Acta Sci. Math. 47 (1984), 487—489

A formula for the solution of the difference equation $x_{n+1} = ax_n^2 + bx_n + c$

DIETMAR DORNINGER and HELMUT LÄNGER

There are many papers dealing with the qualitative behaviour of the solution of the difference equation $x_{n+1} = ax_n^2 + bx_n + c$, but up to now no explicit formula for the solution is known. (For a survey of results cf. [2].) In the following we deduce such a formula in a graph-theoretic context.

By a graph (V, E) with the vertex-set V and the set of edges E we mean an undirected graph without loops and without multiple edges. Thus the set E of edges of a graph (V, E) can be considered as a set of unordered pairs $\{v, w\}$, where v, w belong to the set V. A graph (V, E), wherein a certain vertex v_0 is distinguished as the "root" of the graph, will be called a rooted graph and will be denoted by (V, E, v_0). A rooted graph S which is a subgraph of a rooted graph G will be called a rooted subgraph of G, if the roots of S and G coincide.

Definition 1. For any non-negative integer n let T_n denote the rooted graph

$$(P(\{1, \ldots, n\}), \{\{M, M \setminus \{\max M\}\} | \emptyset \neq M \subseteq \{1, \ldots, n\}\}, \emptyset)$$

where $P(\{1, \ldots, n\})$ denotes the power set of $\{1, \ldots, n\}$ and $\max M$ the maximum number occurring within the subset M of $\{1, \ldots, n\}$.

Remark. T_n can be easily constructed inductively by observing $T_0 = (\{\emptyset\}, \emptyset, \emptyset)$ and

$$T_{n+1} = (V(T_n) \cup \{M \cup \{n+1\} | M \in V(T_n)\}, E(T_n) \cup \{\{M, M \cup \{n+1\}\} | M \in V(T_n)\}, \emptyset)$$

for all $n \geqq 0$.

We say that a vertex M of T_n has cardinality k if the cardinality $|M|$ of the set M is k.

Lemma. *For any non-negative integer n, T_n is a rooted tree.*

Received March 1, 1983.

Proof. Let n be some fixed non-negative integer. Studying the definition of T_n one can see easily that there are no loops and that there always exists a path connecting an arbitrary vertex of T_n with the root $\emptyset$. Thus T_n is a connected graph (without loops). If T_n would contain a circle C, then C would have to have at least three vertices, since there are no loops and no double edges in T_n. Assume, M is a vertex of maximal cardinality of C. Then, by the definition of T_n, the vertices of C being adjacent to M would have to coincide, which is a contradiction. Hence T_n is a tree.

Definition 2. For a graph $G=(V(G), E(G))$ and for any subgraph $S=(V(S), E(S))$ of G let S_G denote the complete subgraph of G which has the vertex-set

$$V(S_G) = V(S)\cup\{x\in V(G)\,|\,\text{there exists some } y\in V(S) \text{ such that } \{x, y\}\in E(G)\}.$$

For a rooted graph G and for any rooted subgraph S of G the rooted subgraph S_G of G is defined analogously.

Theorem. *Let I be an arbitrary integral domain. Then the solution of the difference equation $x_{n+1}=ax_n^2+bx_n+c$ $(a, b, c\in I;\ n\geqq 0)$ is given by $x_n=x_0+nc$ if $(a, b)=(0, 1)$ and*

$$x_n = \bar{x}+\sum a^{|V(S)|-1}(f'(\bar{x}))^{|V(S_{T_n})\setminus V(S)|}(x_0-\bar{x})^{|V(S)|}$$

otherwise. Thereby $f(x)$ denotes the polynomial function ax^2+bx+c, $\bar{x}$ is an arbitrary fixed point of f (which in case $(a, b)\neq(0, 1)$ exists in a suitable extension field of I) and the sum is taken over all rooted subtrees S of T_n. (By definition $0^0:=1$.)

Proof. The solution in case $(a, b)=(0,1)$ is obvious. Therefore assume $(a, b)\neq(0, 1)$.

Then within the algebraic closure K of the quotient field of I there exists some fixed point of f, say $\bar{x}$. Performing the substitution $x_n=\bar{x}+y_1^{(n)}$ the difference equation $x_{n+1}=f(x_n)$ is transformed into the difference equation

$$y_1^{(n+1)} = y_1^{(n)}(ay_1^{(n)}+f'(\bar{x})). \tag{1}$$

Now consider the system

$$\begin{aligned} y_1^{(n+1)} &= y_1^{(n)}(ay_1^{(n)}+f'(\bar{x})\,y_2^{(n)}) \\ y_2^{(n+1)} &= y_2^{(n)}(0y_1^{(n)}+1y_2^{(n)}) \end{aligned} \tag{2}$$

of difference equations over K. As one can see easily, $y_1^{(n)}$ is a solution of (1) with the initial value $y_1^{(0)}$ if and only if $(y_1^{(n)}, 1)$ is a solution of (2) with the initial value $(y_1^{(0)}, 1)$. To solve the system (2) one can apply the formula

$$y_1^{(n)} = y_1^{(0)} \sum_{\substack{g:P(\{1,\dots,n\})\to\{1,2\}\\ g(\emptyset)=1}} \prod_{M:\emptyset\neq M\subseteq\{1,\dots,n\}} (a_{g(M\setminus\{\max M\}),g(M)}\,y_{g(M)}^{(0)})$$

(which was proved in [1]) where in our case $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & f'(\bar{x}) \\ 0 & 1 \end{pmatrix}$.

Performing the index transformation $g \leftrightarrow V:=g^{-1}(\{1\})$ we get

$$y_1^{(n)} = \sum a^{|V|-1}(f'(\bar{x}))^{|\{M\in P(\{1,\dots,n\})\setminus V \mid M\setminus\{\max M\}\in V\}|}(y_1^{(0)})^{|V|}$$

where the sum is taken over all subsets V of $P(\{1, \dots, n\})$ which contain the empty set as an element and have the property $\emptyset\neq M\in V\Rightarrow M\setminus\{\max M\}\in V$. We claim that the sets V are exactly the vertex-sets of the rooted subtrees of T_n. Given a set V one can see immediately that within the complete subgraph of T_n with vertex-set V there exists a path connecting each element of V with $\emptyset$. Thus the complete subgraph of T_n having V as its set of vertices is connected and hence is a rooted subtree of T_n. Conversely, let S be a rooted subtree of T_n. Then from each vertex M of S with $|M|\geqq 1$ to the root $\emptyset$ we can find a path $M=M_0, M_1, \dots, M_k=\emptyset$ $(k\geqq 1)$ within S. $|M_1|>|M_0|$ would imply $k>1$ and $|M_{m-1}|=|M_{m+1}|$ and hence $M_{m-1}=$ $=M_{m+1}$ for $m:=\min\{i \mid 1\leqq i<k,\ |M_{i+1}|<|M_i|\}$ contradicting the definition of a path. Therefore $|M_1|<|M_0|$ which implies $M_0\setminus\{\max M_0\}=M_1\in V(S)$. This shows that with every non-empty vertex M, S also contains the vertex $M\setminus\{\max M\}$ wherefrom we can conclude

$$y_1^{(n)} = \sum a^{|V(S)|-1}(f'(\bar{x}))^{|V(S_{T_n})\setminus V(S)|}(y_1^{(0)})^{|V(S)|},$$

the sum being taken over all rooted subtrees S of T_n. Replacing $y_1^{(n)}$ by $x_n-\bar{x}$ yields the result of the theorem.

Remark. If $a=f'(\bar{x})=1$, then $x_n=\bar{x}+\sum_{i=1}^{2^n} b_{ni}(x_0-\bar{x})^i$ where for all $n\geqq 0$ and for all i with $1\leqq i\leqq 2^n$, b_{ni} denotes the number of all rooted subtrees of T_n with exactly i vertices.

References

[1] H. LÄNGER, A formula for the solution of a system of difference equations over a commutative ring, in: *Contributions to General Algebra 2* (Proc. Klagenfurt Conf. 1982, ed. G. Eigenthaler, H. K. Kaiser, W. B. Müller and W. Nöbauer), Hölder—Pichler—Tempsky and Teubner (Wien and Stuttgart, 1983), 233—238.

[2] CH. PRESTON, *Iterates of maps on an interval*, Lecture Notes in Mathematics 999, Springer-Verlag (Berlin, 1983).

TECHNISCHE UNIVERSITÄT WIEN
INSTITUT FÜR ALGEBRA UND DISKRETE MATHEMATIK
WIEDNER HAUPTSTRASSE 8-10
1040 WIEN, AUSTRIA