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Embedding theorems and strong approximation

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1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. We denote by $s_n = s_n(x) = s_n(f;\, x)$ the n-th partial sum of (1), the usual supremum norm by $\|\cdot\|$ and by $E_n = E_n(f)$ the best approximation of f by trigonometric polynomials of order at most n. Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$. Such a function is called a modulus of continuity.

In order to quote the result of [3], which has initiated our present investigation, we define two classes of functions:

$$H^{\omega} := \{f:\ \omega(f;\, \delta) = O(\omega(\delta))\}$$

and

$$S_p(\lambda) := \{f:\ \|\sum_{n=0}^{\infty} \lambda_n |s_n - f|^p\| < \infty\},$$

where $\lambda = \{\lambda_n\}$ is a monotonic sequence of positive numbers and $0<p<\infty$. V. G. Krotov and L. Leindler [3] proved the following result.

Theorem A. *If $\{\lambda_n\}$ is a monotonic sequence, ω is a modulus of continuity and $0<p<\infty$, then*

$$\sum_{k=1}^{n} (k\lambda_k)^{-1/p} = O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2}$$

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implies

(3) $$S_p(\lambda) \subset H^\omega.$$

Conversely, if there exists a number Q *such that* $0 \leqq Q < 1$ *and*

(4) $$n^Q \lambda_n \uparrow,$$

then (3) *implies* (2).

It is well known that the classical de la Vallée Poussin means

$$\tau_n = \tau_n(f;\, x) := \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x), \quad n = 1, 2, \ldots$$

usually approximate the function f, in the sup norm, better than the partial sums do. Hence, if in analogy to $S_p(\lambda)$ we consider the class of functions

$$V_p(\lambda) := \{f\colon \|\sum_{n=0}^{\infty} \lambda_n |\tau_n - f|^p\| < \infty\},$$

we may expect that under reasonable conditions the following embedding relations will hold

(5) $$S_p(\lambda) \subset V_p(\lambda) \subset H^\omega.$$

In the present paper we establish that condition (2) does imply the inclusion $V_p(\lambda) \subset H^\omega$ for all positive p. We further show that the embedding relation $S_p(\lambda) \subset V_p(\lambda)$ also holds if $p \geqq 1$ and the sequence $\{\lambda_n\}$ satisfies the mild restriction

(6) $$\frac{\lambda_n}{\lambda_{2n}} \leqq K, \quad n = 1, 2, \ldots,$$

with a fixed positive K ($K, K_1, K_2, \ldots$ will denote positive constants, not necessarily the same at each occurrence).

We were unable to decide whether $S_p(\lambda) \subset V_p(\lambda)$ holds when $0 < p < 1$; it is left as an open problem.

2. We shall establish the following results.

Theorem 1. *If* $p \geqq 1$ *and* $\{\lambda_n\}$ *is a monotonic (nondecreasing or nonincreasing) sequence of positive numbers satisfying* (6), *then*

(7) $$S_p(\lambda) \subset V_p(\lambda)$$

holds.

Theorem 2. *Let* $\{\lambda_n\}$ *be a monotonic sequence of positive numbers, furthermore let* ω *be a modulus of continuity and* $0 < p < \infty$. *Then condition* (2) *implies*

(8) $$V_p(\lambda) \subset H^\omega.$$

If $p \geqq 1$ *and there exists a number* Q *such that* $0 \leqq Q < 1$ *and* (4) *holds, then, conversely,* (8) *implies* (2).

3. To prove our theorems we require the following lemmas.

Lemma 1 ([1, p. 534]). *For any continuous function* f *we have the following inequality*

$$\omega\left(f; \frac{1}{n}\right) \leqq K n^{-1} \sum_{k=1}^{n} E_k(f). \tag{9}$$

Lemma 2. *Let* $a = \{a_n\}_0^\infty$ *be a nonincreasing sequence of positive numbers,* $q > 0$ *and* $\gamma > 0$. *Then there exists a positive constant* $C = C(a, \gamma, q)$ *such that for every* m

$$\sum_{n=0}^{m} q^n a_n \leqq C \cdot \sum_{n=0}^{m} q^n a_n \left(\frac{a_{n+1}}{a_n}\right)^\gamma. \tag{10}$$

Proof. We let $\beta = \min(a_1/a_2, 1/2q)$. We define the (possibly finite) sequence of integers $N_0 < N_1 < \dots$ as follows. Let $N_0 = 0$. For $i \geqq 1$ let N_i be the smallest integer such that $N_i > N_{i-1}$ and $a_{N_i+1} \geqq \beta a_{N_i}$; if no such integer exists we set $N_i = \infty$. Now, if $N_i < n < N_{i+1}$, then $a_{n+1} < \beta a_n$ and so $a_{N_i+r} \beta^{r-1}$ for $r = 1, 2, \dots, N_{i+1} - N_i$. Therefore, we have for $i = 0, 1, \dots$

(11)

$$\sum_{n=N_i+1}^{N_{i+1}} q^n a_n \leqq q^{N_i+1} a_{N_i+1} \cdot (1 + q\beta + q^2\beta^2 + \dots) \leqq 2q^{N_i+1} a_{N_i+1} \leqq 2q^{N_i+1} a_{N_i} \leqq$$

$$\leqq 2q\beta^{-\gamma} q^{N_i} a_{N_i} \left(\frac{a_{N_i+1}}{a_{N_i}}\right)^\gamma,$$

on using, in the last inequality, the definition of the sequence $\{N_i\}$. Now, for any given integer m, let j be the largest integer so that $N_j < m$. We then have, on using (11), and the fact that $\beta \leqq a_1/a_0$,

$$\sum_{n=0}^{m} q^n a_n \leqq \beta^{-\gamma} a_0 \left(\frac{a_1}{a_0}\right)^\gamma + 2q\beta^{-\gamma} \sum_{i=0}^{j} q^{N_i} a_{N_i} \left(\frac{a_{N_i+1}}{a_{N_i}}\right)^\gamma \leqq C \cdot \sum_{n=0}^{m} q^n a_n \left(\frac{a_{n+1}}{a_n}\right)^\gamma,$$

with $C = \beta^{-\gamma}(1 + 2q)$.

This completes the proof of Lemma 2.

4. Proof of Theorem 1. For $p \geqq 1$ we have, by the "power sum inequality",

$$|\tau_n - f|^p \leqq \frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f|^p.$$

Hence

$$(12)\qquad \begin{aligned}\sum_{n=1}^{\infty}\lambda_n|\tau_n-f|^p &\leqq \sum_{n=1}^{\infty}(\lambda_n/n)\sum_{k=n+1}^{2n}|s_k-f|^p \leqq\\ &\leqq \sum_{k=2}^{\infty}|s_k-f|^p\sum_{n=k/2}^{k-1}(\lambda_n/n) \leqq K\sum_{k=2}^{\infty}\lambda_k|s_k-f|^p,\end{aligned}$$

where the last inequality follows from (6). Inequality (12) clearly implies (7).

Proof of Theorem 2. First we consider the case $p\geqq 1$. Suppose $f\in V_p(\lambda)$. Then we have for $n=1,2,\ldots$

$$(13)\qquad E_{4n}(f)\leqq\left\|\frac{1}{n}\sum_{k=n+1}^{2n}|\tau_k-f|\right\|\leqq\left\|\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|\tau_k-f|^p\right\}^{1/p}\right\|\leqq K_1(n\lambda_n^*)^{-1/p},$$

where $\lambda_n^*=\min(\lambda_{n+1},\lambda_{2n})$ and the last inequality follows from the assumption $f\in V_p(\lambda)$. Now, from (13), both if $\{\lambda_n\}$ is increasing or decreasing we can deduce that

$$(14)\qquad \sum_{\nu=1}^{m}4^{\nu}E_{4^{\nu}}(f)\leqq K_2\sum_{\nu=0}^{m}4^{\nu}(4^{\nu}\lambda_{4^{\nu}})^{-1/p},$$

with a suitable $K_2>0$.

Hence, by Lemma 1 and (2), for $m=1,2,\ldots$

$$(15)\qquad \omega(f;\,4^{-m})\leqq K_3\omega(4^{-m}),$$

which proves that $f\in H^{\omega}$.

We turn now to the case $0<p<1$. We have for $n=1,2,\ldots$

$$(16)\qquad nE_{4n}(f)\leqq\left\|\sum_{k=n+1}^{2n}|\tau_k-f|\right\|=\left\|\sum_{k=n+1}^{2n}|\tau_k-f|^p\cdot|\tau_k-f|^{1-p}\right\|.$$

It is known [see e.g. [2], p. 58] that $\|\tau_k-f\|\leqq KE_k(f)$ for all k; hence, in particular, for $n+1\leqq k\leqq 2n$, $\|\tau_k-f\|\leqq KE_n(f)$. Therefore, from (16) we obtain that

$$nE_{4n}(f)\leqq K(E_n(f))^{1-p}\left\|\sum_{k=n+1}^{2n}|\tau_k-f|^p\right\|$$

which, since $f\in V_p(\lambda)$, implies that $E_{4n}(f)\leqq K_1(E_n(f))^{1-p}(n\lambda_n^*)^{-1}$, with $\lambda_n^*=\min(\lambda_{n+1},\lambda_{2n})$. If we rewrite the last inequality as

$$E_n(f)\cdot\left(\frac{E_{4n}(f)}{E_n(f)}\right)^{1/p}\leqq K_2(n\lambda_n^*)^{-1/p},$$

and use it for $n=4^{\nu}$, $\nu=0,1,\ldots,m$, we see that

$$\sum_{\nu=0}^{m}4^{\nu}E_{4^{\nu}}(f)\left(\frac{E_{4^{\nu+1}}(f)}{E_{4^{\nu}}(f)}\right)^{1/p}\leqq K_3\sum_{\nu=0}^{m}4^{\nu}(4^{\nu}\lambda_{4^{\nu}})^{-1/p}$$

holds. Applying Lemma 2 now with $a_n = E_{4^n}$, $q=4$ and $\gamma=1/p$, we get that (14) is satisfied in this case as well. Hence, as before, f satisfies (15) and so $f\in H^\omega$.

This completes the proof of (8) for all positive p.

In order to prove that, under the stated assumptions, (8) implies (2), it is sufficient to note that, because of (7), relation (3) of Theorem A is satisfied; therefore Theorem A provides the proof of the required assertion.

References

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