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Some discrete inequalities of Opial's type

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1. Introduction

Let us given an index set $I=\{1, 2, \dots, n\}$ and weight sequences $\mathbf{r}=(r_k)_{k\in I}=$ $=(r_1, \dots, r_n)$ and $\mathbf{p}=(p_k)_{k\in I}=(p_1, \dots, p_n)$. For a sequence $\mathbf{x}=(x_k)_{k\in I}=(x_1, \dots, x_n)$

(1) $$\|\mathbf{x}\|_{\mathbf{r}} = \Big(\sum_{k=1}^{n} r_k x_k^2\Big)^{1/2}$$

and

(2) $$(\mathbf{x}, \nabla\mathbf{x}) = \sum_{n=1}^{n} p_k x_k \nabla x_k,$$

where the sequence $\nabla\mathbf{x}$ is given by $\nabla\mathbf{x}=(x_1, x_2-x_1, \dots, x_n-x_{n-1})$. If we put $x_0=0$ and $\nabla x_k=x_k-x_{k-1}$ $(k=1, \dots, n)$, then the sequence $\nabla\mathbf{x}$ can be expressed in the form $\nabla\mathbf{x}=(\nabla x_1, \nabla x_2, \dots, \nabla x_n)$.

In this paper we determine the best constants A_n and B_n in the inequalities

(3) $$A_n\|\mathbf{x}\|_{\mathbf{r}}^2 \leqq (\mathbf{x}, \nabla\mathbf{x}) \leqq B_n\|\mathbf{x}\|_{\mathbf{r}}^2,$$

which are a discrete analogue of inequalities of Opial's type (see, for example, [1, pp. 154—162]). The idea for this paper came from the papers [2] and [3].

2. Main results

Theorem. *Define a sequence* $(Q_k(x))$ *of polynomials for the given weight sequences* $\mathbf{r}$ *and* $\mathbf{p}$ *using the recursive relation*

(4) $$xQ_{k-1}(x) = b_kQ_k(x)+a_kQ_{k-1}(x)+b_{k-1}Q_{k-2}(x) \quad (k = 1, 2, \dots),$$
$$Q_0(x) = Q_0 \neq 0, \quad Q_{-1}(x) \overset{\text{def}}{=} 0,$$

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where

(5) $a_k = (p_k/r_k)$ $(k = 1, \ldots, n)$ *and* $b_k = -(p_{k+1}/2\sqrt{r_k r_{k+1}})$ $(k = 1, \ldots, n-1)$.

For each sequence $\mathbf{x}=(x_k)_{k\in I}$ *of real numbers the inequalities* (3) *hold, where* A_n *and* B_n *are the minimum and the maximum zeros of polynomial* $Q_n(x)$, *respectively.*

Equality holds in the left-hand (right-hand) inequality in (3) *if and only if* $x_k = (C/\sqrt{r_k})Q_{k-1}(\lambda)$ $(k=1, \ldots, n)$, *where* $\lambda=A_n$ $(\lambda=B_n)$ *and* C *is an arbitrary real constant different from zero.*

Proof. Let X be an n-dimensional euklidean space with scalar product $(\vec{z}, \vec{w}) = \sum_{k=1}^{n} z_k w_k$, where $\vec{z}=[z_1, \ldots, z_n]^{\mathrm{T}}$ and $\vec{w}=[w_1, \ldots, w_n]^{\mathrm{T}}$. Let, further, $\mathbf{a}=(a_1, \ldots, a_n)$, $\mathbf{b}=(b_1, \ldots, b_{n-1})$, and define a three-diagonal matrix by

$$H_n(\mathbf{a}, \mathbf{b}) = \begin{bmatrix} a_1 & b_1 & 0 & \ldots & 0 & 0 \\ b_1 & a_2 & b_2 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & & b_{n-1} & a_n \end{bmatrix}.$$

Introducing $z_k=\sqrt{r_k}\,x_k$ $(k=1, \ldots, n)$, from (1) and (2) we get

$$\|\mathbf{x}\|_r^2 = \sum_{k=1}^{n} r_k x_k^2 = \sum_{k=1}^{n} z_k^2 = (\vec{z}, \vec{z}),$$

and

$$(\mathbf{x}, \nabla\mathbf{x}) = \sum_{k=1}^{n} p_k x_k \nabla x_k = \sum_{k=1}^{n} (p_k z_k/\sqrt{r_k})\nabla(z_k/\sqrt{r_k}) =$$

$$= (p_1 z_1^2/r_1) + \sum_{k=2}^{n} (p_k z_k/r_k\sqrt{r_{k-1}})(\sqrt{r_{k-1}}\, z_k - \sqrt{r_k}\, z_{k-1}).$$

Thus by (5),

$$(\mathbf{x}, \nabla\mathbf{x}) = (H_n(\mathbf{a}, \mathbf{b})\vec{z}, \vec{z}).$$

On the other hand, let us consider the sequence $(Q_k(x))$ of polynomials defined by (4). For $k=1, 2, \ldots, n$, we obtain from (4) the equality

$$x\vec{v} = H_n(\mathbf{a}, \mathbf{b})\vec{v} + b_n Q_n(x)\vec{e}, \tag{6}$$

where $\vec{v}=[Q_0(x), Q_1(x), \ldots, Q_{n-1}(x)]^{\mathrm{T}}$ and $\vec{e}=[0, 0, \ldots, 0, 1]^{\mathrm{T}}$. Setting $x=\lambda$ in (6), we conclude: If λ is such that $Q_n(\lambda)=0$, then λ is an eigenvalue of the matrix $H_n(\mathbf{a}, \mathbf{b})$ and $\vec{v}=[Q_0(\lambda), Q_1(\lambda), \ldots, Q_{n-1}(\lambda)]^{\mathrm{T}}$ is the corresponding eigenvector of the matrix $H_n(\mathbf{a}, \mathbf{b})$, and conversely, according to (6), if λ is an eigenvalue of the matrix $H_n(\mathbf{a}, \mathbf{b})$, then $Q_n(\lambda)=0$, i.e. λ is a zero of the polynomial $Q_n(x)$.

Thus, the eigenvalues of the matrix $H_n(\mathbf{a}, \mathbf{b})$ are exactly the zeros the of polynomial $Q_n(x)$. Since $H_n(\mathbf{a}, \mathbf{b})$ is a three-diagonal matrix ($b_i^2>0$, $i=1, \ldots, n-1$) all its eigenvalues λ_i $(i=1, \ldots, n)$ are real and distinct, and

$$A_n(\vec{z}, \vec{z}) \leqq \left(H_n(\mathbf{a}, \mathbf{b})\vec{z}, \vec{z}\right) \leqq B_n(\vec{z}, \vec{z})$$

hold, with equality for eigenvectors corresponding to the eigenvalues $A_n=\min \lambda_i$, $B_n=\max \lambda_i$.

This completes the proof of the theorem.

Corollary 1. *Let the sequences* $\mathbf{r}$ *and* $\mathbf{p}$ *be given recursively by*

$$r_{k+1} = \left(4k(k+s)/(2k+s+1)^2\right)r_k \quad (k = 1, \ldots, n-1),$$
$$p_k = (2k+s-1)r_k \quad (k = 1, \ldots, n),$$

with $r_1=1$ *and* $s>-1$. *Then for every sequence* $\mathbf{x}=(x_k)_{k\in I}$ *of real numbers the inequalities* (3) *hold, where* A_n *and* B_n *are the minimal and the maximal zeros of the normalized generalized Laguerre polynomials* $\bar{L}_n^s(x)=L_n^s(x)/\|L_n^s\|$. *Here*

$$L_n^s(x) = \sum_{m=0}^{n} \binom{n+s}{n-m}\left((-x)^m/m!\right) \quad \text{and} \quad \|L_n^s\| = \sqrt{\Gamma(n+s+1)/n!}.$$

Equality holds in the left-hand (right-hand) inequality in (3) *if and only if* $x_k= =(C_k/\sqrt{r_k})L_{k-1}^s(\lambda)$ $(k=1, \ldots, n)$, *where* $\lambda=A_n$ $(\lambda=B_n)$ *and* $C(\neq 0)$ *is an arbitrary constant.*

Proof. For the proof of this result it is enough to show that in this case (4) reduces to the recurrence relation for generalized Laguerre polynomials. Since

$$a_k = (p_k/r_k) = 2k+s-1 \quad \text{and} \quad b_k = -\left(p_{k+1}/2\sqrt{r_k r_{k+1}}\right) = -\sqrt{k(k+s)},$$

(4) becomes

$$xQ_{k-1}(x) = -\sqrt{k(k+s)}Q_k(x)+(2k+s-1)Q_{k-1}(x)-\sqrt{(k-1)(k+s-1)}Q_{k-2}(x),$$

which is the recurrence relation for normalized generalized Laguerre polynomials $(Q_k(x)=\bar{L}_k^s(x))$.

In the special case $p_k=r_k=1$ $(k=1, \ldots, n)$, we have the following result:

Corollary 2. *For every sequence* $\mathbf{x}=(x_k)_{k\in I}$ *of real numbers and for* $x_0=0$, *the inequalities*

$$2\sin^2\left(\pi/2(n+1)\right)\sum_{k=1}^{n} x_k^2 \leqq \sum_{k=1}^{n} x_k(x_k-x_{k-1}) \leqq 2\cos^2\left(\pi/2(n+1)\right)\sum_{k=1}^{n} x_k^2, \tag{7}$$

are valid.

Equality holds in the left-hand inequality if and only if $x_k=C\sin(k\pi/(n+1))$ $(k=1,\ldots,n)$, *where* $C=const\neq 0$, *and in the right-hand inequality if and only if* $x_k=(-1)^{k-1}C\sin(k\pi/(n+1))$, $(k=1,\ldots,n)$, *where* $C=const\neq 0$.

Proof. In this case, we have $a_k=1$, $b_k=-1/2$ and

$$xQ_{k-1}(x)=-(1/2)Q_k(x)+Q_{k-1}(x)-(1/2)Q_{k-2}(x), \tag{8}$$

where $Q_0(x)$ can be $Q_0(x)=1$. If we put $t=1-x$, one can easily obtain the solution of the difference equation (8), for example for $|t|<1$, i. e. $0<x<2$,

$$Q_k(x)=\left(\sin(k+1)\theta/\sin\theta\right)\quad(k=1,\ldots,n), \tag{9}$$

where $e^{i\theta}=t+i\sqrt{1-t^2}$. Then, from $Q_n(x)=0$ it follows $\lambda_k=2\sin^2(k\pi/2(n+1))$ $(k=1,\ldots,n)$, implying

$$A_n=\min_k\lambda_k=2\sin^2(\pi/2(n+1))\quad\text{and}\quad B_n=\max_k\lambda_k=2\cos^2(\pi/2(n+1)).$$

Using (9) the conditions for equality are simply obtained.

Also we note that the inequalities (7) can be written in the form

$$-\cos(\pi/(n+1))\sum_{k=1}^{n}x_k^2\leqq\sum_{k=2}^{n}x_kx_{k-1}\leqq\cos(\pi/(n+1))\sum_{k=1}^{n}x_k^2,$$

i.e.,

$$\Big|\sum_{k=2}^{n}x_kx_{k-1}\Big|\leqq\cos(\pi/(n+1))\sum_{k=1}^{n}x_k^2. \tag{10}$$

Remark. The inequality (10) is related to an extremal problem occurring in the investigation of approximative properties of positive polynomial operators. Namely, let C_m be the class of all nonnegative trigonometric polynomials of order m

$$T_m(t)=1+2a_1\cos t+\ldots+2a_m\cos mt. \tag{11}$$

The problem is to determine a polynomial $T_m^*\in C_m$ which has the greatest coefficient a_1 (see, for example, [4, pp. 113—115]). If the polynomial (11) is written in the form

$$T_m(t)=|x_1+x_2e^{it}+\ldots+x_{m+1}e^{imt}|=\sum_{k=1}^{m+1}x_k^2+2\Big(\sum_{k=2}^{m+1}x_kx_{k-1}\Big)\cos t+\ldots,$$

where x_k $(k=1,\ldots,m+1)$ are real numbers, the determination of T_m^* is reduced to finding

$$\sup a_1=\sup\sum_{k=2}^{m+1}x_kx_{k-1},\quad\sum_{k=1}^{m+1}x_k^2=1.$$

Putting $n=m+1$ in (10), we have $\sup a_1=\cos(\pi/(m+2))$.

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