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On the unicity of best Chebyshev approximation of differentiable functions

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Let X be a normed linear space, U_n an n-dimensional subspace of X. The problem of best approximation consists in determining for each $x\in X$ its best approximations in U_n, i.e. such elements $p\in U_n$ for which $\|x-p\|=\text{dist}\,(x, U_n)=\inf\{\|x-g\|: g\in U_n\}$. Let us denote by $G(x)=\{p\in U_n: \|x-p\|=\text{dist}\,(x, U_n)\}$ the set of best approximations of x. Evidently, for every $x\in X$ the set $G(x)$ is nonempty and convex. Recall that the convex set $G(x)$ is said to have dimension k if there exist $k+1$ elements $p_0, \dots, p_k\in G(x)$ such that p_i-p_0, $1\leqq i\leqq k$, are linearly independent and $G(x)$ does not contain $k+2$ elements satisfying this property ($k\geqq 0$). The subspace U_n is called k-Chebyshev if the dimension of $G(x)$ is at most k for any $x\in X$ $(0\leqq k\leqq n-1)$. In particular when $k=0$, i.e. each $x\in X$ possesses a unique best approximation in U_n, we say that U_n is a Chebyshev subspace of X.

Let us consider the classical case of Chebyshev approximation when $X=C(Q)$ is the space of complex valued continuous functions on the compact Hausdorff space Q endowed with the supremum norm $\|f\|_C=\sup\{|f(x)|: x\in Q\}$. (The subspace of real valued functions in $C(Q)$ will be denoted by $C_0(Q)$.) The characterization of Chebyshev subspaces of $C(Q)$ is given by the celebrated Haar—Kolmogorov theorem: the n-dimensional subspace U_n is a Chebyshev subspace of $C(Q)$ if and only if each $p\in U_n\setminus\{0\}$ has at most $n-1$ distinct zeros at Q. (This theorem was given at first by HAAR [3] in the real case $X=C_0(Q)$ and then by KOLMOGOROV [4] in the complex case $X=C(Q)$.) Later RUBINSTEIN [8] gave the characterization of k-Chebyshev subspace of $C_0(Q)$ and ROMANOVA [7] generalized it for $C(Q)$. Their result reads as follows: U_n is a k-Chebyshev subspace of $C(Q)$ if and only if any $k+1$ linearly independent elements of U_n have at most $n-k-1$ common zeros at Q $(0\leqq k\leqq n-1)$. (For $k=0$ this result immediately implies the Haar—Kolmogorov theorem.)

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In the present paper we shall investigate the unicity of best Chebyshev approximation in the spaces of differentiable functions. This problem was posed by S. B. Stechkin and considered in the real case by GARKAVI [2].

Let $C^r[a, b]$ ($C_0^r[a, b]$) denote the space of r-times continuously differentiable complex (resp. real) functions on $[a, b]$ endowed with the supremum norm, $1 \leqq r \leqq \infty$. (In what follows $c \in [a, b]$ will be called a special zero of $f \in C_0^1[a, b]$ if either $f'(c) = f(c) = 0$ or $f(c) = 0$ and c coincides with one of the endpoints of the interval $[a, b]$.) GARKAVI [2] gave the following characterization of k-Chebyshev subspaces of $C_0^r[a, b]$: $U_n \subset C_0^r[a, b]$ $(r \geqq 1)$ is a k-Chebyshev subspace of $C_0^r[a, b]$ if and only if for any s linearly independent elements $p_1, \ldots, p_s \in U_n$ $(k+1 \leqq s \leqq n)$ among their common zeros there are not more than $n-s$ special zeros common to any $k+1$ of the elements $p_1, \ldots, p_s$. In particular in order that U_n be a Chebyshev subspace of $C_0^r[a, b]$ it is necessary and sufficient that for any s linearly independent elements $p_1, \ldots, p_s$ of U_n $(1 \leqq s \leqq n)$ among their common zeros there are at most $n-s$ special zeros of any of $p_1, \ldots, p_s$. (Remark, that the characterization of k-Chebyshev subspaces of $C_0^r[a, b]$ turns out to be independent of $1 \leqq r \leqq \infty$.)

In this paper we shall present another approach to the study of k-Chebyshev and Chebyshev subspaces of differentiable functions. This approach is based on the so-called "extremal sets" which are essential in the study of unicity of best Chebyshev approximation of *complex valued* differentiable functions. Our method gives a possibility to generalize Garkavi's result to the complex case. In the last sections of the paper we shall give several applications for the study of unicity of best Chebyshev approximation of differentiable functions by real and complex lacunary polynomials.

1. First of all let us formulate a lemma characterizing best approximants. Recall, that the sign of a complex number $c \in \mathbf{C}$ is given by $\bar{c}/|c|$ if $c \neq 0$ and 0 if $c = 0$.

Lemma 1 ([9], p. 178). *Let U_n be an n-dimensional subspace of $C(Q)$ $(C_0(Q))$. Then $p \in U_n$ is a best approximation of $f \in C(Q)$ $(C_0(Q))$ if and only if there exist m points $x_1, \ldots, x_m \in Q$, where $1 \leqq m \leqq n+1$ in the real case and $1 \leqq m \leqq 2n+1$ in the complex case, and m numbers $a_1, \ldots, a_m \neq 0$ such that*

$$\sum_{j=1}^{m} a_j g(x_j) = 0 \tag{1}$$

for each $g \in U_n$ and

$$f(x_j) - p(x_j) = \operatorname{sign} a_j \|f-p\|_C \quad (1 \leqq j \leqq m).$$

This lemma suggests the following definition.

Definition. The set of m distinct points $x_1, \ldots, x_m \in Q$, where $1 \leqq m \leqq n+1$ in the real case and $1 \leqq m \leqq 2n+1$ in the complex case, is called an extremal set of

$U_n \subset C(Q)$ if there exist nonzero complex numbers $a_1, \ldots, a_m \neq 0$ (real if $U_n \subset C_0(Q)$) such that (1) holds for any $g \in U_n$.

If $\{x_i\}_{i=1}^m$ is an extremal set of U_n then the corresponding numbers $\{a_i\}_{i=1}^m$ are called the coefficients of this extremal set. Evidently the coefficients of an extremal set are defined in general nonuniquely (even with a normalization). Note that extremal sets are closely related to the set Q, on which the functions of U_n are considered. (The idea of the above definition comes essentially from REMEZ [6] who was the first to give a proposition like Lemma 1.)

Using the notion of extremal sets we can formulate the Rubinstein—Romanova (and in particular the Haar—Kolmogorov) theorem in the following way: U_n is a k-Chebyshev subspace of $C(Q)$ if and only if the points of an extremal set of U_n cannot be common zeros of $k+1$ linearly independent elements of U_n $(0 \leqq k \leqq n-1)$. In particular U_n is Chebyshev if and only if no $p \in U_n \setminus \{0\}$ can vanish on an extremal set of U_n. The proof is left to the reader. Similar characterizations of Chebyshev subspaces of $C(Q)$ were also given by CHENEY—WULBERT [1] and PHELPS [5].

The next theorem characterizing the k-Chebyshev (in particular Chebyshev) subspaces of $C^r[a, b]$ is our principal result. This characterization is essentially based on extremal sets since it also involves the coefficients of extremal sets.

Theorem 1. *Let U_n be a subspace of $C^r[a, b]$, $1 \leqq r \leqq \infty$, $0 \leqq k \leqq n-1$. Then U_n is a k-Chebyshev subspace of $C^r[a, b]$ if and only if there does not exist an extremal set $\{x_i\}_{i=1}^m \subset [a, b]$ of U_n with coefficients $\{a_i\}_{i=1}^m$ and $k+1$ linearly independent elements $p_1, \ldots, p_{k+1} \in U_n$ such that $p_j(x_i)=0$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq k+1)$ and $\operatorname{Re} a_i p_j'(x_i) = 0$ for each $1 \leqq j \leqq k+1$ and $x_i \in (a, b)$.*

In particular U_n is a Chebyshev subspace of $C^r[a, b]$ if and only if there does not exist an extremal set $\{x_i\}_{i=1}^m$ of U_n with coefficients $\{a_i\}_{i=1}^m$ and $p \in U_n \setminus \{0\}$ such that $p(x_i)=0$ $(1 \leqq i \leqq m)$ and $\operatorname{Re} a_i p'(x_i)=0$ for each $x_i \in (a, b)$.

In the real case the coefficients of the extremal set do not appear in the characterization theorem and therefore its formulation is much simpler.

Corollary 1. *In order that U_n be a k-Chebyshev subspace of $C_0^r[a, b]$ it is necessary and sufficient that the points of an extremal set of U_n cannot be common special zeros of $k+1$ linearly independent elements of U_n.*

In particular U_n is a Chebyshev subspace of $C_0^r[a, b]$ if and only if the points of an extremal set of U_n cannot be special zeros of a nontrivial element of U_n.

The above corollary is equivalent to Garkavi's result. It also follows from a result of BROSOWSKI—STOER [11] where an extension of Garkavi's result for *real* rational families was given.

Proof of Theorem 1. Sufficiency. Assume that U_n is not a k-Chebyshev subspace of $C^r[a, b]$ $(1 \leqq r \leqq \infty)$. Then there exists $f \in C^r[a, b]$ with best approximants

$0, p_1, \ldots, p_{k+1} \in U_n$, where $p_1, \ldots, p_{k+1}$ are linearly independent. Since 0 is a best approximation of f it follows from Lemma 1 that we can find an extremal set $\{x_i\}_{i=1}^m$ of U_n with coefficients $\{a_i\}_{i=1}^m$ such that

$$f(x_j) = \operatorname{sign} a_j \|f\|_C, \quad 1 \leqq j \leqq m. \tag{2}$$

Moreover, $\|f-p_s\|_C = \|f\|_C$ for each $1 \leqq s \leqq k+1$. Hence and by (2) we have

$$\begin{gathered} \|f\|_C^2 \geqq |f(x_j)-p_s(x_j)|^2 = \left|\|f\|_C - \operatorname{sign} \bar{a}_j p_s(x_j)\right|^2 = \\ = \left(\|f\|_C - \frac{1}{|a_j|} \operatorname{Re} a_j p_s(x_j)\right)^2 + \left(\frac{1}{|a_j|} \operatorname{Im} a_j p_s(x_j)\right)^2 \\ (1 \leqq j \leqq m, \quad 1 \leqq s \leqq k+1). \end{gathered} \tag{3}$$

We can easily derive from (3) that $\operatorname{Re} a_j p_s(x_j) \geqq 0$ for each $1 \leqq j \leqq m$ and $1 \leqq s \leqq \leqq k+1$. On the other hand by the definition of extremal sets $\sum_{j=1}^m a_j p_s(x_j) = 0$ for every $1 \leqq s \leqq k+1$. Hence $\operatorname{Re} a_j p_s(x_j) = 0$ $(1 \leqq j \leqq m,\ 1 \leqq s \leqq k+1)$. Moreover, this and (3) imply that $\operatorname{Im} a_j p_s(x_j) = 0$, too. Since all coefficients $a_j \neq 0$ we finally obtain that

$$p_s(x_j) = 0 \quad (1 \leqq j \leqq m,\ 1 \leqq s \leqq k+1). \tag{4}$$

Now we shall use the differentiability of the functions involved. Consider an arbitrary $x_j \in (a, b)$ and set $\tilde{f}(x) = (1/|a_j|) \operatorname{Re} a_j f(x)$, $\tilde{p}_s(x) = (1/|a_j|) \operatorname{Re} a_j p_s(x)$, $1 \leqq s \leqq \leqq k+1$. Obviously, $\tilde{f}, \tilde{p}_1, \ldots, \tilde{p}_{k+1} \in C_0^r[a, b]$; $\tilde{f}(x_j) = \|\tilde{f}\|_C = \|f\|_C$, $\tilde{p}_s(x_j) = 0$ $(1 \leqq s \leqq \leqq k+1)$ and $\|\tilde{f} - \tilde{p}_s\|_C = \|\tilde{f}\|_C$ $(1 \leqq s \leqq k+1)$. Since $x_j \in (a, b)$ is an extremum point of the real function $\tilde{f}$ it follows that $\tilde{f}'(x_j) = 0$. Therefore for any $h \in \mathbf{R}$ such that $|h| < \min\{x_j - a, b - x_j\}$ we have

$$\begin{gathered} \tilde{p}_s(x_j+h) \geqq \tilde{f}(x_j+h) - \|\tilde{f}\|_C = \tilde{f}(x_j+h) - \tilde{f}(x_j) \geqq \\ \geqq -|h|\omega(\tilde{f}', |h|) \quad (1 \leqq s \leqq k+1). \end{gathered} \tag{5}$$

(Here and in what follows we denote by $\omega(F, \delta) = \max\{|F(x_1) - F(x_2)|: x_1, x_2 \in \in [a, b],\ |x_1 - x_2| \leqq \delta\}$ the modulus of continuity of $F \in C[a, b]$.) Combining (4) and (5) we can easily derive, that $\tilde{p}_s'(x_j) = 0$, i.e.

$$\operatorname{Re} a_j p_s'(x_j) = 0 \quad (1 \leqq s \leqq k+1)$$

if $x_j \in (a, b)$. This and (4) imply that for the extremal set $\{x_i\}_{i=1}^m$ with coefficients $\{a_i\}_{i=1}^m$ and $k+1$ linearly independent elements $p_1, \ldots, p_{k+1} \in U_n$ the condition of the theorem is violated, which proves its sufficiency.

Necessity. Assume that the condition of theorem does not hold, i.e. there exists an extremal set $\{x_i\}_{i=1}^m$ of U_n with coefficients $\{a_i\}_{i=1}^m$ and $k+1$ linearly independent functions $p_1, \ldots, p_{k+1} \in U_n$ such that $p_j(x_i) = 0$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq k+1)$ and

$\operatorname{Re} a_i p_j'(x_i)=0$ for any $1 \leqq j \leqq k+1$ and $x_i \in (a, b)$. Without loss of generality we may assume that $\|p_j'\|_C \leqq 1$ for each $1 \leqq j \leqq k+1$.

Let $0<h$ be small enough so that $[-h+x_i, x_i+h] \cap [-h+x_j, x_j+h] = \emptyset$ if $i \neq j$, and set $A_h = [a, b] \cap (\bigcup_{i=1}^{m} (-h+x_i, x_i+h))$. Evidently, there exists a function $g \in C^\infty[a, b]$ such that $\|g\|_C = 1$ and $g(x) = \operatorname{sign} a_i$ for $x \in [-h+x_i, x_i+h] \cap [a, b]$ $(1 \leqq i \leqq m)$. (This function can be chosen real if $a_i \in \mathbf{R}$.)

Consider at first the case $r=1$. Set

$$\varphi(\delta) = \delta + \sum_{j=1}^{k+1} \omega(p_j', \delta) \quad (0 \leqq \delta \leqq b-a); \tag{6}$$

$$\psi_i(x) = \begin{cases} \displaystyle\int_0^{|x-x_i|} \varphi(t)\,dt, & \text{if} \quad x_i \in (a, b) \\ |x-x_i|, & \text{if} \quad x_i = a \quad \text{or } b \end{cases} \quad (1 \leqq i \leqq m); \tag{7}$$

$$\psi(x) = \prod_{i=1}^{m} \psi_i(x) \quad (x \in [a, b]). \tag{8}$$

It is easy to see that $\psi_i \in C_0^1[a, b]$ $(1 \leqq i \leqq m)$. Furthermore, we have by (6) and (7) that if $x_i \in (a, b)$ then for any $x \in [a, b]$

$$\begin{aligned} \psi_i(x) &= \int_0^{|x-x_i|} \varphi(t)\,dt \geqq \frac{|x-x_i|}{2}\, \varphi\left(\frac{|x-x_i|}{2}\right) \geqq \\ &\geqq \frac{|x-x_i|}{2}\, \omega\left(p_j', \frac{|x-x_i|}{2}\right) \geqq \frac{|x-x_i|}{4}\, \omega(p_j', |x-x_i|) \quad (1 \leqq j \leqq k+1). \end{aligned} \tag{9}$$

On the other hand, since $p_j(x_i)=0$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq k+1)$ and $\operatorname{Re} a_i p_j'(x_i)=0$ for any $1 \leqq j \leqq k+1$ if $x_i \in (a, b)$ we have by (9) and (7) that

$$|\operatorname{Re} a_i p_j(x)|/|a_i| \leqq |x-x_i|\,\omega(p_j', |x-x_i|) \leqq 4\psi_i(x)$$

if $x_i \in (a, b)$ and

$$|\operatorname{Re} a_i p_j(x)|/|a_i| \leqq |x-x_i|\, \|p_j'\|_C \leqq \psi_i(x)$$

if $x_i = a$ or b $(x \in [a, b],\ 1 \leqq j \leqq k+1)$. Thus for any $1 \leqq i \leqq m$, $1 \leqq j \leqq k+1$ and $x \in [a, b]$

$$|\operatorname{Re} a_i p_j(x)|/|a_i| \leqq 4\psi_i(x). \tag{10}$$

Furthermore the function ψ/ψ_i is positive on $[a, b] \cap [-h+x_i, x_i+h]$ $(1 \leqq i \leqq m)$, hence $\psi(x)/\psi_i(x) \geqq c_0 > 0$ for any $x \in [a, b] \cap [-h+x_i, x_i+h]$ and $1 \leqq i \leqq m$. This and (10) imply that for each $x \in [a, b] \cap [-h+x_i, x_i+h]$

$$\psi(x) = \prod_{i=1}^{m} \psi_i(x) \geqq \frac{c_0 |\operatorname{Re} a_i p_j(x)|}{4|a_i|} = \frac{K_1}{|a_i|} |\operatorname{Re} a_i p_j(x)| \tag{11}$$
$$(1 \leqq i \leqq m, \quad 1 \leqq j \leqq k+1).$$

In addition we can derive from (6) and (7) that

$$\psi_i(x) \geqq \min\{|x-x_i|,\ (x-x_i)^2/2\} \geqq (x-x_i)^2 \min\{1/(b-a),\ 1/2\} = c_1(x-x_i)^2.$$

Hence estimating as in (11) we have for $x\in[a,b]\cap[-h+x_i, x_i+h]$

$$\psi(x) \geqq c_0c_1(x-x_i)^2 = K_2(x-x_i)^2 \quad (1 \leqq i \leqq m). \tag{12}$$

Let us consider now the function

$$f(x) = g(x)\big(1-\lambda\psi(x)\big) \quad (x\in[a,b]), \tag{13}$$

where $\lambda=1/2\|\psi\|_C$. Obviously $f\in C^1[a,b]$ and $\|f\|_C=1$. Moreover $f(x_i)= =g(x_i)\big(1-\lambda\psi(x_i)\big)=g(x_i)=\operatorname{sign} a_i$ and $|f(x)|<1$ if $x\neq x_i$ $(1\leqq i\leqq m)$. Since $\{x_i\}_{i=1}^m$ is an extremal set of U_n with coefficients $\{a_i\}_{i=1}^m$ it follows from Lemma 1 that 0 is a best approximation of f. We state that $\varepsilon p_1, \dots, \varepsilon p_{k+1}$ are also best approximants of f for $\varepsilon>0$ small enough. Using that $|f(x)|<1$ if $x\neq x_i$, and $x_i\in A_h$, $1\leqq i\leqq m$, we can find a constant $n>0$ such that $|f(x)|\leqq 1-\eta$ if $x\in[a,b]\setminus A_h$. Then if $0<\varepsilon\leqq\eta/M$, where $M=\max_{1\leqq j\leqq k+1}\|p_j\|_C$ we have for $x\in[a,b]\setminus A_h$

$$|f(x)-\varepsilon p_j(x)| \leqq 1-\eta+\varepsilon M \leqq 1 \quad (1 \leqq j \leqq k+1). \tag{14}$$

Assume now that $x\in A_h$, i.e. $x\in(-h+x_i, x_i+h)\cap[a,b]$ for some $1\leqq i\leqq m$. In this case $g(x)=\operatorname{sign} a_i$, hence and by (13)

$$\begin{aligned} |f(x)-\varepsilon p_j(x)|^2 &= |\operatorname{sign} a_i\big(1-\lambda\psi(x)\big)-\varepsilon p_j(x)|^2 = |1-\lambda\psi(x)-\varepsilon(a_i/|a_i|)\,p_j(x)|^2 = \\ &= \big(1-\lambda\psi(x)-(\varepsilon/|a_i|)\operatorname{Re} a_ip_j(x)\big)^2+\big((\varepsilon/|a_i|)\operatorname{Im} a_ip_j(x)\big)^2 \quad (1 \leqq j \leqq m). \end{aligned} \tag{15}$$

Since $p_j(x_i)=0$ $(1\leqq j\leqq k+1)$ it follows that

$$|\operatorname{Im} a_ip_j(x)|/|a_i| \leqq \|p_j'\|_C|x-x_i| \leqq |x-x_i| \quad (1 \leqq j \leqq m). \tag{16}$$

Assume now in addition that $\varepsilon<\lambda K_1/2$. Then (11) yields that for any $1\leqq j\leqq m$

$$0 \leqq 1-(3\lambda/2)\psi(x) \leqq 1-\lambda\psi(x)-(\varepsilon/|a_i|)\operatorname{Re} a_ip_j(x) \leqq 1-(\lambda/2)\psi(x). \tag{17}$$

Applying inequalities (17), (16) and (12) in (15) we have

$$\begin{aligned} |f(x)-\varepsilon p_j(x)|^2 &\leqq \big(1-(\lambda/2)\psi(x)\big)^2+\varepsilon^2(x-x_i)^2 \leqq 1-(\lambda/2)\psi(x)+\varepsilon^2(x-x_i)^2 \leqq \\ &\leqq 1-(\lambda K_2/2)(x-x_i)^2+\varepsilon^2(x-x_i)^2 \leqq 1 \quad (1 \leqq j \leqq k+1), \end{aligned}$$

if we assume also that $\varepsilon<\sqrt{\lambda K_2/2}$. Hence and by (14) we finally obtain that for ε small enough $\|f-\varepsilon p_j\|_C\leqq 1$ $(1\leqq j\leqq k+1)$. This means that $k+1$ linearly independent elements $p_1, \dots, p_{k+1}\in U_n$ are also best approximants of f (in addition to 0), i.e. U_n is not k-Chebyshev.

If $r\geqq 2$ then setting $\varphi(\delta)=\delta$ and constructing f by (7), (8) and (13) we can analogously verify that U_n is not k-Chebyshev.

The proof of Theorem 1 is completed.

The corollary follows immediately from Theorem 1 since in the real case the coefficients of extremal sets and therefore the function (13) are real.

Let us now show that the characterization of k-Chebyshev subspaces of $C_0^r[a, b]$ given by Corollary 1 is equivalent to Garkavi's characterization.

Proposition 1. *Let U_n be a subspace of $C_0^1[a, b]$. Then for any $0 \leqq k \leqq n-1$ the following statements are equivalent:*

(i) *for any extremal set of U_n its points cannot be common special zeros of $k+1$ linearly independent elements of U_n;*

(ii) *for any s linearly independent elements $p_1, \ldots, p_s$ in U_n $(k+1 \leqq s \leqq n)$ among their common zeros there are at most $n-s$ special zeros common to any $k+1$ of the elements $p_1, \ldots, p_s$.*

Proof. (i)$\Rightarrow$(ii). Let $\{\varphi_i\}_{i=1}^n$ be a basis in U_n and consider the matrix

$$M = M(x_1, \ldots, x_{n-s+1}) = \begin{pmatrix} \varphi_1(x_1) & \ldots & \varphi_n(x_1) \\ \vdots & & \\ \varphi_1(x_{n-s+1}) & \ldots & \varphi_n(x_{n-s+1}) \end{pmatrix}$$

where $x_1, \ldots, x_{n-s+1}$ are arbitrary distinct points at $[a, b]$. If $x_1, \ldots, x_{n-s+1}$ are common zeros of s linearly independent elements in U_n then it follows that $\operatorname{rank} M \leqq n-s$. Therefore for some $b_i \in \mathbf{R}$, $\sum_{i=1}^{n-s+1} |b_i| = 1$, we have $\sum_{i=1}^{n-s+1} b_i \varphi_j(x_i) = 0 \quad (1 \leqq j \leqq n)$. This means that the set $\{x_i\}_{i=1}^{n-s+1}$ or a proper subset of it is an extremal set of U_n. Hence if s linearly independent elements in U_n have $n-s+1$ common zeros $x_1, \ldots, x_{n-s+1}$ then the set $\{x_i\}_{i=1}^{n-s+1}$ or a proper subset of it is an extremal set of U_n. This observation proves the implication (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i). Assume that (i) is not true. Then there exists an extremal set $\{x_i\}_{i=1}^m$ and $k+1$ linearly independent elements $p_1, \ldots, p_{k+1} \in U_n$ such that each x_i is a special zero of p_j $(1 \leqq i \leqq m,\ 1 \leqq j \leqq k+1)$. Consider the matrix $M^* = M(x_1, \ldots, x_m)$. Then the set of functions in U_n vanishing on $\{x_i\}_{i=1}^m$ is a subspace of dimension $s = n - \operatorname{rank} M^*$. Evidently, $s \geqq k+1$. Then we can find elements $p_{k+2}, \ldots, p_s \in U_n$ such that $p_1, \ldots, p_s$ are linearly independent and $p_{k+2}, \ldots, p_s$ also vanish on $\{x_i\}_{i=1}^m$. It follows from (ii) that $m \leqq n-s = \operatorname{rank} M^*$, i.e. $\operatorname{rank} M^* = m$. But since $\{x_i\}_{i=1}^m$ is an extremal set of U_n the rows of M^* are linearly dependent. This implies that $\operatorname{rank} M^* \leqq m-1$, a contradiction.

2. In [2] there are given different examples of real polynomial spaces which are Chebyshev subspaces of $C_0^1[a, b]$ but do not satisfy this property with respect to $C_0[a, b]$. Let P_n denote the space of real algebraic polynomials of degree at most $n-1$. It is shown in [2] that if for the subspace U the embeddings $P_{[n/2]} \subset U \subset P_n$ hold then U is a Chebyshev subspace of $C_0^1[a, b]$. In this section applying Theorem 1 we shall give a similar statement for complex polynomials. (Since the characterization of Chebyshev subspaces of $C^r[a, b]$ does not depend on $r \geqq 1$, in what follows we shall consider only the case $r=1$.)

Let $T_n = \left\{ \sum_{s=0}^{n-1} c_s e^{isx},\ c_s \in \mathbf{C} \right\}$ be the space of complex polynomials of degree at most $n-1$ and real variable $x \in [a, b]$, where $0 \leqq a < b < 2\pi$. Evidently, each extremal set of T_n consists of at least $n+1$ points (and at most $2n+1$ points by definition). In order to apply Theorem 1 we shall also need some information on the coefficients of extremal sets of T_n.

Lemma 2 (Vidensky [10]). *Let $\{x_j\}_{j=1}^m \subset [a, b]$ be an extremal set of T_n with coefficients $\{a_j\}_{j=1}^m$ $(n+1 \leqq m \leqq 2n+1)$. Then there exists $u \in T_{m-n-1}$ such that for any $j = 1, 2, \ldots, m$*

$$a_j = u(x_j) \Big/ \prod_{\substack{s=1 \\ s \neq j}}^{m} (e^{ix_j} - e^{ix_s}). \tag{18}$$

Let now $0 = r_0 < r_1 < \ldots < r_{p-1}$ be a sequence of integers and set $U_p = \left\{ \sum_{s=0}^{p-1} c_s e^{ir_s x},\ c_s \in \mathbf{C} \right\}$. Since $g(x) = 1 - e^{ir_{p-1}x} \in U_p$ may have r_{p-1} distinct zeros at $[a, b]$ it follows by the Haar—Kolmogorov theorem that U_p is in general a Chebyshev subspace of $C[a, b]$ only if $r_{p-1} = p-1$ and thus $U_p = T_p$. But for the space $C^1[a, b]$ we have a much more general statement.

Theorem 2. *Assume that $T_r \subset U_p \subset T_n$, where $r \leqq p \leqq n$ and $r = [2n/3]$ $(n \geqq 4)$. Then U_p is a Chebyshev subspace of $C^1[a, b]$ for any $0 \leqq a < b < 2\pi$.*

Proof. Assume that U_p is not a Chebyshev subspace of $C^1[a, b]$. Then by Theorem 1 there exists an extremal set $\{x_j\}_{j=1}^m$ of U_p with coefficients $\{a_j\}_{j=1}^m$ and $g \in U_p \setminus \{0\}$ such that $g(x_j) = 0$ $(1 \leqq j \leqq m)$ and $\operatorname{Re} a_j g'(x_j) = 0$ for each $x_j \in (a, b)$. Without loss of generality we may assume that $x_j \in (a, b)$ for every $2 \leqq j \leqq m-1$. Since $U_p \supset T_r$, $\{x_j\}_{j=1}^m$ is an extremal set of T_r, too. Hence $m \geqq r+1$. On the other hand, $g \in T_n \setminus \{0\}$ vanishes on x_j, $1 \leqq j \leqq m$. Thus $r+1 \leqq m \leqq n-1 \leqq 2r+1$. Therefore by Lemma 2 we can find a polynomial $u \in T_{m-r-1}$ such that for any $j = 1, 2, \ldots, m$ (18) holds. Furthermore, using that $g(x_j) = 0$ $(1 \leqq j \leqq m)$ we can write

$$g(x) = \prod_{j=1}^{m} (e^{ix} - e^{ix_j})\, \tilde{g}(x),$$

where $\tilde{g} \in T_{n-m}$. This yields that

$$g'(x_j) = ie^{ix_j} \prod_{\substack{s=1 \\ s \neq j}}^{m} (e^{ix_j} - e^{ix_s})\, \tilde{g}(x_j), \quad 1 \leqq j \leqq m. \tag{19}$$

Since $\{x_j\}_{j=2}^{m-1}\subset(a, b)$, $\operatorname{Re} a_j g'(x_j)=0$ for $2\leqq j\leqq m-1$. This together with (18) and (19) imply that for each $2\leqq j\leqq m-1$

$$0 = \operatorname{Re} a_j g'(x_j) = \operatorname{Re} u(x_j)\, ie^{ix_j}\, \tilde{g}(x_j) = \operatorname{Re} \tilde{u}(x_j), \tag{20}$$

where $\tilde{u}(x)=ie^{ix}u(x)\tilde{g}(x)\in T_{n-r-1}$ and $\tilde{u}$ does not contain the constant term. Moreover (20) yields that $t(x)=\operatorname{Re}\tilde{u}(x)$ has $m-2$ distinct zeros at (a, b), where $m-2\geqq \geqq r-1$. On the other hand t is a trigonometric polynomial of degree at most $n-r-2$. Thus either t is identically zero or it has not more than $2n-2r-4$ distinct zeros at (a, b). But since $r=[2n/3]$ it follows that $2n-2r-4<3r+3-2r-4=r-1$. Hence $t(x)=\operatorname{Re}\tilde{u}(x)$ is the zero function. Using that $\tilde{u}\in T_{n-r-1}$ does not contain the constant term we finally obtain that $\tilde{u}$ is identically zero, a contradiction. The theorem is proved.

3. In this final section of our paper we shall solve some extremal problems connected with the unicity of best Chebyshev approximation of real differentiable functions by lacunary polynomials. Consider the space $C_0[-1, 1]$. Then $P_n= =\operatorname{span}\{1, x, ..., x^{n-1}\}$ is a simple example of a Chebyshev subspace of $C_0[-1, 1]$. Here and in what follows we denote by span $\{...\}$ the real linear span of functions specified in the brackets. Let us now omit the basis function x^k $(0<k<n-1)$ and consider the resulting space of lacunary polynomials $P_{n-1}^{(k)}=\operatorname{span}\{1, ..., x^{k-1}, x^{k+1}, ..., x^{n-1}\}$. The polynomials in $P_{n-1}^{(k)}$ may still have $n-1$ distinct zeros at $[-1, 1]$, while the dimension of this space is only $n-1$. Thus $P_{n-1}^{(k)}$ is not a Chebyshev subspace of $C_0[-1, 1]$. On the other hand it was shown in [2] that $P_{n-1}^{(k)}$ is a Chebyshev subspace of $C_0^1[-1, 1]$, if $n\geqq 4$. Analogously, if we add to P_n an arbitrary power function x^r $(r\in\mathbf{N}, r\geqq n+1)$ then the resulting space $\bar{P}_{n+1}^{(r)}=\operatorname{span}\{1, x, ..., x^{n-1}, x^r\}$ is Chebyshev in $C_0[-1, 1]$ only if $r-n$ is even but nevertheless it is Chebyshev in $C_0^1[-1, 1]$ for any r (see [2]). Thus deleting from P_n or adding to P_n a power function we in general violate the Haar property and hence obtain nonuniqueness of best Chebyshev approximation in $C_0[-1, 1]$. On the other hand the unicity with restriction to the space $C_0^1[-1, 1]$ still holds. This observation raises the following questions:

A) Determine the maximal integer $\gamma=\gamma(n)$ such that omitting from P_n arbitrary γ basis functions $x^{r_1}, ..., x^{r_\gamma}$ $(1\leqq r_1<...<r_\gamma\leqq n-2,\ r_i\in\mathbf{N})$ the resulting set of lacunary polynomials $P_{n-\gamma}^{*}=\operatorname{span}\{x^i,\ 0\leqq i\leqq n-1,\ i\neq r_j,\ 1\leqq j\leqq\gamma\}$ is still a Chebyshev subspace of $C_0^1[-1, 1]$.

B) Determine the maximal integer $\mu=\mu(n)$ such that adding to P_n arbitrary μ powers $x^{t_1}, ..., x^{t_\mu}$ $(n+1\leqq t_1<...<t_\mu,\ t_i\in\mathbf{N})$ the resulting set of lacunary polynomials $P'_{n+\mu}=\operatorname{span}\{1, x, ..., x^{n-1}, x^{t_1}, ..., x^{t_\mu}\}$ is still a Chebyshev subspace of $C_0^1[-1, 1]$.

We shall verify in this section that $\gamma(n)=[n/4]$ and $\mu(n)=[n/2]$. Thus omitting (or adding) from P_n a considerable number of power functions we can still guarantee the unicity of best Chebyshev approximation in $C_0^1[-1, 1]$.

In what follows the finite-dimensional Chebyshev subspaces of $C_0[-1,1]$ will be called Haar spaces.

We shall need the following simple lemma.

Lemma 3. *Let $r\in\mathbf{N}$ and $0=m_0<m_1<\ldots<m_r$ be a sequence of integers such that m_j-m_{j-1} is odd for each $j=1,2,\ldots,r$. Then the space $P^*_{r+1}=\mathrm{span}\,\{1=x^{m_0}, x^{m_1},\ldots,x^{m_r}\}$ is a Haar space.*

Proof. We shall prove the lemma by induction. For $r=1$ the statement is evident. Assume that it holds for $r-1$ $(r\geqq 2)$. For any $p\in P^*_{r+1}$ which is not a constant function $p'(x)=x^{m_1-1}\tilde{p}(x)$, where $\tilde{p}\in\tilde{P}_r=\mathrm{span}\,\{1,x^{m_2-m_1},\ldots,x^{m_r-m_1}\}$. By our assumption $\tilde{P}_r$ is an r-dimensional Haar space, hence $\tilde{p}$ has at most $r-1$ distinct zeros at $[-1,1]$. Moreover, m_1-1 is even, therefore p' has at most $r-1$ points of change of sign at $[-1,1]$. This yields that p has not more than r distinct zeros at $[-1,1]$. The lemma is proved.

By the well-known interpolatory property of Haar spaces it follows that each extremal set of an n-dimensional Haar space consists of exactly $n+1$ points on $[-1,1]$. In particular if U_n contains a k-dimensional Haar subspace $(k\leqq n)$ then each extremal set of U_n consists of at least $k+1$ points $(U_n\subset C_0[-1,1])$. We shall frequently use this simple observation.

Theorem 3. *For any $n\geqq 4$, $\gamma(n)=[n/4]$.*

Proof. Let us prove at first that $\gamma(n)\geqq[n/4]$. Set $m=[n/4]$ and let $1\leqq r_1<\ldots<r_m\leqq n-2$ be arbitrary integers. Omitting from P_n the basis functions x^{r_i} $(1\leqq i\leqq m)$ we obtain the space $P^*_{n-m}=\mathrm{span}\,\{x^{t_0},x^{t_1},\ldots,x^{t_{n-m-1}}\}$, where $0=t_0<t_1<\ldots<t_{n-m-1}=n-1$ and $t_i\neq r_j$ for every $0\leqq i\leqq n-m-1$, $1\leqq j\leqq m$. Set $c_j=t_j-t_{j-1}$, $1\leqq j\leqq n-m-1$. Evidently, at most m of these $n-m-1$ integers are even. Deleting from the sequence $0=t_0<t_1<\ldots<t_{n-m-1}=n-1$ those integers t_j for which c_j is even we obtain a sequence $0=t'_0<t'_1<\ldots<t'_s\leqq n-1$, where $s\geqq n-2m-1$. Let us prove that for any $1\leqq j\leqq s$, $t'_j-t'_{j-1}$ is odd. Indeed, we have for some $q<r$, that $t'_{j-1}=t_q<t_{q+1}<\ldots<t_r=t'_j$, where c_i is even for every $q+1\leqq i\leqq r-1$, while c_r is odd. Therefore $t'_j-t'_{j-1}=t_r-t_q=\sum_{i=q+1}^{r}c_i$ is odd. Applying Lemma 3 we can conclude that span $\{x^{t'_0},\ldots,x^{t'_s}\}$ is a Haar space. Thus P^*_{n-m} contains a Haar space of dimension $s+1\geqq n-2m$. Therefore each extremal set of P^*_{n-m} consists of at least $n-2m+1$ points. If the points of an extremal set of P^*_{n-m} are special zeros of $g\in P^*_{n-m}$ then g has at least $(n-2m+1)+(n-2m-1)=2n-4m\geqq n$ zeros counting double zeros twice. Since $g\in P_n$ it follows that g is identically zero. Thus we obtain by Corollary 1 that P^*_{n-m} is a Chebyshev subspace of $C^1_0[-1,1]$, i.e. $\gamma(n)\geqq m=[n/4]$.

Now we shall verify that $\gamma(n)\leqq m=[n/4]$. We have $n=4m+i$ $(i=0,1,2,3)$. Assume that in contrary $\gamma(n)\geqq m+1$, i.e. omitting from P_n arbitrary $m+1$ basis functions we still have a Chebyshev subspace of $C_0^1[-1, 1]$. Set

$$P_{n-m-1}^* = \operatorname{span}\{1, x, \dots, x^{2m+i-3}, x^{2m+i-1}, \dots, x^{4m+i-1}\}.$$

P_{n-m-1}^* is generated from $P_{4m+i}=P_n$ by deleting $m+1$ powers $x^{2m+i-2+2s}$, $0\leqq s\leqq m$. Thus by our assumption P_{n-m-1}^* is a Chebyshev subspace of $C_0^1[-1, 1]$. Consider the function $f(x)=x^{2m+i-2}$ $(x\in[-1, 1])$.

Case 1: $i=1$ or 3. Then f is odd. Since f possesses a unique best approximation q in P_{n-m-1}^*, q is odd, too. But the powers $x^{2m+i-1+2s}$ $(0\leqq s\leqq m)$ are even, hence $q\in P_{2m+i-3}$. Therefore $f-q=t_{2m+i-2}$, where $t_k(x)=2^{-k+1}\cos k \arccos x$ denotes the Chebyshev polynomial of degree k. Consider the extremas of t_{2m+i-2}: $x_j= =\cos j\pi/(2m+i-2)$ $(0\leqq j\leqq 2m+i-2)$. Since q is the best approximation of f in P_{n-m-1}^* it follows from Lemma 1 that the set $\{x_j, 0\leqq j\leqq 2m+i-2\}$ or a proper subset of it is an extremal set of P_{n-m-1}^*. On the other hand $P_{n-m-1}^*\supset P_{2m+i-2}$, hence each extremal set of P_{n-m-1}^* contains at least $2m+i-1$ points. Thus the set $\{x_j, 0\leqq j\leqq 2m+i-2\}$ is an extremal set of P_{n-m-1}^*. Consider now the polynomial

$$\tilde{p}(x) = (1-x^2)\prod_{i=1}^{2m+i-3}(x-x_i)^2. \tag{21}$$

Evidently, each x_j is a special zero of $\tilde{p}$ $(0\leqq j\leqq 2m+i-2)$ and $\deg\tilde{p}=4m+2i-4\leqq \leqq 4m+i-1$. Furthermore, since $x_j=-x_{2m+i-2-j}$ $(0\leqq j\leqq 2m+i-2)$ it follows that $\tilde{p}$ is even. Thus finally we obtain that $\tilde{p}\in P_{n-m-1}^*$, which contradicts Corollary 1.

Case 2: $i=0$ or 2. In this case instead of polynomial $\tilde{p}$ given by (21) we should consider the polynomial $p^*(x)=x\tilde{p}(x)$. Then we can derive a contradiction analogously to Case 1, the details are left to the reader.

Thus the assumption $\gamma(n)\geqq m+1$ leads to a contradiction. This completes the proof of the equality $\gamma(n)=[n/4]$.

Theorem 4. *For any* $n\geqq 2$, $\mu(n)=[n/2]$.

Proof. Let us verify that $\mu(n)\geqq m=[n/2]$. Take arbitrary integers $n+1\leqq \leqq t_1<t_2<\dots<t_m$ and consider the space $P'_{n+m}=\operatorname{span}\{1, x, \dots, x^{n-1}, x^{t_1}, \dots, x^{t_m}\}$. Obviously, each extremal set of P'_{n+m} consists of at least $n+1$ points. We state that P'_{n+m} is a Chebyshev subspace of $C_0^1[-1, 1]$. Assume the contrary. Then for some extremal set of P'_{n+m} and some $g\in P'_{n+m}\setminus\{0\}$ the points of the extremal set are special zeros of g, hence g' has at least $2n-1$ distinct zeros at $[-1, 1]$. Furthermore $g'\in\operatorname{span}\{1, x, \dots, x^{n-2}, x^{t_1-1}, \dots, x^{t_m-1}\}\setminus\{0\}$. By Lemma 3 the space $\operatorname{span}\{1, x, \dots \dots, x^{n-2}, x^{t_1-1}, \dots, x^{t_m-1}\}$ can be imbedded to a Haar space of dimension at most $n+2m-1\leqq 2n-1$. This means that each element of this space, in particular g', may have at most $2n-2$ distinct zeros at $[-1, 1]$, a contradiction. By this contradic-

tion we obtain that P'_{n+m} is a Chebyshev subspace of $C_0^1[-1, 1]$, i.e. $\mu(n)\geqq m=$ $=[n/2]$.

Assume now that $\mu(n)\geqq m+1$. Set $n=2m+i$ $(i=0, 1)$,

$$P'_{n+m+1} = \text{span}\,\{1, x, \ldots, x^{2m+i-1}, x^{2m+i+1}, \ldots, x^{4m+i+1}\}.$$

Then P'_{n+m+1} is generated from $P_{2m+i}=P_n$ by adding $m+1$ basis functions $x^{2m+i+1+2s}$ $(0\leqq s\leqq m)$. Since $\mu(n)\geqq m+1$, it follows that P'_{n+m+1} is a Chebyshev subspace of $C_0^1[-1, 1]$. Now we can derive a contradiction analogously to the proof of Theorem 3. We omit the details.

This completes the proof of Theorem 4.

Consider now the general case of lacunary polynomials. Let $0=m_0<m_1<\ldots<m_r$ be arbitrary integers and set

$$\bar{P} = \bar{P}_{r+1} = \text{span}\,\{1 = x^{m_0}, x^{m_1}, \ldots, x^{m_r}\} \quad (r\in\mathrm{N}). \tag{22}$$

Furthemore denote by $\delta(\bar{P})$ the number of those j-s for which m_j-m_{j-1} is even, $1\leqq j\leqq r$. Then $0\leqq\delta(\bar{P})\leqq r=\dim(\bar{P})-1$. By Lemma 3 if $\delta(\bar{P})=0$ then $\bar{P}$ is a Haar space on $[-1, 1]$. It can be easily shown that this condition is also necessary for the Haar property. The next theorem gives a sufficient condition for $\bar{P}$ to be a Chebyshev subspace of $C_0^1[-1, 1]$.

Theorem 5. *Let* $\dim(\bar{P})\geqq 4$ *and assume that*

$$\delta(\bar{P}) \leqq \left[(\dim(\bar{P})-1)/3\right] \tag{23}$$

holds. Then $\bar{P}$ is a Chebyshev subspace of $C_0^1[-1, 1]$.

Proof. Consider the space $\bar{P}^*$ which results from $\bar{P}$ after deleting in (22) all basis functions x^{m_j} such that m_j-m_{j-1} is even. Obviously, $\bar{P}^*$ is a space of dimension $\dim(\bar{P})-\delta(\bar{P})$. Moreover, similarly as in the proof of Theorem 3 we can show that $\delta(\bar{P}^*)=0$, thus by Lemma 3 $\bar{P}^*$ is a Haar space. Therefore each extremal set of $\bar{P}$ consists of at least $\dim(\bar{P}^*)+1=\dim(\bar{P})-\delta(\bar{P})+1$ points. Assume that (23) holds but $\bar{P}$ is not a Chebyshev subspace of $C_0^1[-1, 1]$. Then there exists a $p\in\bar{P}\setminus\{0\}$ such that the set of special zeros of p contains at least $\dim(\bar{P})-\delta(\bar{P})+1$ points. This means that p' has at least $2\dim(\bar{P})-2\delta(\bar{P})-1$ distinct zeros at $[-1, 1]$. Furthermore, $p'=x^{m_1-1}g$, where $g\in\text{span}\,\{1, x^{m_2-m_1}, \ldots, x^{m_r-m_1}\}=\tilde{P}^*$ and g is not identically zero. It is evident, that $\delta(\tilde{P}^*)\leqq\delta(\bar{P})$. Hence adding to $\tilde{P}^*$ at most $\delta(\bar{P})$ power functions we can obtain (by Lemma 3) a Haar space. This means that $\tilde{P}^*$ can be enbedded to a Haar space of dimension at most $\dim(\bar{P})+\delta(\bar{P})-1$. Hence $g\in\tilde{P}^*\setminus\{0\}$ can have not more than $\dim(\bar{P})+\delta(\bar{P})-2$ zeros, i.e. p' has at most $\dim(\bar{P})+$ $+\delta(\bar{P})-1$ distinct zeros at $[-1, 1]$. Since we have shown that p' has at least $2\dim(\bar{P})-2\delta(\bar{P})-1$ distinct zeros, it follows that $2\dim(\bar{P})-2\delta(\bar{P})-1\leqq\dim(\bar{P})+$ $+\delta(\bar{P})-1$, i.e. $\dim(\bar{P})\leqq 3\delta(\bar{P})$. But this contradicts (23). The theorem is proved.

Remark. The converse of Theorem 5 is not true in general. There exist Chebyshev subspaces of $C_0^1[-1, 1]$ of the form (22) such that (23) does not hold. Indeed, let $n=2k$ and add to P_n k odd powers greater than $n-1$. Then for the resulting space $\bar{P}$ the realtion $\delta(\bar{P})=k$ holds. By Theorem 4 $\bar{P}$ is a Chebyshev subspace of $C_0^1[-1, 1]$. On the other hand $\delta(\bar{P})=k>k-1=[(\dim(\bar{P})-1)/3]$.

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