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Lacunarity with respect to orthogonal polynomial sequences

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Lacunarity has been studied in a variety of settings: on the circle group T with dual $\mathbf{Z}$, on compact abelian groups G with dual $\hat{G}$, on compact (nonabelian) groups resp. on the space of conjugacy classes of compact groups and on compact hypergroups with dual Σ. For references we recommend [11]. In view of the classical case T and $\mathbf{Z}$ a most natural setting to study lacunarity are orthogonal polynomial sequences. In fact to many orthogonal polynomial sequences there corresponds a hypergroup structure on $\mathbf{N}_0=\mathbf{N}\cup\{0\}$ having as dual a compact subset D_S of $\mathbf{R}$, see [8]. In this way a set $E\subseteq\mathbf{N}_0$ is a Sidon set if each bounded sequence can be represented on E as a (generalized) Fourier—Stieltjes transform. We emphasize that D_S is in general not a hypergroup under pointwise operations. Thus only $\mathbf{N}_0$ bears an algebraic structure in constrast to the situations above.

Combining two recent results, Theorem 3.2 of [14] and Chapter 4, ad(a) of [8], we can deduce that with respect to Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $\alpha\geqq\beta>-1$ and in addition $\beta\geqq-1/2$ or $\alpha+\beta\geqq 0$, but $\alpha\neq-1/2$, a set E is a Sidon set if and only if E is finite. This result suggests to perform further investigations on the subject.

We assume that the polynomial sequences satisfy a certain positivity property. This property and its consequences are presented in Section I. Sidonicity is the subject of II. In III there is shown that $\mathbf{N}_0$ is never a Sidon set. The fact that some orthogonal polynomial sequences admit only finite Sidon sets is established in IV. The existence of infinite Sidon sets is studied in section V.

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I. Property (P)

At first we have to set up some notation. Let $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$, $(c_n)_{n\in\mathbf{N}}$ be three real-valued sequences such that $a_n>0$, $c_n>0$, $b_n\geqq 0$ and $a_n+b_n+c_n=1$. Further fixing $a_0>0$, $b_0\in\mathbf{R}$ such that $a_0+b_0=1$ define

$$P_0(x)=1,\quad P_1(x)=\frac{1}{a_0}x-\frac{b_0}{a_0},$$

$$P_{n+1}(x)=\frac{1}{a_n}P_1(x)P_n(x)-\frac{b_n}{a_n}P_n(x)-\frac{c_n}{a_n}P_{n-1}(x),\quad n\in\mathbf{N}.$$

Then $(P_n(x))$ is an orthogonal polynomial sequence. Write the linearization of the products $P_m(x)P_n(x)$, $1\leqq m\leqq n$, by

$$P_m(x)P_n(x)=\sum_{k=0}^{2m} g(m,n,n+m-k)P_{n+m-k}(x).$$

The coefficients $g(m,n,n+m-k)$ are uniquely determined by the sequences $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$, $(c_n)_{n\in\mathbf{N}}$. We require throughout this paper that the positivity property

(P) $$g(m,n,n+m-k)\geqq 0$$

is satisfied.

This assumption yields that $(P_n(x))$ is closely related to a commutative hypergroup structure on $\mathbf{N}_0=\mathbf{N}\cup\{0\}$. The convolution on $\mathbf{N}_0$ is defined by

$$p_m*p_n=\sum_{k=0}^{2m} g(m,n,n+m-k)p_{n+m-k},\quad 1\leqq m\leqq n,$$

where p_n is the point measure of $n\in\mathbf{N}_0$. The involution is the identity on $\mathbf{N}_0$ and the zero is the unit element. Each character on $\mathbf{N}_0$ is given by $\alpha_x\colon \mathbf{N}_0\to\mathbf{R}$, where $x\in D_s$,

$$D_S=\{x\in\mathbf{R}\colon (P_n(x))_{n\in\mathbf{N}} \text{ is bounded}\}\quad\text{and}\quad \alpha_x(n)=P_n(x).$$

Further the character space $\hat{\mathbf{N}}_0$ is homeomorphic to D_S. For details we refer to [8]. Many prominent examples of $(P_n(x))$ satisfying property (P) can be found in [8], [9], [10].

The Haar measure h on the hypergroup $\mathbf{N}_0$ is given by

$$h(0)=1,\quad h(1)=\frac{1}{c_1},\quad h(n)=\prod_{k=1}^{n-1}a_k\Big/\prod_{k=1}^{n}c_k,\quad n=2,3,\ldots.$$

The Plancherel measure π on D_S is the orthogonalization measure of $(P_n(x))$:

$$\int_{D_S}P_n(x)P_m(x)\,d\pi(x)=\begin{cases}1/h(n) & \text{if } n=m\\ 0 & \text{if } n\neq m.\end{cases}$$

We have $\operatorname{supp}\pi\subseteq D_S\subseteq[1-2a_0,1]$.

For an absolutely convergent function $f\in l^1(\mathbf{N}_0)=l^1(\mathbf{N}_0,h)$ define the Fourier transform $\hat{f}$ on D_S by

$$\hat{f}(x)=\sum_{n\in\mathbf{N}_0} f(n)P_n(x)h(n).$$

For a Radon measure $\mu\in M(D_S)$ denote the inverse Fourier—Stieltjes transform $\check{\mu}$ on $\mathbf{N}_0$ by

$$\check{\mu}(n)=\int_{D_S} P_n(x)\,d\mu(x).$$

We shall say that D_S is a hypergroup with respect to pointwise multiplication, if for $x,y\in D_S$ there exists a probability measure $p_x*p_y\in M(D_S)$ such that

(i) $\quad P_n(x)P_n(y)=\int_{D_S} P_n(z)\,dp_x*p_y(z)$ for each $n\in\mathbf{N}_0$, and

(ii) D_S is a hypergroup with this convolution, the identity as involution and $1\in D_S$ as unit;

compare Chapters 1, 4 of [8]. We recall that D_S is in general not a hypergroup with respect to pointwise multiplication.

II. Sidon sets

We assume throughout this paper that $(P_n(x))$ is an orthogonal polynomial sequence defined by (a_n), (b_n), (c_n) satisfying property (P). We abbreviate $S=\operatorname{supp}\pi$ and interpret $M(S)$ as a subspace of $M(D_S)$ and $L^1(D_S)=L^1(D_S,\pi)$ as a subspace of $M(S)$. We write $\|f\|_S=\sup\{|f(x)|: x\in S\}$ for a function $f\in C(D_S)$. Let E be a subset of $\mathbf{N}_0$. As usual $l^\infty(E)$ denotes the space of all bounded functions on E, $c_0(E)$ the space of all functions on E vanishing at infinity, $M(E)$ the space of all bounded measures on E and $\operatorname{Trig}_E(D_S)$ the linear span of $\{P_n(x): n\in E\}$. We shall call E a *Sidon set* if $l^\infty(E)=M(S)^\vee|E$.

Proposition 1. *Let* $E\subseteq\mathbf{N}_0$. *The following are equivalent:*

(a) *E is a Sidon set.*

(b) $L^1(D_S)^\vee|E=c_0(E)$.

(c) *$M(E)^\wedge|S$ is sup-norm closed in $C(S)$.*

(d) *There exists a constant $B>0$ such that $\|\mu\|\leqq B\|\hat{\mu}\|_S$ for each $\mu\in M(E)$.*

(e) *There exists a constant $B>0$ such that $\|\check{f}\|_1\leqq B\|f\|_S$ for each $f\in\operatorname{Trig}_E(D_S)$.*

(f) *Given $\varphi: E\to\{-1,1\}$, there exists some $\mu\in M(S)$ such that* $\sup\{|\check{\mu}(n)-\varphi(n)|: n\in E\}<1$.

Proof. Since the set of measures having finite support in E is norm-dense in $M(E)$, property (d) is equivalent to (e). Using Proposition 1 of [10] define the operator

$A\colon L^1(D_S)\to c_0(E)$, $A(g)=\check{g}|E$, $g\in L^1(D_S)$. The adjoint operator $A^*\colon M(E)\to \to L^\infty(D_S)$ satisfies for $g\in C(D_S)$ and $\mu\in M(E)$:

$$\int_{D_S} A^*(\mu)(x)\cdot g(x)\,d\pi(x) = \sum_{n\in E} A(g)(n)\mu(n) = \int_{D_S} \hat{\mu}(x)\cdot g(x)\,d\pi(x).$$

Thus $A^*(\mu)|S=\hat{\mu}|S$. By Lemma 12.2B of [6] the operator A^* is injective. Now Theorem (E.9) of [5] yields the equivalence of (b), (c) and (d). Define $B\colon M(E)\to C(S)$, $B(\mu)=\hat{\mu}|S$. For the adjoint operator $B^*\colon M(S)\to l^\infty(E)$ we deduce that $B^*(\nu)= =\check{\nu}|E$ for $\nu\in M(S)$. The injectivity of B, Corollary (E.8) and Theorem (E.10) of [5] imply the equivalence of (a) and (c). There remains to prove that (f) implies (d). First deduce that in (d) it is sufficient to consider only real-valued measures $\mu\in M^{\mathbf{R}}(E)$ having finite support. Now assume that (d) does not hold. Then there exists for each $n\in\mathbf{N}$ a measure $\lambda_n\in M^{\mathbf{R}}(E)$ such that $\|\lambda_n\|=1$, $\|\hat{\lambda}_n\|_S<1/n$, and the sets $F_n=\operatorname{supp}\lambda_n$ are finite and pairwise disjoint. In fact having already chosen appropriate $\lambda_1, \dots, \lambda_m\in \in M^{\mathbf{R}}(E)$ observe that $E'=E\setminus\bigcup_{k=1}^{m} F_k$ is not a Sidon set, too. Hence there exists a measure $\lambda_{m+1}\in M^{\mathbf{R}}(E')\subseteq M^{\mathbf{R}}(E)$ such that $\|\lambda_{m+1}\|=1$, $\|\lambda_{m+1}\|_S<1/(m+1)$ and $F_{m+1}=\operatorname{supp}\lambda_{m+1}$ finite. Define $\varphi\colon E\to\{-1, 1\}$ by $\varphi(k)\lambda_n(K)=|\lambda_n(k)|$ for $k\in F_n$ and $\varphi(k)=1$ for k elsewhere. By (f) there exist $\mu\in M(S)$ and $\delta>0$ such that $|\check{\mu}(k)-\varphi(k)|\leqq 1-\delta$ for each $k\in E$. We may assume that $\check{\mu}(k)\in\mathbf{R}$. One obtains that

$$\big|\check{\mu}\lambda_n(k)-|\lambda_n(k)|\big| = |\check{\mu}(k)-\varphi(k)|\,|\lambda_n(k)| \leqq (1-\delta)\,|\lambda_n(k)|$$

and then $0\leqq\delta|\lambda_n(k)|\leqq\check{\mu}(k)\lambda_n(k)$ for each $k\in E$, $n\in\mathbf{N}$. Hence for each $n\in\mathbf{N}$ we have

$$\int_S \hat{\lambda}_n(x)\,d\mu(x) = \sum_{k\in F_n} \lambda_n(k)\check{\mu}(k) \geqq \delta \sum_{k\in F_n} |\lambda_n(k)| = \delta.$$

This is in contradiction to

$$\Big|\int_S \hat{\lambda}_n(x)\,d\mu(x)\Big| \leqq \int_S |\hat{\lambda}_n(x)|\,d|\mu|(x) \leqq \frac{|\mu|(S)}{n},$$

and we have shown that (f) implies (d).

Remark. There holds an appropriate version of Proposition 1 for any discrete hypergroup K.

If $f\in C(D_S)$ satisfies $\check{f}(n)=0$ for each $n\notin E$ we write $f\in C_E(D_S)$. Comparing with the group case, see e.g. Theorem 1.3 of [11], one might notice the failure of the following property $(*)$ in the above list of equivalences

$$(*)\qquad C_E(D_S)^\vee \subseteq l^1(\mathbf{N}_0).$$

We know the following partial results:

Proposition 2. *Let* $E \subseteqq \mathbf{N}_0$.

(a) *If E satisfies property* $(*)$ *then E is a Sidon set.*

(b) *If* D_S *is a hypergroup with respect to pointwise multiplication, then E is a Sidon set if and only if E fulfils property* $(*)$.

Proof. (a) Using $(*)$ the map $f|S \to \check{f}$, $C_E(D_S)|S \to l^1(E)$ is an isomorphism such that $\|f\|_S \leqq \|\check{f}\|_1$. By the open mapping theorem there exists a constant $B>0$ such that $\|\check{f}\|_1 \leqq B\|f\|_S$ for each $f \in C_E(D_S)$. In particular condition (e) of Proposition 1 is valid.

(b) We refer to Theorem 2.2 of [13]. Note that $\hat{D}_S = \mathbf{N}_0$, see Proposition 2 of [8].

For $n \in \mathbf{N}$ and $f \in C(D_S)$ denote $S_n(f)(x) = \sum_{k=0}^{n} \check{f}(k) P_k(x) h(k)$. Further for $E \subseteqq \mathbf{N}_0$ let

$$U_E(D_S) = \{f \in C_E(D_S)\colon\ S_n f \to f \text{ uniformly on } S\}.$$

Proposition 3. *Let* $E \subseteqq \mathbf{N}_0$. *The following are equivalent:*

(a) *E is a Sidon set.*

(b) $U_E(D_S)^{\vee} \subseteqq l^1(\mathbf{N}_0)$.

Proof. At first assume that E is a Sidon set. Let $f \in U_E(D_S)$. Since $S_n f \in M(E)^{\wedge}$ we have that $f|S$ is an element of the uniform closure of $M(E)^{\wedge}|S$. By condition (c) of Proposition 1 it follows that $f|S \in M(E)^{\wedge}|S$. Hence $\check{f} \in l^1(\mathbf{N}_0)$.

Now assume that (b) is valid and E is not a Sidon set. Write $E = \{n_1, n_2, \ldots\}$. Let $N_0 = 0$. For $j \in \mathbf{N}$ there exist $N_j \in \mathbf{N}$, $\lambda_j \in M(E)$ such that $\lambda_j = \sum_{k=N_{j-1}+1}^{N} c_k p_{n_k}$, $\|\lambda_j\| = 1/j$, $\|\hat{\lambda}_j\|_S \leqq 1/2^j$. Define $g(x) = \sum_{j=1}^{\infty} \hat{\lambda}_j(x)$ for $x \in S$, and let f be a continuous extension of g to D_S. Then $\check{f}(n) = 0$ for $n \notin E$ and $\check{f}(n_k) = c_k / h(n_k)$. Hence $S_{N_j}(f) \xrightarrow{j} f$ uniformly on S. For $N_j < n \leqq N_{j+1}$ we obtain

$$\|S_n(f) - S_{N_j}(f)\|_S \leqq \Big\| \sum_{k=N_j+1}^{n} \check{f}(k) P_k h(k) \Big\|_S \leqq \sum_{k=N_j+1}^{n} |c_k| \leqq 1/(j+1).$$

Thus $S_n(f) \xrightarrow{n} f$ uniformly on S, i.e. $f \in U_E(D_S)$. But $\sum_{n=0}^{\infty} |\check{f}(n)| h(n) = \infty$, a contradiction.

III. $\mathbf{N}_0$ is not a Sidon set

If we assume that D_S is a hypergroup with respect to pointwise multiplication, Theorem 2.11 of [13] or Theorem 2.5 of [10] yield that $\mathbf{N}_0$ is not a Sidon set. We shall show that this is true without any assumption on D_S. Our proof is motivated by [3]. In $l^\infty(\mathbf{N}_0)^*$ let τ denote the weak-$*$ topology. Let j be the canonical embedding of

$l^1(\mathbf{N}_0)$ into $l^\infty(N_0)^*$. The set

$$\mathcal{M} = \{\varphi \in l^\infty(\mathbf{N}_0)^*: \varphi(f) \geqq 0 \quad \text{for} \quad f \geqq 0, \ \varphi(1) = 1\}$$

is convex and τ-compact. $M^1(\mathbf{N}_0) = \{g \in l^1(\mathbf{N}_0): g(n) \geqq 0, \|g\|_1 = 1\}$ acts as a commutative semigroup of τ-continuous operators of $\mathcal{M}$ in $\mathcal{M}$, where

$$g * \varphi(f) = \varphi(g * f), \quad f \in l^\infty(\mathbf{N}_0), \quad g \in M^1(\mathbf{N}_0), \quad \varphi \in \mathcal{M}.$$

The Markov—Kakutani fixed point theorem yields $\psi \in \mathcal{M}$ such that $g * \psi = \psi$, i.e. $\psi(g * f) = \psi(f)$ for each $g \in M^1(\mathbf{N}_0)$, $f \in l^\infty(\mathbf{N}_0)$. Using the notation of means we have shown that there exists a mean on $l^\infty(\mathbf{N}_0)$ which is invariant under $f \mapsto g * f$, $g \in M^1(\mathbf{N}_0)$.

Lemma 1. *There exists a sequence* (g_k), $g_k \in M^1(\mathbf{N}_0)$, *such that* $\hat{g}_k(x) \xrightarrow{k} 0$ *for each* $x \in D_S$, $x \neq 1$.

Proof. Let ψ be an invariant mean according to the above arguments. By Goldstine's theorem [2, p. 424], there is a sequence (h_k), $h_k \in l^1(\mathbf{N}_0)$, $\|h_k\|_1 \leqq 1$ such that $jh_k \xrightarrow{k} \psi$ in the τ-topology. Note that $l^1(\mathbf{N}_0)$ is separable. Consider the Jordan decompositions $h_k = h_{1k} - h_{2k} + ih_{3k} - ih_{4k}$. Since $\psi \in \mathcal{M}$ we may assume that $h_{3k} = h_{4k} = 0$. Further $1 \geqq \|h_{1k} - h_{2k}\|_1 = \hat{h}_{1k}(1) + \hat{h}_{2k}(1)$ and $\hat{h}_k(1) = \hat{h}_{1k}(1) - \hat{h}_{2k}(1) \xrightarrow{k} 1$ imply that $\hat{h}_{1k}(1) \xrightarrow{k} 1$ and $\hat{h}_{2k}(1) \xrightarrow{k} 0$. Let $g_k = h_{1k}/\hat{h}_{1k}(1)$, k sufficiently large. Fix $x \in D_S$, $x \neq 1$. Then

$$\sum_{n \in \mathbf{N}_0} g_k(n) p_1 * \alpha_x(n) \xrightarrow{k} \psi(p_1 * \alpha_x) = \psi(\alpha_x)$$

and

$$\sum_{n \in \mathbf{N}_0} g_k(n) p_1 * \alpha_x(n) = P_1(x) \sum_{n \in \mathbf{N}_0} g_k(n) \alpha_x(n) \xrightarrow{k} P_1(x) \psi(\alpha_x).$$

Since $P_1(x) \neq 1$, we have $\hat{g}_k(x) \xrightarrow{k} 0$.

Proposition 4. *Let* $\varphi \in \mathcal{M}$ *be invariant on* $M(D_S)^\vee$, *i.e.* $\varphi(g * \check{v}) = \varphi(\check{v})$ *for each* $g \in M^1(\mathbf{N}_0)$, $v \in M(D_S)$. *Then* $\varphi(\check{v}) = v(\{1\})$ *for* $v \in M(D_S)$.

Proof. The argument in Lemma 1 yields a sequence (h_k), $h_k \in M^1(\mathbf{N}_0)$, such that $\sum_{k \in \mathbf{N}_0} h_k(n) f(n) \xrightarrow{k} \varphi(f)$, $f \in l^\infty(\mathbf{N}_0)$. In particular $\hat{h}_k(1) = 1$ and $\hat{h}_k(x) \to 0$ for $x \neq 1$. Since

$$\sum_{n \in \mathbf{N}_0} h_k(n) \check{v}(n) = \int_{D_S} \hat{g}_k(x) \, dv(x),$$

we obtain by the dominated convergence theorem $\varphi(\check{v}) = v(\{1\})$.

Now given $f \in l^\infty(\mathbf{N}_0)$ let $\mathcal{O}(f)$ be the weak-$*$ closure of $\{g * f: g \in M^1(\mathbf{N}_0)\}$.

Proposition 5. *Let $f \in l^\infty(\mathbf{N}_0)$ such that the constant function $c \in \mathcal{O}(f)$. Then there exists $\psi \in \mathcal{M}$ such that ψ is invariant on $M(D_S)^\vee$ and $\psi(f) = c$.*

Proof. Let (h_k) be a sequence such that $h_k * f \xrightarrow{k} c$ in the weak-$*$ topology. By Lemma 1 we may assume that in addition $\hat{h}_k(x) \xrightarrow{k} 0$ for $x \neq 1$. Let ψ be a τ-cluster point of (jh_k) in $l^\infty(\mathbf{N}_0)^*$. Then $\psi \in \mathcal{M}$ and for $g \in M^1(\mathbf{N}_0)$, $\nu \in M(D_S)$ we obtain

$$\psi(g * \check{\nu}) = \lim \sum_{n \in \mathbf{N}_0} h_k(n) g * \check{\nu}(n) = \lim \sum_{n \in \mathbf{N}_0} h_k * g(n) \check{\nu}(n) =$$

$$= \lim \int_{D_S} \hat{h}_k(x) \hat{g}(x) d\nu(x) = \nu(\{1\}) = \psi(\check{\nu}).$$

Further $\psi(f) = \lim \sum_{n \in \mathbf{N}_0} h_k(n) f(n) = \lim h_k * f(0) = c$.

Theorem 1. *$M(D_S)^\vee$ is a proper subspace of $l^\infty(\mathbf{N}_0)$, i.e. $\mathbf{N}_0$ is not a Sidon set.*

Proof. We present a function $f \in l^\infty(\mathbf{N}_0)$ such that $\mathcal{O}(f)$ contains the two constants 1 and 0. Then by Proposition 4 and 5 the assertion follows. Let

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 5^i,\ \ 5^i+1, \dots, 5^i+2\cdot 5^i-1 \quad \text{and} \quad n = 0 \\ 0 & \text{if} \quad n = 5^i+2\cdot 5^i, \dots, 5^{i+1}-1, \quad \text{where} \quad i \in \mathbf{N}_0. \end{cases}$$

Let $n_i = 2\cdot 5^i$. One easily obtains that $p_{n_i} * f(m) \xrightarrow{i} 1$. For $n_i = 4 \cdot 5^i$ we have $p_{n_i} * f(m) \xrightarrow{i} 0$. In fact choose $i \in \mathbf{N}$ such that $m+1 \leqq 5^i$.

IV. Orthogonal polynomial sequences admitting only finite Sidon sets

Let A be a finite subset of D_S. Denote $M_A(D_S) = \{\mu \in M(D_S)\colon |\mu|(A) = 0\}$. Obviously $M(D_S) = M(A) \oplus M_A(D_S)$.

Proposition 6. *Assume that there exists a finite subset A of D_S such that $M_A(D_S)^\vee \subseteq c_0(\mathbf{N}_0)$. Then the Sidon sets are exactly the finite subsets of $\mathbf{N}_0$.*

Proof. Assume that $E \subseteq \mathbf{N}_0$ is an infinite Sidon set. Since $M(D_S) = M(A) \oplus \oplus M_A(D_S)$ we obtain that

$$l^\infty(E) = M(S)^\vee | E \subseteq V + c_0(E),$$

where V is a space with dimension at most $|A|$. But E being infinite, $c_0(E)$ has infinite codimension in $l^\infty(E)$.

Assume that for each $x, y \in D_S$ there exists a (not necessarily positive) measure $\mu_{x,y} \in M(D_S)$ such that

(i) $P_n(x) P_n(y) = \int_{D_S} P_n(z) d\mu_{x,y}(z)$,

(ii) $\|\mu_{x,y}\| \leqq M$, M a constant independent of x, y.

Using conditions (i) and (ii) we can show that given $f\in C(D_S)$ the map $(x, y)\to\mu_{x,y}(f)$ is continuous, compare e.g. Proposition 1 of [8]. Hence we can define a "quasi-convolution" of two measures $\mu, \nu\in M(D_S)$ by

$$\mu*\nu(f) = \int_{D_S}\int_{D_S} \mu_{x,y}(f)\,d\mu(x)\,d\nu(y).$$

By (i) $\mu*\nu(P_n)=\mu(P_n)\nu(P_n)$ is valid for each $n\in\mathbf{N}_0$.

We present now examples for which Proposition 6 applies. The Jacobi-polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on $D_S=[-1, 1]$ with respect to $d\pi(x)=(1-x)^\alpha\cdot(1+x)^\beta dx$ (up to normalization). The sequences $(P_n^{(\alpha,\beta)}(x))$ satisfy property (P) for $\alpha\geqq\beta>-1$, $\alpha+\beta+1\geqq 0$, see Chapter 3(a) of [8]. The generalized Tchebichef polynomials $T_n^{(\alpha,\beta)}(x)$ are orthogonal on $D_S=[-1, 1]$ with respect to $d\pi(x)=(1-x^2)^\alpha|x|^{2\beta+1}dx$ (up to normalization) and satisfy property (P) for $\beta>-1$, $\alpha\geqq\beta+1$, see Chapter 3(f) of [8]. Finally we consider polynomials $G_n^a(x)$ studied by Geronimus. They are orthogonal on $D_S=[-1, 1]$ with respect to $(1-x^2)^{1/2}/(1-\mu x^2)$, $\mu=a-a^2/4$ and satisfy property (P) for $a\geqq 2$, see Chapter 3(g)(i) of [8].

Theorem 2. *The set $E\subseteqq\mathbf{N}_0$ is a Sidon set if and only if E is finite in case*

(a) $P_n(x)=P_n^{(\alpha,\beta)}(x)$ *the Jacobi polynomials with* $\alpha\geqq\beta>-1$, $\alpha+\beta+1\geqq 0$ *and* $\alpha\neq -1/2$.

(b) $P_n(x)=T_n^{(\alpha,\beta)}(x)$ *the generalized Tchebichef polynomials with* $\beta>-1$, $\alpha\geqq\beta+1$.

(c) $P_n(x)=G_n^a(x)$ *with* $a>2$.

Proof. (a) Fix $\alpha,\beta\in\mathbf{R}$ such that $\alpha\geqq\beta>-1$, $\alpha+\beta+1\geqq 0$ and choose $A=\{-1, 1\}$. By Gasper's theorem of [4] and (2.3), (2.4) of [7] there exist for $x, y\in D_S=[-1, 1]$ measures $\mu_{x,y}\in M(D_S)$ such that the above conditions (i) and (ii) are satisfied. First consider the case $\alpha+\beta+1>0$. If $x, y\in]-1, 1[$ then $d\mu_{x,y}(z)=K(x, y, z)\,d\pi(z)$, see [4]. Let $\mu\in M_A(D_S)$. We show that $\mu*\mu\in L^1(D_S, \pi)$. Let $B\subseteqq D_S$ be a Borel set such that $\pi(B)=0$. Then

$$|\mu*\mu(B)| \leqq \int_{D_S\setminus A}\int_{D_S\setminus A} |\mu_{x,y}(B)|\,d|\mu|(x)\,d|\mu|(y)+$$
$$+\int_A\int_{D_S}|\mu_{x,y}(B)|\,d|\mu|(x)\,d|\mu|(y)+\int_{D_S}\int_A|\mu_{x,y}(B)|\,d|\mu|(x)\,d|\mu|(y).$$

Since $\mu\in M_A(D_S)$, the second and third integrals are zero. Since the measures $\mu_{x,y}$, $-1<x, y<1$, are absolutely continuous, the first integral is zero. Hence $\check\mu\check\mu=(\mu*\mu)\check{}\in c_0(\mathbf{N}_0)$ by Proposition 1 of [10]. Then obviously $\check\mu\in c_0(\mathbf{N}_0)$ and Proposition 6 applies. If $\beta>-1$, $\alpha>-1/2$ and $\alpha+\beta+1=0$ then given $x, y\in]-1, 1[$ we have $d\mu_{x,y}(z)=K(x, y, z)\,d\pi(z)+d\nu_{x,y}(z)$, where $\nu_{x,y}=0$ if $x\neq -y$ and $\nu_{x,y}=p_{-1}/2$ if $x=-y$. If $\mu\in M_A(D_S)$ and $\pi(B)=0$ we obtain now $\mu*\mu(B)-$

$-cp_{-1}(B)=0, \quad c=\int\limits_{D_S\setminus A}\int\limits_{D_S\setminus A} v_{x,y}(\{-1\})d\mu(x)d\mu(y)$. Hence $\check{\mu}\check{\mu}-c\alpha_{-1}\in c_0(\mathbf{N}_0)$.
Using the recurrence formula of $P_n(x)$ and $\alpha+\beta+1=0$ an induction argument shows that

$$\alpha_{-1}(n)=P_n(-1)=\prod_{k=0}^{n-1}(\alpha-k)/\prod_{k=1}^{n}(\alpha+k), \quad n\in\mathbf{N}.$$

Hence $\lim|\alpha_{-1}(n)|=(\Gamma(1-|\alpha|)/\Gamma(|\alpha|))\lim\Gamma(|\alpha|+n)/\Gamma(1-|\alpha|+n)=0$, because of $-1/2<\alpha<0$. Thus $\alpha_{-1}\in c_0(\mathbf{N}_0)$ and consequently $\check{\mu}\in c_0(\mathbf{N}_0)$.

(b) Choose again $A=\{-1, 1\}$. Using Theorem 1 of [7] an argument as in (a) yields that $M_A(D_S)^{\vee}\subseteq c_0(\mathbf{N}_0)$.

(c) Derive from [8] or from Chapter VI, (13.4) of [1] that

$$G_n^a(x)=(a/(n(a-2)+2))P_n^{(-1/2,-1/2)}(x)+(((a-2)(n-1))/(n(a-2)+2))P_n^{(1/2,1/2)}(x).$$

For $A=\{-1, 1\}$ and $\mu\in M_A(D_S)$ we have

$$\check{\mu}(n)=(a/(n(a-2)+2))\int\limits_{D_S}P_n^{(-1/2,-1/2)}(x)d\mu(x)+$$
$$+(((a-2)(n-1))/(n(a-2)+2))\int\limits_{D_S}P_n^{(1/2,1/2)}(x)d\mu(x).$$

Since $|\int\limits_{D_S}P_n^{(-1/2,-1/2)}(x)d\mu(x)|\leqq\|\mu\|$ for $n\in\mathbf{N}_0$ and $\int\limits_{D_S}P_n^{(1/2,1/2)}(x)d\mu(x)\xrightarrow{n}0$ by (a), we have $\check{\mu}\in c_0(\mathbf{N}_0)$ provided $a>2$.

Remark. The assertion of Theorem 2(a) follows by Theorem 3.2 of [14] and Chapter 4(a) of [8] provided we require that in addition $\beta\geqq-1/2$ or $\alpha+\beta\geqq0$.

V. Infinite Sidon sets

Finally we consider orthogonal polynomial sequences $(P_n(x))$ having infinite Sidon sets. Let $E=\{n_1, n_2, \dots\}\subseteq\mathbf{N}_0$ and m, $N\in\mathbf{N}$, $N\geqq m$. Denote by

$$E_N^m=\{p_{n_{i_1}}*p_{n_{i_2}}*\dots*p_{n_{i_m}}\in M^1(\mathbf{N}_0)\colon 1\leqq i_1<i_2<\dots<i_m\leqq N\}$$

and call E a *Rider set* if there exists a constant $B\geqq1$ such that

$$\lim_N\sum_{\mu\in E_N^m}\mu(\{0\})\leqq B^m \quad \text{for each } m\in\mathbf{N}.$$

Lemma 2. *Let E be a Rider set. There exists a constant $C\geqq1$ such that $\lim_N\sum_{\mu\in E_N^m}\mu(\{k\})\leqq C^m h(k)$ for each $k\in\mathbf{N}_0$, $m\in\mathbf{N}$.*

Proof. Write $E=\{n_1, n_2, \dots\}$. Let $\beta=1/(2B)$, where B is the constant of the Rider set E. Consider for $N\in\mathbf{N}$ the Riesz products

$$R_N(x) = \prod_{k=1}^{N} (1+\beta P_{n_k}(x)).$$

Obviously $R_N(x)=1+\sum_{k=0}^{\infty} c_N(k) P_k(x)$, where $c_N(k)=\sum_{m=1}^{N} \Big(\sum_{\mu\in E_N^m} \mu(\{k\})\Big)\beta^m$. Now conclude as in the proof of Lemma 3.1 of [12], compare also [11, p. 28], that

$$\|R_N\|_1 = 1+c_N(0) \leqq 1+\sum_{m=1}^{\infty} 2^{-m} = 2.$$

Hence for $k\in\mathbf{N}$, $c_N(k)/h(k)=|R_N^{\vee}(k)|\leqq\|R_N\|_1\leqq 2$ and then $\lim_N \sum_{\mu\in E_N^m} \mu(\{k\})\leqq$ $\leqq(2B)^m 2h(k)\leqq C^m h(k)$, where $C=4B$.

Lemma 3. *Let E be a Rider set with $0\notin E$, and let $C\geqq 1$ be the corresponding constant of Lemma 2. Let $0<\varepsilon<1$. Given $\varphi\colon E\to\mathbf{R}$, $\|\varphi\|_E\leqq 1$ there exists a positive measure $\mu\in M(S)$ such that $\|\mu\|\leqq\varepsilon+2C^2/\varepsilon$, $|\check{\mu}(k)|\leqq\varepsilon$ for each $k\notin E\cup\{0\}$ and $|\check{\mu}(k)-\varphi(k)/h(k)|\leqq\varepsilon$ for $k\in E$.*

Proof. We have again to modify the arguments of Theorem 3.2 in [12] or of [11, p. 28—29]. Write $E=\{n_1, n_2, \dots\}$. Let $\beta=\varepsilon/(2C^2)$ and

$$R_N(x) = \prod_{k=1}^{N} (1+\beta\varphi(n_k) P_{n_k}(x)) = 1+\sum_{k=1}^{N} \beta\varphi(n_k) P_{n_k}(x)+\sum_{k=0}^{\infty} d_N(k) P_k(x),$$

where $|d_N(k)|\leqq \sum_{m=2}^{N} \Big(\sum_{\mu\in E_N^m} \mu(\{k\})\Big)\beta^m$. Then

$$\|R_N\|_1 \leqq 1+|d_N(0)| \leqq 1+\sum_{m=2}^{\infty} (C\beta)^m \leqq 1+\varepsilon\beta.$$

Further for $n_k\in E$ and $N\geqq k$

$$|h(n_k) R_N^{\vee}(n_k)-\beta\varphi(n_k)| \leqq |d_N(n_k)| \leqq \sum_{m=2}^{\infty} (C\beta)^m h(n_k) \leqq \varepsilon\beta h(n_k).$$

For $n\notin E\cup\{0\}$ we have $|R_N^{\vee}(n)|=|d_N(n)|/h(n)\leqq\varepsilon\beta$. Alaoglu's theorem and a normalization by $1/\beta$ yields the appropriate positive measure μ.

Theorem 3. *Let E be a finite union of Rider sets and assume that* $\sup\{h(k)\colon k\in E\}<\infty$. *Then E is a Sidon set.*

Proof. Assume $0\notin E$. As in [11, pp. 29—30] one obtains that Lemma 3 is valid for a finite union of Rider sets. Let $M=\sup\{h(k)\colon k\in E\}$. Given $\psi\colon E\to\{-1, 1\}$ consider $\varphi\colon E\to\mathbf{R}$, $\varphi(n)=\psi h(n)/M$, $n\in E$. There exists a positive measure $\mu\in M(S)$

such that

$$|\check{\mu}(k)-\psi(k)/M| \leqq 1/(2M) \quad \text{for} \quad k\in E.$$

Define $\nu=M\mu$. By Proposition 1(f) E is a Sidon set. Finally let $0\in E$. Given $\psi\colon E\to\{-1, 1\}$ we know that there exists a measure such that $|\check{\mu}(k)-\psi(k)|\leqq 1/2$ for $k\in E\setminus\{0\}$. Define $\alpha=\psi(0)-\check{\mu}(0)$. Replace μ by $\mu+\alpha\pi$ establishing that E is a Sidon set.

Corollary. *Assume that* $\sup\{h(k)\colon k\in\mathbf{N}\}<\infty$. *If* $E=\{n_1, n_2, \dots\}\subseteq\mathbf{N}$ *satisfies* $n_{k+1}/n_k\geqq q$ *for* $k\in\mathbf{N}$, *where* $q>1$ *is a constant, then E is a Sidon set.*

Proof. It is sufficient to consider the case $q\geqq 3$, compare e.g. [11, p. 23]. Then for $m, N\in\mathbf{N}$, $2\leqq m\leqq N$ and $1\leqq n_{i_1}<n_{i_2}<\dots<n_{i_m}\leqq N$ we obtain $n_{i_m}-(n_{i_{m-1}}+\dots+n_{i_1})\geqq n_{i_m}(q-2)/(q-1)\geqq 0$. Hence $0\notin\operatorname{supp} p_{n_{i_1}}*\dots*p_{n_{i_m}}$. Consequently E is a Rider set.

We present now examples with bounded Haar function h (and property (P)). Of course the Tchebichef polynomials of first kind, $T_n(x)=P_n^{(-1/2, -1/2)}(x)=\cos n\varphi$, $\cos\varphi=x$, $\varphi\in[0, \pi]$, have the Haar function $h(0)=1$, $h(n)=2$, $n\in\mathbf{N}$. A class containing $(T_n(x))$ is studied in Chapter 3(g) (ii) of [8]. For $a\geqq 2$ these polynomials $T_n(x; a)$ have the representation

$$T_1(x; a) = x, \quad T_n(x; a) = \big(a/2(a-1)\big)T_n(x)+\big((a-2)/2(a-1)\big)T_{n-2}(x), \quad n \geqq 2.$$

The Haar function is $h(0)=1$, $h(1)=a$, $h(n)=2(a-1)$, $n\geqq 2$. We introduce an extension depending on two parameters $a, b\geqq 2$. Let $a_1=(a-1)/a$, $c_1=1/a$, $a_2=(b-1)/b$, $c_2=1/b$, $a_n=c_n=1/2$ if $n=3, 4, \dots$ and $b_n=0$, $n\in\mathbf{N}$. Further let $a_0=1$, $b_0=0$. By the recursion formula, see Section I, there is defined an orthogonal polynomial sequence $(T_n(x; a, b))$ with the representation

$$T_1(x; a, b) = x, \quad T_2(x; a, b) = \big(a/2(a-1)\big)T_2(x)+\big((a-2)/2(a-1)\big)T_0(x),$$

$$T_3(x; a, b) = \big(ab/4(a-1)(b-1)\big)T_3(x)+$$

$$+\big(((a-2)(b-2)+(a-2)b+(b-2)a)/4(a-1)(b-1)\big)T_1(x),$$

$$T_n(x; a, b) = \big(ab/4(a-1)(b-1)\big)T_n(x)+\big(2(a-2)/4(a-1)\big)T_{n-2}(x)+$$

$$+\big(a(b-2)/4(a-1)(b-1)\big)T_{n-4}(x) \quad \text{if} \quad n = 3, 4, \dots$$

$(T_n(x; a, b))$ satisfies property (P). In fact the coefficients $g(m, n, n+m-k)$, $m\leqq n$, $0\leqq k\leqq 2m$, can be computed directly using formula (1) of [8]. Obviously $g(m, n, n+m-k)=0$ for $k=1, 3, \dots, 2m-1$. We omit the coefficients $g(m, n, n+m-k)$

for $m=2,3,4$ noting only their positivity. The general formulas for $5\leqq m\leqq n$ are

$$8(a-1)(b-1)\cdot g(m,n,n+m-2k)=\begin{cases} ab & \text{if } k=0\\ (a-2)(b-2)+b(a-2) & \text{if } k=1\\ a(b-2) & \text{if } k=2\\ 0 & \text{if } k=3,\dots,m-3\end{cases}$$

and for $k=m-2$

$$8(a-1)(b-1)\cdot g(m,n,n-m+4)=\begin{cases} 2a(b-2) & \text{if } n=m\\ a(b-2) & \text{if } n=m+1,\dots,\end{cases}$$

for $k=m-1$

$$8(a-1)(b-1)\cdot g(m,n,n-m+2)=$$

$$=\begin{cases} 2b(a-2) & \text{if } n=m\\ (a-2)(b-2)+b(a-2)+a(b-2) & \text{if } n=m+1\\ (a-2)(b-2)+b(a-2) & \text{if } n=m+2,\dots,\end{cases}$$

and for $k=m$

$$8(a-1)(b-1)\cdot g(m,n,n-m)=\begin{cases} 4 & \text{if } n=m\\ 2a & \text{if } n=m+1\\ ab & \text{if } n=m+2,\dots.\end{cases}$$

The Haar function is

$$h(0)=1,\quad h(1)=a,\quad h(2)=b(a-1),\quad h(n)=2(a-1)(b-1)\quad \text{if}\quad n=3,4,\dots.$$

Remark. The above example suggests the way how to define a general class of polynomials depending on an arbitrary number of parameters and having bounded Haar function. A study of this class, such as representations and orthogonality relations, will be given in another paper.

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials,* Gordon and Breach (New York, 1978).

[2] N. Dunford and J. T. Schwartz, *Linear Operators. I: General Theory,* Interscience (New York, 1958).

[3] C. F. Dunkl and D. E. Ramirez, Existence and nonuniqueness of invariant means on $\mathcal{L}^\infty(\hat{G})$, *Proc. Amer. Math. Soc.,* **32** (1972), 525—530.

[4] G. Gasper, Banach algebras for Jacobi series and positivity of a kernel, *Ann. of Math.,* **95** (1972), 261—280.

[5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, II,* Springer-Verlag (Berlin, 1970).

[6] R. I. Jewett, Spaces with an abstract convolution of measures, *Adv. in Math.,* **18** (1975), 1—101.

[7] T. P. Laine, The product formula and convolution structure for the generalized Chebyshev polynomials, *SIAM J. Math. Anal.*, **11** (1980), 133—146.
[8] R. Lasser, Orthogonal polynomials and hypergroups, *Rend. Mat.*, **3** (1983), 185—209.
[9] R. Lasser, Bochner theorems for hypergroups and their applications to orthogonal polynomial expansions, *J. Approx. Theory*, **37** (1983), 311—325.
[10] R. Lasser, Fourier—Stieltjes transforms on hypergroups, *Analysis*, **2** (1982), 281—303.
[11] J. M. Lopez and K. A. Ross, *Sidon sets*, Marcel Dekker (New York, 1975).
[12] W. A. Parker, Central Sidon and central Λ_p sets, *J. Austral. Math. Soc.*, **14** (1972), 62—74.
[13] R. C. Vrem, Lacunarity on compact hypergroups, *Math. Z.*, **164** (1978), 93—104.
[14] R. C. Vrem, Continuous measures and lacunarity on hypergroups, *Trans. Amer. Math. Soc.*, **269** (1982), 549—556.

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