setminus Q_{N_{p-1}} \quad (p = 1, 2, \ldots).$$

The first assertion in (16) is clearly satisfied. On the other hand, using the triangle inequality and (4),

$$\|\lambda\mathfrak{a}\| = \lim_{q\to\infty} \|\lambda\mathfrak{a}(Q_{N_q})\| \leqq \lim_{q\to\infty} \sum_{p=1}^{q} \|\lambda\mathfrak{a}(Q_{N_p} \setminus Q_{N_{p-1}})\| =$$

$$= \lim_{q\to\infty} \sum_{p=1}^{q} p\|\mathfrak{a}(Q_{N_p} \setminus Q_{N_{p-1}})\| \leqq \mathfrak{a}(Q_{N_1})\| + \sum_{p=1}^{\infty} p^{-2} < \infty.$$

This is the second assertion in (16).

If $\mathfrak{a} \notin \mathfrak{M}$, then by (4), (5) and the triangle inequality there exists a sequence $(0=)N_0 < N_1 < \ldots < N_p < \ldots$ of integers such that

$$\|\mathfrak{a}(Q_{N_p} \setminus Q_{N_{p-1}})\| \geqq p^2 \quad (p = 1, 2, \ldots).$$

Now we set

$$\mu_k = p^{-1} \quad \text{for} \quad k \in Q_{N_p} \setminus Q_{N_{p-1}} \quad (p = 1, 2, \ldots).$$

The fulfilment of the first assertion in (17) is obvious. Applying Theorem 2, we find that

$$\|\mu\mathfrak{a}\| \geqq \|\mu\mathfrak{a}(Q_{N_p} \setminus Q_{N_{p-1}})\| \geqq p \quad (p = 1, 2, \ldots),$$

which implies $\mu\mathfrak{a} \notin \mathfrak{M}$.

3. Two convergence notions for multiple series

Let us consider a d-multiple series

$$\sum_{k\in Z_+^d} u_k = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} u_{k_1,\ldots,k_d} \tag{18}$$

of real numbers, with the rectangular partial sums

$$s_n = \sum_{k\leqq n} u_k = \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} u_{k_1,\ldots,k_d} \quad (n \in Z_+^d).$$

More generally, given a rectangle R in Z_+^d with edges of finite length and parallel to the coordinate axis, i.e. $R = \{k \in Z_+^d:\ m \leqq k \leqq n\}$, set

$$s(R) = \sum_{k\in R} u_k = \sum_{m\leqq k\leqq n} u_k =$$

$$= \sum_{k_1=m_1}^{n_1} \cdots \sum_{k_d=m_d}^{n_d} u_{k_1,\ldots,k_d} \quad (m, n \in Z_+^d;\ m \leqq n).$$

It is clear that $s(R)=s_n$ in the special case $m=1$. On the other hand, it will be useful to notice that

$$s(R) = \sum_{\delta_1=0}^{1} \cdots \sum_{\delta_d=0}^{1} (-1)^{\delta_1+\dots+\delta_d} s_{\delta_1(m_1-1)+(1-\delta_1)n_1, \dots, \delta_d(m_d-1)+(1-\delta_d)n_d} \tag{19}$$

with the agreement $s_{k_1,\dots,k_d}=0$ if $k_j=0$ for at least one j.

We remind that series (18) is said to be convergent in *Pringsheim's sense* if there exists a finite number s with the following property: for every $\varepsilon>0$ there exists a number $N=N(\varepsilon)$ so that

$$|s_n-s| < \varepsilon \quad \text{whenever} \quad \min_{1\le j\le d} n_j \geqq N.$$

The number s is said to be the sum of (18). It is well-known that a necessary and sufficient condition that series (18) converge in Pringsheim's sense is that for every $\varepsilon>0$ there exist a number $M=M(\varepsilon)$ so that

$$|s_m-s_n| < \varepsilon \quad \text{whenever} \quad \min_{1\le j\le d} m_j \geqq M \quad \text{and} \quad \min_{1\le j\le d} n_j \geqq M$$

(the Cauchy convergence principle).

It is also known from the literature that series (18) is said to be *regularly convergent* if for every $\varepsilon>0$ there exists a number $N=N(\varepsilon)$ so that for every rectangle $R=\{k\in Z_+^d: m\leqq k\leqq n\}$

$$|s(R)| < \varepsilon \quad \text{whenever} \quad \max_{1\le j\le d} m_j > N \quad \text{and} \quad n \geqq m,$$

i.e. $m\in Z_+^d\setminus Q_N$ and $n\geqq m$.

It is an exercise to show that convergence in Pringsheim's sense follows from regular convergence, but the converse statement is not true.

The notion of regular convergence is due to Hardy [1]. Much later this kind of convergence was rediscovered by the first author and called in [3] convergence in a restricted sense. (As to a relatively complete history of the question, we refer to [4], where some of the results of the present paper were already stated.)

4. The main results

One of our main results is that the class $\mathfrak{M}$ introduced in Section 1 contains exactly those d-multiple sequences $\mathfrak{a}=\{a_k: k\in Z_+^d\}$ of coefficients for which the orthogonal series (1) regularly converges a.e. for every ONS φ on I.

Theorem 5. *If* $\mathfrak{a}\in\mathfrak{M}$, *then series* (1) *regularly converges a.e. for every d-multiple ONS* φ *on* I.

Proof. Let us fix an ONS φ on I and set

$$G_N(x) = \Big(\sup_{m,n\in Z_+^d\setminus Q_N: m\leqq n} \Big| \sum_{m\leqq k\leqq n} a_k\varphi_k(x)\Big|\Big)^2 \quad (N = 1, 2, \ldots).$$

It is plain that

$$G_N(x) \geqq G_{N+1}(x) \geqq 0 \quad (N = 1, 2, \ldots).$$

Since

$$\int_0^1 G_N(x)\,dx \leqq \|\mathfrak{a}(Z_+^d\setminus Q_N)\|^2,$$

Corollary 2 yields

$$\lim_{N\to\infty} \int_0^1 G_N(x)\,dx = 0.$$

Hence, via Fatou's lemma, we obtain that

$$\lim_{N\to\infty} G_N(x) = 0 \quad \text{a.e.}$$

and this is equivalent to the a.e. regular convergence of series (1).

Theorem 6. *If* $\mathfrak{a}\notin\mathfrak{M}$, *then there exists an ONS* $\Phi=\{\Phi_k(x)\colon k\in Z_+^d\}$ *of step functions on I such that series* (1) *for* $\varphi=\Phi$ *does not converge regularly a.e. on I; even we have*

$$\limsup \Big|\sum_{k\leqq n} a_k\Phi_k(x)\Big| = \infty \quad \text{a.e. as} \quad \max_{1\leqq j\leqq d} n_j \to \infty. \tag{20}$$

Proof. By (4) and (5) there exists a sequence $(0=)N_0<N_1<\ldots<N_p<\ldots$ of integers such that

$$\|\mathfrak{a}(Q_{N_p}\setminus Q_{N_{p-1}})\| \geqq p \quad (p = 1, 2, \ldots).$$

For each p we consider the sequence $\mathfrak{a}(Q_{N_p}\setminus Q_{N_{p-1}})$ and apply Lemma 2. As a result we obtain an ONS $\{\psi_k(p;x)\colon k\in Q_{N_p}\}$ of step functions and a simple set E_p for each $p=1,2,\ldots$ with the properties stated in Lemma 2.

By induction we will define an ONS $\Phi=\{\Phi_k(x)\colon k\in Z_+^d\}$ of step functions and a sequence $\{H_p\colon p=1,2,\ldots\}$ of stochastically independent, simple subsets of I having the following properties:

$$\max_{m,n\in Q_{N_p}\setminus Q_{N_{p-1}}: m\leqq n} \Big| \sum_{m\leqq k\leqq n} a_k\Phi_k(x)\Big| \geqq 2^{-d}p \quad \text{for} \quad x\in H_p \tag{21}$$

and

$$\operatorname{mes} H_p \geqq C_1 \quad (p = 1, 2, \ldots) \tag{22}$$

with the same constant as in Lemma 2.

For $p=1$ we set

$$H_1 = E_1 \quad \text{and} \quad \Phi_k(x) = \psi_k(1;x) \quad (k\in Q_{N_1}).$$

Then (21) and (22) are obviously satisfied $(Q_0=\emptyset)$.

Now let p_0 be a positive integer and assume that the step functions $\Phi_k(x)$ for $k\in Q_{N_{p_0}}$ and the simple sets $H_1, H_2, ..., H_{p_0}$ have been defined in such a way that these functions constitute an ONS on I, these sets are stochastically independent and relations (21) and (22) are satisfied for $p=1, 2, ..., p_0$. Then there exists a partition $\{J_r: r=1, 2, ..., \varrho\}$ of the interval I into disjoint subintervals such that each function $\Phi_k(x)$, $k\in Q_{N_{p_0}}$, assumes a constant value on each J_r, $r=1, 2, ..., \varrho$, and each set H_p, $p=1, 2, ..., p_0$, is the union of a certain number of J_r. Let us divide each J_r into two subintervals J_r' and J_r'' of equal length.

We shall use the following notations. Given a function $f(x)$ defined on I, a subset H and a subinterval $J=(a, b)$ of I, we define

$$f(J; x) = \begin{cases} f\left(\dfrac{x-a}{b-a}\right) & \text{for } x\in J, \\ 0 & \text{for } x\in I\setminus J; \end{cases}$$

and $H(J)$ to be the set, into which H is carried over by the linear transformation $y=(b-a)x+a$.

Now we define the functions $\Phi_k(x)$ for $k\in Q_{N_{p_0+1}}\setminus Q_{N_{p_0}}$ and the set H_{p_0+1} as follows:

$$\Phi_k(x) = \sum_{r=1}^{\varrho} [\psi_k(p_0+1; J_r'; x)-\psi_k(p_0+1; J_r''; x)]$$

and

$$H_{p_0+1} = \bigcup_{r=1}^{\varrho} [E_{p_0+1}(J_r')\cup E_{p_0+1}(J_r'')].$$

Obviously, these $\Phi_k(x)$, $k\in Q_{N_{p_0+1}}\setminus Q_{N_{p_0}}$, are step functions and H_{p_0+1} is a simple set. It is a routine to verify that the functions $\Phi_k(x)$, $k\in Q_{N_{p_0+1}}$, form an ONS on I, the sets H_p, $p=1, 2, ..., p_0+1$, are stochastically independent, and relations (21) and (22) are satisfied for $p=p_0+1$. (To deduce (21) from (10) one has to use a representation similar to (19).)

The above induction scheme shows that the ONS $\Phi=\{\Phi_k(x): k\in Z_+^d\}$ and the sequence $\{H_p: p\in Z_+^1\}$ of stochastically independent sets can be defined in such a way that conditions (21) and (22) hold true.

We set

$$H = \limsup_{p\to\infty} H_p.$$

By (22), the second Borel—Cantelli lemma implies that $\operatorname{mes} H=1$. If $x\in H$, then $x\in H_p$ and consequently (21) holds true for an infinite number of p. In other words, this means that

$$\limsup \Big|\sum_{m\le k\le n} a_k\Phi_k(x)\Big| = \infty \quad \text{a.e. as} \quad \max_{1\le j\le d} m_j \to \infty.$$

Hence it is clear that series (1) for $\varphi=\Phi$ does not converge regularly a.e. Taking into account the representation of $\sum_{m\leq k\leq n} a_k \Phi_k(x)$ corresponding to (19), assertion (20) also follows.

Theorems 5 and 6 immediately yield the following two corollaries.

Corollary 4. *A necessary and sufficient condition that a d-multiple sequence* $\mathfrak{a}$ *of numbers be such that series* (1) *regularly converge a.e. for every ONS* φ *on I is that* $\mathfrak{a}\in\mathfrak{M}$.

Corollary 5. *If a d-multiple sequence* $\mathfrak{a}$ *of numbers is such that series* (1) *regularly converges a.e. for every ONS* φ *on I, then for every ONS* φ *the rectangular partial sums* $s_n(x)$ *of series* (1) *are majorized by a square integrable function* $F(x)=F(x;\ \mathfrak{a},\varphi)$ *on I, the square integral of which depends only on* $\mathfrak{a}$, *but not on* φ.

Indeed, the condition of Corollary 5 is equivalent to the fact that $\mathfrak{a}\in\mathfrak{M}$. In this case, setting

$$F(x)=\sup_{m,n\in Z_+^d:\, m\leq n}\Big|\sum_{m\leq k\leq n} a_k\varphi_k(x)\Big|,$$

we have

$$\int_0^1 F^2(x)\,dx \leqq \|\mathfrak{a}\|^2<\infty,$$

as stated in Corollary 5.

Using a previous result of the second author, we are able to prove a stronger assertion than that is stated in Theorem 6. This makes possible to deduce our second main result; if the a.e. convergence of series (1) is considered for every ONS on I, then regular convergence and convergence in Pringsheim's sense are equivalent, up to a set of measure zero. This will be a corollary of the following

Theorem 7. *If* $\mathfrak{a}\notin\mathfrak{M}$, *then there exist an ONS* $\Phi=\{\Phi_k(x):\ k\in Z_+^d\}$ *of step functions on I such that*

$$\limsup\Big|\sum_{k\leq n} a_k\Phi_k(x)\Big|=\infty \quad \text{a.e. as} \quad \min_{1\leq j\leq d} n_j\to\infty. \tag{23}$$

Consequently, series (1) *for* $\varphi=\Phi$ *does not converge a.e. even in Pringsheim's sense.*

Proof. It will be done by induction with respect to d. If $d=1$, Theorem 7 is a result of the second author [7].

For the sake of simplicity, we present the induction step from $d=1$ to $d+1=2$. In this case we write (k,l) instead of (k_1,k_2). For given positive integers k_0 and l_0 let us put

$$T_{k_0}^{(1)}=\{(k_0,l):\ l=1,2,\ldots\}\quad\text{and}\quad T_{l_0}^{(2)}=\{(k,l_0):\ k=1,2,\ldots\}$$

and consider the norms $\|\mathfrak{a}(T_{k_0}^{(1)})\|$ and $\|\mathfrak{a}(T_{l_0}^{(2)})\|$, respectively. We distinguish two cases.

Case (i). For all positive integers k_0 and l_0 we have respectively

$$\|\mathfrak{a}(T_{k_0}^{(1)})\| < \infty \quad \text{and} \quad \|\mathfrak{a}(T_{l_0}^{(2)})\| < \infty.$$

Applying the above mentioned theorem of the second author, we obtain that for every positive integer k_0 the single series

$$\sum_{l=1}^{\infty} a_{k_0,l}\,\varphi_l(x)$$

(a so-called "column") converges a.e. on I for every ONS $\{\varphi_l(x)\colon l=1,2,\ldots\}$; and for every positive integer l_0 the single series

$$\sum_{k=1}^{\infty} a_{k,l_0}\,\varphi_k(x)$$

(a so-called "row") converges a.e. on I for every ONS $\{\varphi_k(x)\colon k=1,2,\ldots\}$. Consequently, for every double ONS $\varphi=\{\varphi_{kl}(x)\colon k,l=1,2,\ldots\}$ and for every positive integer N we have

(24)
$$\limsup \Big|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}\varphi_{kl}(x)\Big| < \infty \quad \text{a.e.} \quad \text{as} \quad \max(m,n)\to\infty \quad \text{and} \quad \min(m,n)\le N.$$

In virtue of Theorem 6, there exists a double ONS $\Phi=\{\Phi_{kl}(x)\colon k,l=1,2,\ldots\}$ such that relation (20) holds true. Taking into account observation (24) we can strengthen (20) as follows:

$$\limsup \Big|\sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}\Phi_{kl}(x)\Big| = \infty \quad \text{a.e. as} \quad \min(m,n)\to\infty.$$

This is statement (23) for $d=2$.

Case (ii). There exists at least one positive integer k_0 or l_0, for which

$$\|\mathfrak{a}(T_{k_0}^{(1)})\| = \infty \quad \text{or} \quad \|\mathfrak{a}(T_{l_0}^{(2)})\| = \infty.$$

For definiteness, let us assume the fulfilment of the first relation. Again applying the theorem of the second author [7], we can find an ONS $\{\Psi_l(x)\colon l=1,2,\ldots\}$ of step functions on I such that the single series

$$\sum_{l=1}^{\infty} a_{k_0,l}\,\Psi_l(x)$$

diverges a.e. on I in the sense that

$$\limsup_{N\to\infty} \left|\sum_{l=1}^{N} a_{k_0,l}\,\Psi_l(x)\right| = \infty \quad \text{a.e.}$$

From here it follows that there exist a sequence $(0=)N_0<N_1<\ldots<N_p<\ldots$ of integers and a sequence $\{E_p\colon p=1,2,\ldots\}$ of simple subsets of I such that

$$\max_{N_{p-1}<N\leqq N_p} \left|\sum_{l=N_{p-1}+1}^{N} a_{k_0,l}\,\Psi_l(x)\right| \geqq p \quad \text{for} \quad x\in E_p \tag{25}$$

and

$$\operatorname{mes} E_p \geqq 1-2^{-p-1} \quad (p=1,2,\ldots). \tag{26}$$

We may assume that $N_1\geqq k_0$.

We are going to construct a double ONS $\Phi=\{\Phi_{kl}(x)\colon k,l=1,2,\ldots\}$ of step functions and another sequence $\{H_p\colon p=1,2,\ldots\}$ of simple subsets of I in such a way that

$$\max_{N_{p-1}<N\leqq N_p} \left|\sum_{k=N_{p-1}+1}^{N}\;\sum_{l=N_{p-1}+1}^{N} a_{kl}\,\Phi_{kl}(x)\right| \geqq p \quad \text{for} \quad x\in H_p \tag{27}$$

and

$$\operatorname{mes} H_p \geqq 1-2^{-p} \quad (p=1,2,\ldots). \tag{28}$$

We use again an induction argument, this time with respect to p. If $p=1$, we set for $l=1,2,\ldots,N_1$

$$\Phi_{k_0,l}(x) = \begin{cases} \sqrt{2}\,\Psi_l(2x) & \text{for} \quad x\in(0,1/2), \\ 0 & \text{for} \quad x\in(1/2,1); \end{cases}$$

and define the other functions $\Phi_{kl}(x)$ for $(k,l)\in Q_{N_1}=\{(k,l)\colon k,l=1,2,\ldots,N_1\}$, $k\neq k_0$, in such a way that they be zero on $(0,1/2)$ and they form an ONS on $(1/2,1)$ consisting of step functions. Furthermore, set $H_1=E_1$. It is clear that (27) and (28) are satisfied for $p=1$.

Now let p_0 be a positive integer and suppose that the step functions $\Phi_{kl}(x)$ for $(k,l)\in Q_{N_{p_0}}$ and the simple sets H_p for $p=1,2,\ldots,p_0$ have been defined in such a way that these functions form an ONS on I, and relations (27) and (28) are satisfied for $p=1,2,\ldots,p_0$. Then there exists a partition $\{J_s\colon s=1,2,\ldots,\sigma\}$ of the interval I into disjoint subintervals such that each function $\Phi_{kl}(x)$, $(k,l)\in Q_{N_{p_0}}$, assumes a constant value on each J_s, $s=1,2,\ldots,\sigma$.

Let us divide each J_s into three subintervals J_s', J_s'' and J_s''' with the following lengths:

$$\operatorname{mes} J_s' = \operatorname{mes} J_s'' = 2^{-1}(1-2^{-p_0-2})\operatorname{mes} J_s \tag{29}$$

and

$$\text{mes}\, J_s''' = 2^{-p_0-2}\, \text{mes}\, J_s \quad (s = 1, 2, \ldots, \sigma).$$

Now we define the functions $\Phi_{k_0, l}(x)$ for $l=N_{p_0}+1, N_{p_0}+2, \ldots, N_{p_0+1}$ and the set H_{p_0+1} as follows:

$$\Phi_{k_0, l}(x) = (1-2^{-p_0-2})^{-1/2} \sum_{s=1}^{\sigma} [\Psi_l(J_s'; x) - \Psi_l(J_s''; x)]$$

and

$$H_{p_0+1} = \bigcup_{s=1}^{\sigma} [E_{p_0+1}(J_s') \cup E_{p_0+1}(J_s'')].$$

Relation (27) follows from (25), while (28) follows from (26) and (29). It is clear that each function $\Phi_{k_0, l}(x)$, $N_{p_0} < l \leqq N_{p_0+1}$, vanishes on $\bigcup_{s=1}^{\sigma} J_s'''$ and H_{p_0+1} is also disjoint from $\bigcup_{s=1}^{\sigma} J_s'''$. Finally, we define the other functions $\Phi_{kl}(x)$ for $Q_{N_{p_0+1}} \setminus Q_{N_{p_0}}$, $k \neq k_0$, in such a way that they vanish on $\bigcup_{s=1}^{\sigma} (J_s' \cup J_s'')$ and they form an ONS on $\bigcup_{s=1}^{\sigma} J_s'''$, consisting of step functions with zero mean on each interval J_s''', $s=1, 2, \ldots, \sigma$.

By construction, the step functions $\Phi_{kl}(x)$, $(k, l) \in Q_{N_{p_0+1}}$, form and ONS on I, the sets $H_1, H_2, \ldots, H_{p_0+1}$ are simple, and relations (27) and (28) are satisfied for $p=1, 2, \ldots, p_0+1$. This completes the proof of the induction step.

We set

$$H = \limsup_{p\to\infty} H_p.$$

By (28), the first Borel—Cantelli lemma implies that

$$\text{mes}\left[\liminf_{p\to\infty} (I \setminus H_p)\right] = 0, \quad \text{or equivalently,} \quad \text{mes}\, H = 1.$$

If $x \in H$, then (27) holds true for an infinite number of p, consequently,

$$\limsup_{p\to\infty} \max_{N_{p-1} < N \leqq N_p} \Big| \sum_{k=N_{p-1}+1}^{N} \sum_{l=N_{p-1}+1}^{N} a_{kl} \Phi_{kl}(x) \Big| = \infty \quad \text{a.e.}$$

Taking into account that

$$\sum_{k=N_{p-1}+1}^{N} \sum_{l=N_{p-1}+1}^{N} a_{kl} \Phi_{kl}(x) =$$

$$= \left\{ \sum_{k=1}^{N} \sum_{l=1}^{N} - \sum_{k=1}^{N} \sum_{l=1}^{N_{p-1}} - \sum_{k=1}^{N_{p-1}} \sum_{l=1}^{N} + \sum_{k=1}^{N_{p-1}} \sum_{l=1}^{N_{p-1}} \right\} a_{kl} \Phi_{kl}(x),$$

assertion (23) for $d=2$ immediately follows.

The proof of Theorem 7 is complete.

We emphasize the significance of the following two consequences of Theorems 5, 6 and 7.

Corollary 6. *If a d-multiple sequence* $\mathfrak{a}$ *of numbers is such that for every ONS* φ *series* (1) *converges in Pringsheim's sense on a set of positive measure, perhaps depending on* φ, *then series* (1) *for every ONS* φ *regularly converges a.e.*

Corollary 7. *If a d-multiple sequence* $\mathfrak{a}$ *of numbers is such that for an ONS* φ *series* (1) *does not converge regularly on a set of positive measure, then there exists another ONS* Φ *such that series* (1) *for* $\varphi=\Phi$ *does not converge in Pringsheim's sense a.e.*

We note that for an individual ONS the notions of a.e. regular convergence and a.e. convergence in Pringsheim's sense can essentially differ from each other. In [4, pp. 214—215] a double sequence $\{a_{kl}: k, l=1, 2, \ldots\}$ of real numbers and on $I^2=[0; 1]^2$ a double ONS $\{\varphi_{kl}(x): k, l=1, 2, \ldots\}$ are constructed in such a way that

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^2 < \infty,$$

the double orthogonal series

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}\varphi_{kl}(x)$$

converges in Pringsheim's sense a.e on I^2, but does not converge regularly on a set of measure at least 1/2. It is not hard to modify this example so as the resulting orthogonal series converges in Pringsheim's sense a.e. and does not converge regularly a.e.

5. Estimation of the norm $\|\mathfrak{a}\|$

Using the d-multiple generalization of the famous Rademacher—Menšov inequality, it is not hard to give an upper bound for $\|\mathfrak{a}\|$ (see [3, Corollary 2]).

Theorem 8. *For every d-multiple sequence* $\mathfrak{a}$ *we have*

$$\|\mathfrak{a}\| \leqq C_2\Big\{\sum_{k\in Z_+^d} a_k^2 \prod_{j=1}^{d} (\log 2k_j)^2\Big\}^{1/2}, \tag{30}$$

where $C_2=C_2(d)$.

Here and in the sequel the logarithms are to the base 2.

A nontrivial lower bound for $\|\mathfrak{a}\|$ is not known in general. In the special case when $\mathfrak{a}$ is such that $\{|a_k|: k\in Z_+^d\}$ is nonincreasing in the sense that

(31) $$|a_k| \geqq |a_n| \quad \text{whenever} \quad k, n\in Z_+^d \ \text{and} \quad k \leqq n,$$

an opposite inequality to (30) holds also true.

Theorem 9. *If a d-multiple sequence* $\mathfrak{a}$ *is such that* (31) *is satisfied, then we have*

(32) $$\|\mathfrak{a}\| \geqq C_3\{\sum_{k\in Z_+^d} a_k^2 \prod_{j=1}^d (\log 2k_j)^2\}^{1/2},$$

where $C_3=C_3(d)$.

The proof of Theorem 9 is based on the following basic result of MENŠOV [2].

Lemma 3. *For every positive integer N there exist an ONS* $\{\psi_k^{(N)}(x): k=1, 2, \ldots, N\}$ *of step functions on the interval I and a simple subset* $E^{(N)}$ *of I such that*

(33) $$\operatorname{mes} E^{(N)} \geqq C_4$$

and for every $x\in E^{(N)}$ *there exists an integer* $n=n(x)$ *between* 1 *and N such that* $\psi_k^{(N)}(x)\geqq 0$ *for* $k=1, 2, \ldots, n$ *and*

(34) $$\sum_{k=1}^n \psi_k^{(N)}(x) \geqq C_5 \sqrt{N} \log 2N.$$

A trivial consequence of (33) and (34) is that

(35) $$\int_0^1 (\max_{1\leqq n\leqq N} |\sum_{k=1}^n \psi_k^{(N)}(x)|)^2\, dx \geqq C_6 N (\log 2N)^2.$$

This inequality will be enough for our purpose.

Proof of Theorem 9. For the sake of simplicity in notations, we present the proof again for the case $d=2$.

Denote by T a measure-preserving transformation of the square I^2 onto the interval I: $T(y_1, y_2)=x$, where $(y_1, y_2)\in I^2$ and $x\in I$. Given two positive integers N_1 and N_2, we define for $k=1, 2, \ldots, N_1$; $l=1, 2, \ldots, N_2$

$$\varphi_{kl}^{(N_1, N_2)}(x) = \psi_k^{(N_1)}(y_1)\, \psi_l^{(N_2)}(y_2).$$

Then (35) yields

(36) $$\int_0^1 (\max_{1\leqq m\leqq N_1} \max_{1\leqq n\leqq N_2} |\sum_{k=1}^m \sum_{l=1}^n \varphi_{kl}^{(N_1, N_2)}(x)|)^2\, dx \geqq \geqq C_6^2 N_1 N_2 (\log 2N_1)^2 (\log 2N_2)^2.$$

After these preliminaries, let us consider the partition of Z_+^2 into the following "dyadic" rectangles:

$$Q_{mn} = \{(k, l)\in Z_+^2 \colon 2^{m-1} \leqq k < 2^m \quad \text{and} \quad 2^{n-1} \leqq l < 2^n\},$$

where (m, n) runs over Z_+^2. According to this partition we modify the original sequence $\mathfrak{a}$ into another $\mathfrak{a}^*$ so as it should be constant on each Q_{mn}: $\mathfrak{a}^* = \{a_{kl}^* \colon (k, l)\in Z_+^2\}$, where

$$a_{kl}^* = a_{2^m, 2^n} \quad \text{for} \quad (k, l)\in Q_{mn}, \quad (m, n)\in Z_+^2.$$

Due to Theorem 2, inequality (36), and the monotony of $|a_{kl}|$, for every $(m, n)\in Z_+^2$

$$\|\mathfrak{a}(Q_{mn})\| \geqq \|\mathfrak{a}^*(Q_{mn})\| \geqq C_6^2 2^{m-1} 2^{n-1} m^2 n^2 a_{2^m, 2^n}^2 \geqq$$

$$\geqq 3^{-4} C_6^2 \sum_{k=2^m}^{2^{m+1}-1} \sum_{l=2^n}^{2^{n+1}-1} a_{kl}^2 (\log 2k)^2 (\log 2l)^2.$$

Applying Theorem 3, we obtain that

$$\|\mathfrak{a}\|^2 \geqq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \|\mathfrak{a}(Q_{mn})\|^2 \geqq 3^{-4} C_6^2 \sum_{k=2}^{\infty} \sum_{l=2}^{\infty} a_{kl}^2 (\log 2k)^2 (\log 2l)^2. \tag{37}$$

Now we examine the cases $m=1$ or (and) $n=1$ once more. A more accurate calculation gives

$$\|\mathfrak{a}(Q_{1n})\| \geqq \|\mathfrak{a}^*(Q_{1n})\| \geqq C_6 2^{n-1} n^2 a_{1, 2^n}^2 \geqq 3^{-2} C_6 \sum_{l=2^n}^{2^{n+1}-1} a_{1l}^2 (\log 2l)^2,$$

whence we get that

$$\|\mathfrak{a}\|^2 \geqq \sum_{n=1}^{\infty} \|\mathfrak{a}(Q_{1n})\|^2 \geqq 3^{-2} C_6 \sum_{l=2}^{\infty} a_{1l}^2 (\log 2l)^2. \tag{38}$$

Analogously,

$$\|\mathfrak{a}\|^2 \geqq 3^{-2} C_6 \sum_{k=2}^{\infty} a_{k1}^2 (\log 2k)^2. \tag{39}$$

Finally, it is obvious that

$$\|\mathfrak{a}\|^2 \geqq a_{11}^2. \tag{40}$$

Now the statement of Theorem 9 immediately follows from relations (37)—(40).

Remark 4. If one treats each "finite" sequence $\mathfrak{a}(Q_N)$, $N=1, 2, \ldots$, separately instead of the whole sequence $\mathfrak{a}$ and makes use of the fact that all $\psi_k^{(N)}(x)$ are step functions, one can prove Theorem 9 without taking a measure-preserving transformation T of the unit sequare I^2 onto I.

References

[1] G. H. HARDY, On the convergence of certain multiple series, *Proc. Cambridge Philosoph. Soc.*, **19** (1916—1919), 86—95.
[2] D. E. MENCHOFF, Sur les séries de fonctions orthogonales, *Fund. Math.*, **4** (1923), 82—105.
[3] F. MÓRICZ, On the convergence in a restricted sense of multiple series, *Analysis Math.*, **5** (1979), 135—147.
[4] F. MÓRICZ, The regular convergence of multiple series, in: *Functional Analysis and Approximation (Proc. Conf. Oberwolfach, 1980)*, Birkhäuser (Basel, 1981), 203—218.
[5] F. MÓRICZ and K. TANDORI, On the divergence of multiple orthogonal series, *Acta Sci. Math.*, **42** (1980), 133—142.
[6] K. TANDORI, Über die Konvergenz der Orhogonalreihen, *Acta Sci. Math.*, **24** (1963), 135—151.
[7] K. TANDORI, Über die Konvergenz der Orthogonalreihen. II, *Acta Sci. Math.*, **25** (1964), 219—232.

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY