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Which $C_{.0}$ contraction is quasi-similar to its Jordan model?

PEI YUAN WU*

Dedicated to Professor Béla Szőkefalvi-Nagy on his 71st birthday

For certain $C_{.0}$ contractions on a Hilbert space, a Jordan model has been obtained by B. SZ.-NAGY [3] (cf. also [5]). It was shown that a $C_{.0}$ contraction T with the defect index $d_T = \operatorname{rank}(I-T^*T)^{1/2}$ finite is completely injection-similar to a unique *Jordan operator* of the form $J=S(\varphi_1)\oplus\ldots\oplus S(\varphi_k)\oplus S_l$, where φ_j's are non-constant inner functions satisfying $\varphi_j|\varphi_{j-1}$, $S(\varphi_j)$ denotes the compression of the unilateral shift $S(\varphi_j)f=P_j(e^{it}f)$ for $f\in H^2\ominus\varphi_jH^2$, P_j being the (orthogonal) projection onto $H^2\ominus\varphi_jH^2$, $j=1,\ldots,k$, and S_l denotes the unilateral shift on H_l^2. Moreover, if $n=d_T$ and $m=d_{T^*}=\operatorname{rank}(I-TT^*)^{1/2}$, then $k\leqq n$ and $l=m-n$. It is known that in general T is not quasi-similar to J even when $m<\infty$. (For an example, see [5], pp. 321—322.) In this paper, we find necessary and sufficient conditions under which they are quasi-similar at least in the case when both defect indices of T are finite. The main result (Theorem 2 below) is a generalization of the corresponding result for C_{10} contractions (cf. [13], Lemma 1). We also obtain other auxiliary results concerning the invariant subspaces and approximate decompositions of $C_{.0}$ contractions. Our treatments of contractions will be based on the dilation theory developed by B. Sz.-Nagy and C. Foiaş. The main reference is their book [4].

Recall that for operators T_1 and T_2 on H_1 and H_2, respectively, $T_1\overset{\text{i}}{\prec}T_2$ (resp. $T_1\overset{\text{d}}{\prec}T_2$) denotes that there exists an operator $X\colon H_1\to H_2$ which is injective (resp. has dense range) such that $T_2X=XT_1$. If X is both injective and with dense range (called *quasi-affinity*), then we denote this by $T_1\prec T_2$. T_1 is *quasi-similar* to T_2 $(T_1\sim T_2)$ if $T_1\prec T_2$ and $T_2\prec T_1$. $T_1\overset{\text{ci}}{\prec}T_2$ denotes that there exists a family of injections $\{X_\alpha\}$ such that $T_2X_\alpha=X_\alpha T_1$ for each α and $\bigvee_\alpha X_\alpha H_1=H_2$. T_1 is *completely injection-similar* to T_2 $(T_1\overset{\text{ci}}{\sim}T_2)$ if $T_1\overset{\text{ci}}{\prec}T_2$ and $T_2\overset{\text{ci}}{\prec}T_1$.

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We start by proving the following preliminary lemma.

Lemma 1. *Let T and S be $C_{\cdot 0}$ contractions with finite defect indices on H and K, respectively. Let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ on $H=H_1 \oplus H_2$ and $S=\begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ on $K=K_1 \oplus K_2$ be the triangulations of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. If $T \sim S$, then $T_1 \sim S_1$ and $T_2 \sim S_2$.*

Proof. Let $X: H \to K$ be a quasi-affinity which intertwines T and S. Since $H_1=\{x \in H: T^n x \to 0 \text{ as } n \to \infty\}$ and $K_1=\{y \in K: S^n y \to 0 \text{ as } n \to \infty\}$, it is easily seen that $XH_1 \subseteq K_1$. Hence X can be triangulated as $X=\begin{bmatrix} X_1 & * \\ 0 & X_2 \end{bmatrix}$. Note that X_1 is an injection which intertwines T_1 and S_1. Thus $T_1 \overset{i}{\prec} S_1$. On the other hand, X_2 has dense range and intertwines T_2 and S_2 whence $T_2 \overset{d}{\prec} S_2$. Similarly, from $S \prec T$ we infer that $S_1 \overset{i}{\prec} T_1$ and $S_2 \overset{d}{\prec} T_2$. Hence $T_1 \sim S_1$ and $T_2 \sim S_2$ as asserted (cf. [6], Theorem 1 (a) and [11], Theorem 2.11).

Now we are ready to prove our main result.

Theorem 2. *Let T be a $C_{\cdot 0}$ contraction with finite defect indices on H and let $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ on $H=H_1 \oplus H_2$ be the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. Then the following statements are equivalent:*

(1) T is quasi-similar to its Jordan model;

(2) T_2 is quasi-similar to a unilateral shift;

(3) there exists a bounded analytic function Ω such that $\Omega \Theta_{*e}=\delta I$ for some outer function δ, where Θ_{*e} is the $*$-outer factor of the characteristic function Θ_T of T.

Moreover, under these conditions we have $T \sim T_1 \oplus S_{m-n}$ $(m=d_{T^*},\ n=d_T)$ and $T \sim T_1 \oplus T_2$ and there exist quasi-affinities $X: H \to H_1 \oplus H^2_{m-n}$ and $Y: H_1 \oplus H^2_{m-n} \to H$ intertwining T and $T_1 \oplus S_{m-n}$ and quasi-affinities $Z: H \to H_1 \oplus H_2$ and $W: H_1 \oplus H_2 \to H$ intertwining T and $T_1 \oplus T_2$ such that $XY=\delta(T_1 \oplus S_{m-n})$, $YX=\delta(T)$, $ZW=\delta(T_1 \oplus T_2)$ and $WZ=\delta(T)$.

Proof. (1)⇒(2). Let $J=J_1 \oplus J_2$ be the Jordan model of T, where $J_1=S(\varphi_1) \oplus \ldots \oplus S(\varphi_k)$ and $J_2=S_{m-n}$. Certainly, $J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ is the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$. By Lemma 1, $T \sim J$ implies $T_1 \sim J_1$ and $T_2 \sim J_2 = S_{m-n}$.

(2)⇒(3). Let $\Theta_T=\Theta_{*e}\Theta_{*i}$ be the $*$-canonical factorization of Θ_T. Then the characteristic function of T_2 coincides with the purely contractive part Θ^0_{*e} of Θ_{*e}. By [13], Lemma 1, there exists a bounded analytic function Ω^0 and an outer function δ^0 such that $\Omega^0 \Theta^0_{*e}=\delta^0 I$. Condition (3) follows immediately.

(3)⇒(1). Note that Ω must be an outer function since $\overline{\Omega H_m^2} \supseteqq \overline{\Omega \Theta_{*e} H_n^2} = \overline{\delta H_n^2} = H_n^2$ implies that $\overline{\Omega H_m^2} = H_n^2$. Consider the operator Ω_+ from H_m^2 to H_n^2 defined by $\Omega_+ f = \Omega f$ for $f \in H_m^2$. Let $K = \ker \Omega_+$. Then K is an invariant subspace for S_m, the unilateral shift on H_m^2. It follows that $K = \Phi H_l^2$ for some inner function Φ, where $0 \leqq l \leqq m$. We consider the functional model of T, that is, consider T as the operator defined on $H = H_m^2 \ominus \Theta_T H_n^2$ by $Tf = P(e^{it} f)$ for $f \in H$, where P denotes the (orthogonal) projection onto H. Similarly, consider T_1 as defined on $H_1 = H_n^2 \ominus \Theta_{*i} H_n^2$ by $T_1 g = P_1(e^{it} g)$ for $g \in H_1$, where P_1 denotes the (orthogonal) projection onto H_1. (Here T_1 is unitarily equivalent to the $C_{0\cdot}$ part of T.) Now define $X: H \to H_1 \oplus H_l^2$ by

$$Xf = P_1(\Omega f) \oplus \Phi^*(\delta f - \Theta_{*e} \Omega f) \quad \text{for } f \in H.$$

Note that

$$\Omega(\delta f - \Theta_{*e} \Omega f) = \delta \Omega f - \Omega \Theta_{*e} \Omega f = \delta \Omega f - \delta \Omega f = 0.$$

Hence $\delta f - \Theta_{*e} \Omega f$ is in $\ker \Omega_+ = K = \Phi H_l^2$. Thus $\Phi^*(\delta f - \Theta_{*e} \Omega f)$ is indeed in H_l^2. Next define $Y: H_1 \oplus H_l^2 \to H$ by

$$Y(g \oplus h) = P(\Theta_{*e} g + \Phi h) \quad \text{for } g \oplus h \in H_1 \oplus H_l^2.$$

It is easily verified that X and Y intertwine T and $T_1 \oplus S_l$. Moreover, for $g \oplus h \in H_1 \oplus H_l^2$ we have

$$XY(g \oplus h) = XP(\Theta_{*e} g + \Phi h) = X(\Theta_{*e} g + \Phi h - \Theta_T u) =$$
$$= P_1(\Omega \Theta_{*e} g + \Omega \Phi h - \Omega \Theta_T u) \oplus \Phi^*(\delta \Theta_{*e} g + \delta \Phi h - \delta \Theta_T u - \Theta_{*e} \Omega \Theta_{*e} g - \Theta_{*e} \Omega \Phi h +$$
$$+ \Theta_{*e} \Omega \Theta_T u) = P_1(\delta g - \delta \Theta_{*i} u) \oplus \Phi^*(\delta \Phi h) = P_1(\delta g) \oplus \delta h = \delta(T_1 \oplus S_l)(g \oplus h),$$

where $u \in H_n^2$. On the other hand, for $f \in H$ we have

$$YXf = Y[P_1(\Omega f) \oplus \Phi^*(\delta f - \Theta_{*e} \Omega f)] = Y[(\Omega f - \Theta_{*i} v) \oplus \Phi^*(\delta f - \Theta_{*e} \Omega f)] =$$
$$= P[\Theta_{*e} \Omega f - \Theta_{*e} \Theta_{*i} v + \Phi \Phi^*(\delta f - \Theta_{*e} \Omega f)] = P(\Theta_{*e} \Omega f - \Theta_T v + \delta f - \Theta_{*e} \Omega f) =$$
$$= P(\delta f) = \delta(T) f,$$

where $v \in H_n^2$ and we made use of the fact that $\Phi \Phi^* w = w$ for $w \in \Phi H_l^2$ to simplify the expression. That δ is outer implies that both $\delta(T_1 \oplus S_l)$ and $\delta(T)$ are quasi-affinities. We conclude that so are X and Y. Thus $T \sim T_1 \oplus S_l$. As before, let $J = J_1 \oplus J_2$ be the Jordan model of T. Then J_1 is the Jordan model of T_1 (cf. [11], Lemma 2.7). From $T_1 \sim J_1$, we infer that $T \sim J_1 \oplus S_l$. If follows from the uniqueness of the Jordan model of T that $l = m - n$ (cf. [5], Theorem 3) and therefore $T \sim J_1 \oplus S_{m-n} = J_1 \oplus J_2 = J$.

From the proof above and the proof of (2)⇒(1) in [13], Lemma 1, we may deduce that $T \sim T_1 \oplus T_2$ and there are intertwining quasi-affinities Z' and W' such that $Z'W'=\delta^2(T_1\oplus T_2)$ and $W'Z'=\delta^2(T)$. In the following, we show that actually quasi-affinities Z and W can be found for which $ZW=\delta(T_1\oplus T_2)$ and $WZ=\delta(T)$.

As before, consider the functional model of T. Then $H=H^2_m\ominus\Theta_T H^2_n$, $H_1=\Theta_{*e}H^2_n\ominus\Theta_T H^2_n$ and $H_2=H^2_m\ominus\Theta_{*e}H^2_n$. Assume that T has the triangulation $T=\begin{bmatrix} T_1 & R \\ 0 & T_2\end{bmatrix}$ on the decomposition $H=H_1\oplus H_2$. Define $S\colon H_2\to H_1$ by

$$Sf=P(\Theta_{*e}\Omega f)\quad\text{for}\quad f\in H_2,$$

where P denotes the (orthogonal) projection onto H. We first check that $T_1S-ST_2=\delta(T_1)R$. For $f\in H_2$, assume that $T_2f=e^{it}f-\Theta_T u-\Theta_{*e}v$ and $Rf=\Theta_{*e}v$ for some $u, v\in H^2_n$. Then

$$(T_1S-ST_2)f=T_1P(\Theta_{*e}\Omega f)-S(e^{it}f-\Theta_T u-\Theta_{*e}v)=$$
$$=P(e^{it}\Theta_{*e}\Omega f)-P(\Theta_{*e}\Omega e^{it}f-\Theta_{*e}\Omega\Theta_T u-\Theta_{*e}\Omega\Theta_{*e}v)=$$
$$=P(\delta\Theta_T u-\delta\Theta_{*e}v)=P(\delta\Theta_{*e}v).$$

On the other hand,

$$\delta(T_1)Rf=\delta(T_1)(\Theta_{*e}v)=P(\delta\Theta_{*e}v).$$

Hence $T_1S-ST_2=\delta(T_1)R$ as asserted. Now define $Z\colon H\to H_1\oplus H_2$ and $W\colon H_1\oplus H_2\to H$ by

$$Z=\begin{pmatrix}\delta(T_1) & S\\ 0 & 1\end{pmatrix}\quad\text{and}\quad W=\begin{pmatrix}1 & V-S\\ 0 & \delta(T_2)\end{pmatrix},$$

where V is the operator appearing in $\delta(T)=\begin{bmatrix}\delta(T_1) & V\\ 0 & \delta(T_2)\end{bmatrix}$ on $H=H_1\oplus H_2$. The proof that Z and W intertwine T and $T_1\oplus T_2$ and that $ZW=\delta(T_1\oplus T_2)$ and $WZ=\delta(T)$ follows exactly the same as the one for Theorem 2.1 in [12]. We leave the verifications to the readers. This completes the proof.

We remark that the proof of (3)⇒(1) in the preceding theorem is valid even when $d_{T^*}=\infty$. Recall that for an arbitrary operator T, Alg T, $\{T\}''$ and $\{T\}'$ denote the weakly closed algebra generated by T and I, the double commutant and the commutant of T; Lat T, Lat$''$ T and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively.

Corollary 3. *Let T be a $C_{\cdot 0}$ contraction with finite defect indices and let $T=\begin{bmatrix}T_1 & *\\ 0 & T_2\end{bmatrix}$ be the triangulation of type $\begin{bmatrix}C_{0\cdot} & *\\ 0 & C_{1\cdot}\end{bmatrix}$. If T is quasi-similar to its Jordan model, then* Lat $T\cong$ Lat$(T_1\oplus T_2)$ *and* Lat$''$ $T\cong$ Lat$''$ $(T_1\oplus T_2)$.

Proof. Since a $C_{\cdot 0}$ contraction T with $d_T<\infty$ satisfies Alg $T=\{T\}''$ (cf. [10], Theorem 3.2 and [9], Theorem 1), we have Lat $T=$ Lat$''$ T and Lat $(T_1\oplus T_2)=$ Lat$''$ $(T_1\oplus T_2)$. Hence we only need to prove Lat $T\cong$ Lat $(T_1\oplus T_2)$. It is easily verified that the lattice isomorphisms can be implemented by the mappings $K\to\overline{ZK}$ and $L\to\overline{WL}$ for $K\in\operatorname{Lat} T$ and $L\in\operatorname{Lat}(T_1\oplus T_2)$, where Z and W are the quasi-affinities given in Theorem 2.

For the hyperinvariant subspace lattice, more is true. If T is a $C_{\cdot 0}$ contraction with $d_T<\infty$ and $T=\begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ is of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$, then T and $T_1\oplus T_2$ have the same Jordan model (cf. [11], Lemma 2.7) whence $\operatorname{Hyperlat} T\cong \operatorname{Hyperlat}(T_1\oplus T_2)$ (cf. [8], Theorem 2). This is true even without the quasi-similarity of T to its Jordan model.

If T is as above and $K\in\operatorname{Lat} T$, then, unlike the case for the more restrictive class of C_{10} contractions (cf. [13], Corollary 4 (2)), the quasi-similarity of T to its Jordan model does not imply that $T|K$ is quasi-similar to its Jordan model. The next example suffices to illustrate this.

Example 4. Let T be the $C_{\cdot 0}$ contraction $S(uv)\oplus S$, where u is the Blaschke product with zeros $1-1/n^2$, $n=1,2,\dots$, v is the singular inner function $v(\lambda)=\exp((\lambda+1)/(\lambda-1))$ for $|\lambda|<1$, and S is the simple unilateral shift. Then the characteristic function of T is $\Theta_T=\begin{bmatrix} uv \\ 0 \end{bmatrix}$. Let $K\in\operatorname{Lat} T$ correspond to the regular factorization $\Theta_T=\Theta_2\Theta_1$, where

$$\Theta_2=\frac{1}{\sqrt{2}}\begin{pmatrix} v & u \\ v & -u \end{pmatrix} \quad\text{and}\quad \Theta_1=\frac{1}{\sqrt{2}}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Note that T is itself a Jordan operator, but $T|K$ is not quasi-similar to its Jordan model (cf. [5], pp. 321—322).

Since it is known that if T is a C_{10} contraction with finite defect indices which is quasi-similar to its Jordan model or T is a C_0 contraction, then $\operatorname{Lat} T=\operatorname{Lat}'' T=\{\overline{\operatorname{ran} S}\colon S\in\{T\}'\}$ (cf. [13], Corollary 8 and [1], Corollary 2.11), we may be tempted to generalize this to $C_{\cdot 0}$ contractions. As it turns out, this is in general not true. The counterexample is provided by the operator T and its invariant subspace K in Example 4. Indeed, if $K=\overline{\operatorname{ran} S}$ for some $S\in\{T\}'$, then, by the main theorem of [7], there exist bounded analytic functions $\Phi=\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$ and $\Psi=[\psi]$ such that $\Phi\Theta_T=\Theta_1\Psi$ and $H_2^2=(\Phi H_2^2+\Theta_1H^2)^-$. From the first equation we have $\varphi_{11}v=(1/\sqrt{2})\psi$ and $\varphi_{21}u=(1/\sqrt{2})\psi$. Thus $v|\psi$ and $u|\psi$. Since $u\wedge v=1$, these imply that $uv|\psi$. Say, $\psi=uvw$ for some $w\in H^\infty$. We obtain $\varphi_{11}=(1/\sqrt{2})uw$ and $\varphi_{21}=(1/\sqrt{2})vw$. For $\begin{bmatrix} f \\ g \end{bmatrix}\in H_2^2$ and $h\in H^2$,

$$\Phi\begin{pmatrix} f \\ g \end{pmatrix}+\Theta_1 h=\begin{pmatrix} (1/\sqrt{2})uwf+\varphi_{12}g \\ (1/\sqrt{2})vwf+\varphi_{22}g \end{pmatrix}+\frac{1}{\sqrt{2}}\begin{pmatrix} uh \\ vh \end{pmatrix}=\begin{pmatrix} u & \varphi_{12} \\ v & \varphi_{22} \end{pmatrix}\begin{pmatrix} (1/\sqrt{2})(wf+h) \\ g \end{pmatrix}.$$

Since these vectors are dense in H_2^2, we conclude that $\begin{bmatrix} u & \varphi_{12} \\ v & \varphi_{22} \end{bmatrix}$, together with its determinant $u\varphi_{22}-v\varphi_{12}$, is outer (cf. [4], Corollary V.6.3). The latter contradicts the main result proved in [2].

However, in such a situation, we still have something to say.

Theorem 5. *Let T be a $C_{\cdot 0}$ contraction with $d_T<\infty$ on H. Then* $\operatorname{Lat} T = \operatorname{Lat}'' T = \{S_1 H \vee S_2 H : S_1, S_2 \in \{T\}'\}$.

Proof. Let $K \in \operatorname{Lat} T$ and let $J = S(\varphi_1) \oplus \ldots \oplus S(\varphi_n) \oplus S_p$ on H_1 and $J' = S(\psi_1) \oplus \ldots \oplus S(\psi_m) \oplus S_q$ on K_1 be the Jordan models of T and $T|K$, respectively. Since $J' \overset{i}{\prec} T|K \overset{i}{\prec} T \prec J$, we infer that $m \leqq n$, $q \leqq p$ and $\psi_j | \varphi_j$ for $j=1, \ldots, m$ (cf. [5], Theorem 4). Say, $\varphi_j = \psi_j \eta_j$ for each j. Note that $S(\varphi_j)|\overline{\operatorname{ran} \eta_j(S(\varphi_j))} \cong S(\psi_j)$ (cf. [5], pp. 315—316). For each j, let Z_j be the operator which implements this unitary equivalence and let $Z: H_1 \to K_1$ be the operator

$$Z_1 \eta_1(S(\varphi_1)) \oplus \ldots \oplus Z_m \eta_m(S(\varphi_m)) \oplus \underbrace{0 \oplus \ldots \oplus 0}_{n-m} \oplus P,$$

where P denotes the (orthogonal) projection from H_p^2 onto H_q^2. Then Z intertwines J and J' and has dense range. Let $X: H \to H_1$ be the quasi-affinity which intertwines T and J and let $Y_1, Y_2: K_1 \to K$ be the injections which intertwine J' and $T|K$ and are such that $K = Y_1 K_1 \vee Y_2 K_1$. Let $S_1 = Y_1 Z X$ and $S_2 = Y_2 Z X$. Then S_1 and S_2 are in $\{T\}'$ and

$$K = Y_1 K_1 \vee Y_2 K_1 = Y_1 Z H_1 \vee Y_2 Z H_1 = Y_1 Z X H \vee Y_2 Z X H = S_1 H \vee S_2 H.$$

This completes the proof.

It is interesting to know whether the converse of Lemma 1 is true. It may turn out that a stronger assertion is true.

Open problem: If T is a $C_{\cdot 0}$ contraction with $d_T<\infty$ and $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ is the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$, is $T \sim T_1 \oplus T_2$?

In this respect, we have the following partial result.

Theorem 6. *If T is a $C_{\cdot 0}$ contraction with $d_T<\infty$ and $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ on $H = H_1 \oplus H_2$ is the triangulation of type $\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$, then $T_1 \oplus T_2 \overset{ci}{\prec} T \prec T_1 \oplus T_2$.*

Proof. Let $J = J_1 \oplus J_2$ be the Jordan model of T, where $J_1 = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k)$ and $J_2 = S_{m-n}$ ($m = d_{T^*}$, $n = d_T$). Then $J \overset{ci}{\prec} T \prec J$. Since J_1 and J_2 are the Jordan models of T_1 and T_2, respectively (cf. [11], Lemma 2.7), we have $T_1 \sim J_1$ and

$J_2 \overset{ci}{\prec} T_2 \prec J_2$. It follows that $T_1 \oplus T_2 \prec J_1 \oplus J_2 \overset{ci}{\prec} T$ and $T \prec J_1 \oplus J_2 \sim T_1 \oplus J_2$. Let X be a quasi-affinity which intertwines T and $T_1 \oplus J_2$. Then it is easily verified that on the decompositions $H = H_1 \oplus H_2$ and $H_1 \oplus H^2_{m-n}$, X can be triangulated as $X = \begin{bmatrix} X_1 & X_3 \\ 0 & X_2 \end{bmatrix}$. Consider the operator $X' = \begin{bmatrix} X_1 & X_3 \\ 0 & 1 \end{bmatrix}$ on $H = H_1 \oplus H_2$. It is easily seen that X' intertwines T and $T_1 \oplus T_2$. Moreover, since T_1 is a $C_0(N)$ contraction and X_1 is an injection in $\{T_1\}'$, X_1 must be a quasi-affinity (cf. [6], Theorem 2). It follows that X' is a quasi-affinity. This shows that $T \prec T_1 \oplus T_2$, completing the proof.

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DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN, CHINA