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On the convergence of eigenfunction expansions in H^s-norm

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Let $S_k \subset \mathbf{R}^n$ $(n \geqq 3;\ k=1, \dots, l)$ be submanifolds of dimension $\dim S_k = m_k \leqq n-3$ having smooth projection to R^{m_k}, i.e. there exist coordinates $(\xi, y) = (\xi_1, \dots, \xi_{m_k};\ y_1, \dots, y_{n-m_k}) \in \mathbf{R}^n$ and functions $\varphi_j^k \in C^1(R^{m_k} \to R^{n-m_k})$ such that

$$S_k = \{(\xi, y) \in R^n:\ y_j = \varphi_j^k(\xi),\ |\nabla \varphi_j^k(\xi)| \leqq C_j^k\}$$

and

$$S = \bigcup_{k=1}^{l} S_k.$$

Let $q \in C^\infty(\mathbf{R}^n \setminus S)$ be such a real valued function for which

$$|q(x)| \leqq c/\operatorname{dist}(x, S)$$

is fulfilled. Consider the Schrödinger operator $L_0 = -\Delta + q(x)\cdot$ with domain $\mathscr{D}(L_0) = C_0^\infty(\mathbf{R}^n)$. Such operators occur as the Hamiltonian of many body problem (cf. [7, XI]). E.g. in the case of two particles we have $H = -\Delta + q\cdot$, $\mathscr{D}(H) = C_0^\infty(\mathbf{R}^n)$, $n=6$, $m=3$, $q(x, y) = \dfrac{c_1}{|x|} + \dfrac{c_2}{|y|} + \dfrac{c_3}{|x-y|}$; $x \in \mathbf{R}^3$, $y \in \mathbf{R}^3$. In the case of homogeneous and izotropic spaces the manifolds S_k are subspaces in $\mathbf{R}^n$.

It is easy to see that the assumptions $m_k \leqq n-3$ implies $q(x) \in L_2^{\mathrm{loc}}(\mathbf{R}^n)$. Indeed; it is enough to prove this for $S_k = S$, $\dim S = m \leqq n-3$,

$$S = \{(\xi, y) \in R^n:\ y_j = \varphi_j(\xi);\ |\nabla \varphi_j(\xi)| \leqq C_j;\quad j = 1, 2, \dots, n-m\}.$$

Using the coordinata-transformation $(\xi, y) \to (\xi, z)$; $z_j = y_j - \varphi_j(\xi)$ we have for the Jacobian $D(\xi, z)/D(\xi, y) = 1$ and for any $0 \leqq \eta \in C_0^\infty(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} |q(x)|^2 \eta(x)\, dx = \int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |q(\xi, z+\varphi(\xi))|^2 \eta(\xi, z+\varphi(\xi))\, dx,$$

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where

$$\varphi = (\varphi_1, \dots, \varphi_{n-m}) \in C^1(\mathbf{R}^m \to \mathbf{R}^{n-m}).$$

On the other hand, for any $x=(\xi, y)\in \mathbf{R}^n$ and $u=(\tilde{\xi}, \varphi(\tilde{\xi}))\in S$, $|y-\varphi(\xi)| \leqq$ $\leqq |y-\varphi(\tilde{\xi})|+|\varphi(\tilde{\xi})-\varphi(\xi)| \leqq |y-\varphi(\tilde{\xi})|+|\nabla\varphi(\xi^*)|\cdot|\xi-\tilde{\xi}| \leqq c(|y-\varphi(\tilde{\xi})|+|\xi-\tilde{\xi}|)$. Hence $|y-\varphi(\xi)|^2 \leqq 2c^2(|y-\varphi(\tilde{\xi})|^2+|\tilde{\xi}-\xi|^2) \leqq 2c^2|x-u|^2$, i.e. $|y-\varphi(\xi)| \leqq c \operatorname{dist}(x, S)$, consequently

$$\left|q(\xi, z+\varphi(\xi))\right| \leqq c/\operatorname{dist}\left\{(\xi, z+\varphi(\xi)), S\right\} \leqq$$

$$\leqq \frac{c}{|z+\varphi(\xi)-\varphi(\xi)|} = \frac{c}{|z|}.$$

We have

$$\int_{\mathbf{R}^n} |q(x)|^2 \eta(x)\, dx = \int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |q(\xi, z+\varphi(\xi))|^2 \eta(\xi, z+\varphi(\xi))\, dz \leqq$$

$$\leqq c \int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} \frac{1}{|z|^2} \eta(\xi, z+\varphi(\xi))\, dz < \infty.$$

It follows from the Lemma 1 of the present work that the operator L_0 is bounded below, i.e. for any $f\in C_0^\infty(\mathbf{R}^n)$

$$(L_0 f, f) = (-\Delta f, f)+(qf, f) = (\nabla f, \nabla f)+(qf, f) \geqq -c(f, f), \quad (c>0),$$

and hence by K. O. Friedrichs' theorem [6] we obtain: L_0 has selfadjoint extension L further $L \geqq -cI (c>0)$. Denote $L = \int\limits_0^\infty \lambda dE_\lambda$ the spectral expansion of L and for any $f\in L_2(\mathbf{R}^n)$ consider the expansion $E_\lambda f$.

In [8] is proved: for any $f\in H^s(\mathbf{R}^n)$ $(0\leqq s\leqq 1)$ $\|E_\lambda f-f\|_{H^s}\to 0$ as $\lambda\to\infty$. H^s denotes the space of functions from $L_2(\mathbf{R}^n)$ with the norm

$$\|f\|_{H^s} \stackrel{\text{def}}{=} \|(I-\Delta)^{s/2} f\|_{L_2} = \|(1+|\xi|^2)^{s/2}\hat{f}(\xi)\|_{L_2} \quad ([9], 2.3.3).$$

The aim of the present note is to prove the following

Theorem 1. *For any* $f\in H^s(\mathbf{R}^n)$ $(0\leqq s\leqq 2)$ *we have*

$$\|E_\lambda f-f\|_{H^s} \to 0 \quad \textit{as} \quad \lambda\to\infty. \tag{1}$$

A theory of general orthogonal series was developped by K. Tandori in the last twenty years (Cf. e.g. [2—4]). At the same time Il'in [5] found a new theory of spectral expansions which is a special case of the general orthogonal expansions, further proved in [5] the Theorem 1 in the specail case when $q\equiv 0$ and s is integer.

From Lemma 2 below it follows, among others, by the well known Kato—Rellich theorem [7, X.2] that the operator L_0 considered is essentially selfadjoint, further $\mathcal{D}(\bar{L}_0)=\mathcal{D}(L)=H^2$.

2. For the proof of the Theorem 1 we need some lemmas. Define

$$\varrho(x) = [\mathrm{dist}\,(x, S)]^{-1}.$$

Lemma 1. *We have for any* $f \in H^1$

$$\int_{\mathbf{R}^n} \varrho(x)\,|f(x)|^2\,dx \leqq c\,\|f\|_{L_2} \cdot \|f\|_{H^1}. \tag{2}$$

Here and below c denotes a constant, which is independent from f and not necessarily the same in each occurences.

Proof. Using polar coordinates and the identity

$$h(r) = -2\int_r^\infty h(t)h'(t)\,dt$$

we obtain

$$\int_{\mathbf{R}^{n-m}} \frac{|g|(y)^2}{|y|}\,dy = \int_\theta \int_0^\infty r^{n-m-2}|g(r,\theta)|^2\,dr\,d\theta = -2\int_\theta\int_0^\infty \left[\int_r^\infty g(t,\theta)\frac{\partial g(t)\theta}{\partial t}\,dt\right]\times$$

$$\times r^{n-m-2}\,dr\,d\theta = \frac{2}{n-m-1}\int_\theta\int_0^\infty g(r,\theta)\frac{\partial g(r,\theta)}{\partial r}\,r^{n-m-1}\,dr\,d\theta$$

whence for $g(y)=g(\xi,y)=f(\xi, y+\varphi(\xi))$ we obtain

$$\int_{\mathbf{R}^n} \varrho(x)\,|f(x)|^2\,dx \leqq c\int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} \frac{|f(\xi, y+\varphi(\xi))|^2}{|y|}\,dy \leqq$$

$$\leqq c\int_{\mathbf{R}^m} d\xi \int_{\mathbf{R}^{n-m}} |f(\xi, y+\varphi(\xi))|\cdot|\nabla_y f(\xi, y+\varphi(\xi))|\,dy =$$

$$= c\int_{\mathbf{R}^n} |f(x)|\,|\nabla_y f(x)|\,dx \leqq c\,\|f\|_{L_2}\cdot\|f\|_{H^1}.$$

Lemma 1 is proved.

Lemma 2. *We have for any* $f \in H^1$

$$\int_{\mathbf{R}^n} \varrho^2(x)\,|f(x)|^2\,dx \leqq c\,\|f\|_{H^1}^2. \tag{3}$$

Proof. Using polar coordinates and the notation

$$I = I(\xi,\theta) = \int_0^\infty r^{n-m-3}|f(\xi, r, \theta)|^2\,dr \quad (\xi \in \mathbf{R})$$

we obtain

$$I \leqq 2 \int_0^\infty r^{n-m-3} \int_r^\infty \left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| dt\, dr =$$

$$= 2 \int_0^\infty \left(\left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| \int_0^t r^{n-m-3}\, dr \right) dt =$$

$$= c \int_0^\infty \left| f(\xi, t, \theta) \frac{\partial f(\xi, t, \theta)}{\partial t} \right| t^{n-m-2}\, dt \leqq$$

$$\leqq c \left(\int_0^\infty t^{n-m-3} |f(\xi, t, \theta)|^2\, dt \right)^{1/2} \left(\int_0^\infty \left| \frac{\partial f(\xi, t, \theta)}{\partial t} \right|^2 t^{n-m-1}\, dt \right)^{1/2},$$

hence

$$I \leqq c \int_0^\infty \left| \frac{\partial f(\xi, t, \theta)}{\partial t} \right|^2 t^{n-m-1}\, dt \leqq c \|\nabla_y f\|^2_{L_2(\mathbf{R}^{n-m})},$$

i.e.

$$\int_{\mathbf{R}^m} \int_\theta I\, d\theta\, d\xi \leqq c \|\nabla f\|^2_{L_2(\mathbf{R}^n)} \leqq c \|f\|_{H^1}.$$

Lemma 2 is proved.

Corollary. *The operator* L_0 *with* $D(L_0) = C_0^\infty(\mathbf{R}^n)$ *is essentially selfadjoint and* $D(\bar{L}_0) = D(L) = H^2$.

Proof. It follows from Lemma 2 and (8) (below)

$$\|qf\|_{L_2} \leqq c\|f\|_{H^1} \leqq \varepsilon \|f\|_{H^2} + c(\varepsilon)\|f\|_{L_2} = \varepsilon \|(I - \nabla)f\|_{L_2} + c(\varepsilon)\|f\|_{L_2}.$$

Because $I - \Delta$ is essentially selfadjoint and $D(\overline{I - \Delta}) = H^2$, the Corollary follows by Kato—Rellich theorem ([7], X.2.).

Lemma 3. *For any* $f \in H^2$

$$\|Lf\|_{L_2} \leqq c\|f\|_{H^2}. \tag{4}$$

Proof. Using Lemma 1 we have for any $f \in H^2$

$$\|Lf\|_{L_2} = \|-\Delta f + qf\|_{L_2} \leqq \|\Delta f\|_{L_2} + \|qf\|_{L_2} \leqq \|f\|_{H^2} + c\|f\|_{H^1} \leqq c\|f\|_{H^2}.$$

Lemma 3 is proved.

Lemma 4. *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $f \in H^2$

$$\|Lf\|^2_{L_2} \geqq c_1 \|f\|^2_{H^2} - c_2 \|f\|^2_{L_2}. \tag{5}$$

Proof. Using the identity

$$\|Lf\|^2_{L_2} = \|\Delta f\|^2_{L_2} - 2(qf, \Delta f) + \|qf\|^2_{L_2}$$

the ingequality $a\cdot b\leqq\varepsilon a^2+(1/4\varepsilon)\cdot b^2$ $(a, b, \varepsilon>0)$, the Cauchy—Bunyakovski inequality and Lemma 2, we have

$$|(qf, \Delta f)|\leqq\|qf\|_{L_2}\cdot\|\Delta f\|_{L_2}\leqq\varepsilon\|\Delta f\|_{L_2}^2+\frac{1}{4\varepsilon}\|qf\|_{L_2}^2,$$

hence

$$\|Lf\|_{L_2}^2\geqq\|\Delta f\|_{L_2}^2-2|(qf, \Delta f)|+\|qf\|_{L_2}^2\geqq$$

$$\geqq\|\Delta f\|_{L_2}^2-\varepsilon\|\Delta f\|_{L_2}^2-c(\varepsilon)\|qf\|_{L_2}^2\geqq(1-\varepsilon)\|\Delta f\|_{L_2}^2-c(\varepsilon)\|qf\|_{L_2}^2.$$

Using

$$\|qf\|_{L_2}^2\leqq c\|f\|_{H^1}^2\leqq\varepsilon_1\|f\|_{H^2}^2+c(\varepsilon_1)\|f\|_{L_2}^2$$

we obtain

$$\|Lf\|_{L_2}^2\geqq(1-\varepsilon)\|\Delta f\|_{L_2}^2-c(\varepsilon)\varepsilon_1\|f\|_{H^2}^2-c(\varepsilon)c(\varepsilon_1)\|f\|_{L_2}^2.$$

On the other hand

$$\|f-\Delta f\|_{L_2}^2=\|f\|_{H^2}^2,$$

whence

$$\|\Delta f\|_{L_2}=\|\Delta f-f+f\|_{L_2}\geqq\|\Delta f-f\|_{L_2}-\|f\|_{L_2}\geqq\|f\|_{H^2}-\|f\|_{L_2}.$$

At last we obtain

$$\|Lf\|_{L_2}^2\geqq\big(1-\varepsilon-c(\varepsilon)\varepsilon_1\big)\|f\|_{H^2}^2-c(\varepsilon, \varepsilon_1)\|f\|_{L_2}^2,$$

and from this (5) follows if $\varepsilon=1/2$ and ε_1 is small enough. Lemma 4 is proved.

Define $L_\mu=L+\mu I$.

Lemma 5. *There exists* $\mu_0>0$ *such that for every* $f\in C_0^\infty(\mathbf{R}^n)$

(6) $$\|L_\mu f\|_{L_2}\geqq c_\mu\|f\|_{H^2}.$$

The constant C_μ *does not depends on* f.

Proof. Obviously

$$\|L_\mu f\|_{L_2}^2=(Lf+\mu f, Lf+\mu f)=(Lf, Lf)+2\mu(Lf, f, f)+\mu^2(f, f)\geqq$$

$$\geqq(Lf, Lf)-2\mu|(qf, f)|+2\mu\|\nabla f\|_{L_2}^2+\mu\|f\|_{L_2}^2.$$

Using Lemma 1, we obtain

(7) $$|(qf, f)|\leqq\left\|\sqrt{|q|}f\right\|_{L_2}^2\leqq c\|f\|_{L_2}\cdot\|f\|_{H^1}\leqq\varepsilon\|f\|_{H^1}^2+\frac{1}{4\varepsilon}\|f\|_{L_2}^2.$$

Taking into account the definition of the H^s-norm and using the inequality

$$1+|\xi|^2\leqq\varepsilon_1(1+|\xi|^2)^2+\frac{1}{4\varepsilon_1},$$

(which is a special case of $a\leqq\varepsilon a^2+1/4\varepsilon$) we obtain

$$\|f\|^2_{H^1}\leqq\varepsilon_1\int_{\mathbf{R}^n}|\hat f(\xi)|^2(1+|\xi|^2)^2\,d\xi+\frac{1}{4\varepsilon_1}\int_{\mathbf{R}^n}|\hat f(\xi)|^2\,d\xi= \tag{8}$$

$$=\varepsilon_1\|f\|^2_{H^2}+\frac{1}{4\varepsilon_1}\|f\|^2_{L_2}.$$

From (7) and (8) it follows

$$|(qf,f)|\leqq\varepsilon\varepsilon_1\|f\|^2_{H^2}+\frac{\varepsilon}{4\varepsilon_1}\|f\|^2_{L_2}+\frac{1}{4\varepsilon}\|f\|^2_{L_2}.$$

Let $\varepsilon=\varepsilon_1=\alpha/\sqrt{\mu}$, where $\alpha>0$ will be choosen below. Summarising our estimates, we obtain

$$\|L_\mu f\|^2_{L_2}\geqq c_1\|f\|^2_{H^2}-c_2\|f\|^2_{L_2}-2\mu\left\{\frac{\alpha^2}{\mu}\|f\|^2_{H^2}+\frac{1}{4}\|f\|^2_{L_2}+\right.$$

$$\left.+\frac{\sqrt{\mu}}{4\alpha}\|f\|^2_{L_2}\right\}+\mu^2\|f\|^2_{L_2}\geqq(c_1-2\alpha^2)\|f\|^2_{H^2}-$$

$$-\left(\mu^2-\frac{\mu}{2}-\frac{\mu\sqrt{\mu}}{2\alpha}\right)\|f\|^2_{L_2}\geqq c\|f\|^2_{H^2},$$

if α is small enough and $\mu_0=\mu_0(\alpha)$ is large enough. Lemma 5 is proved.

The following Lemma generalizes that of 2.4 in [11].

Lemma 6. *Let A and B be strongly positive selfadjoint operators in the Hilbert space H. Suppose*

$$D(B)\subset D(A) \tag{9}$$

and

$$\|Af\|_H\leqq c\|Bf\|_H\quad(f\in D(B)) \tag{10}$$

are fulfilled. Then for any θ, $0\leqq\theta\leqq1$

$$\|A^\theta f\|_H\leqq c_\theta\|B^\theta f\|_H\quad(f\in D(B)). \tag{11}$$

Proof. Define

$$\|f\|_{D(A)^\theta}\overset{\text{def}}{=}\|A^\theta f\|_H,\quad\theta\in\mathbf{C}.$$

According to the strong positivity of A, this norm is equivalent with that of defined in Triebel [9]. We obtain from (10) (taking into account the definition of the Petree functional)

$$K(t,f,H,D(A))\leqq cK(t,f,H,D(B))\quad(f\in D(B)).$$

Hence, using 1.3.2 of [9], we obtain (11). Lemma 6 is proved.

Lemma 7. *For* $\mu \geqq \mu_0$, $0 \leqq s \leqq 2$ *we have*

$$\|L_\mu^{s/2} f\|_{L_2} \leqq c_s \|f\|_{H^s} (f \in H^s), \quad D(L_\mu^{s/2}) = H^s(\mathbf{R}^n). \tag{12}$$

Proof. (12) is trivial for $s=0$ and it is proved for $s=2$ in Lemma 3. Now apply Lemma 6 for $A=L_\mu$, $B=I-\Delta$, $D(B)=H^2(\mathbf{R}^n)$. We obtain:

$$\|L_\mu^\theta f\|_{L_2} \leqq c\|(I-\Delta)^\theta f\|_{L_2} = c\|f\|_{H^{2\theta}}.$$

According to TRIEBEL [9], 1.18.10 and using the Corollary after Lemma 2, we obtain: $D(L_\mu^\theta)=H^{2\theta}(\mathbf{R}^n)$. Lemma 7 is proved.

Lemma 8. *For* $\mu \geqq \mu_0$, $0 \leqq s \leqq 2$

$$\|L_\mu^{-s/2} f\|_{H^s} \leqq c_s \|f\|_{L_2} \quad (f \in L_2(\mathbf{R}^n)). \tag{13}$$

Proof. First we prove that for every $g \in H^s$

$$\|g\|_{H^s} \leqq c_s \|L_\mu^{s/2} g\|_{L_2} \quad (0 \leqq s \leqq 2). \tag{14}$$

This estimate is trivial for $s=0$ and for $s=2$ it is proved in the Lemma 5. Use Lemma 6 for $B=L_\mu$, $A=I-\Delta$, $D(A)=H^2$, then (14) follows.

From Lemma 7 we obtain $R(L_\mu^{-s/2})=D(L_\mu^{s/2})=H^s(\mathbf{R}^n)$, whence for every $f \in L_2(\mathbf{R}^n)$ we have $L_\mu^{-s/2} f \in H^s$. Now applying (14) for the function $g \overset{\text{def}}{=} L_\mu^{-s/2} f$ $(f \in L_2(\mathbf{R}^n))$ we obtain (13). Lemma 8 is proved.

Proof of Theorem 1. Using (12) and (13) we obtain for any $f \in H^s$:

$$\|f-E_\lambda f\|_{H^s} = \|L_\mu^{-s/2} L_\mu^{s/2}(I-E_\lambda) f\|_{H^s} \leqq$$
$$\leqq c\|L_\mu^{s/2}(I-E_\lambda) f\|_{L_2} = c\|(I-E_\lambda) L_\mu^{s/2} f\|_{L_2} \to 0 \quad (\lambda \to \infty).$$

Theorem 1 is proved.

Now consider the Schrödinger operator

$$L_0 = -\Delta + q(x)\cdot, \quad x \in \mathbf{R}^3, \quad D(L_0) = C_0^\infty(\mathbf{R}^3),$$

and suppose $|q(x)| \leqq c/|x|$. Then the method of the proof of Theorem 1 gives

Theorem 2. *The operator* L_0 *is essentially selfadjoint,* $D(\bar{L}_0)=D(L)=H^2(\mathbf{R}^3)$, *where* $L \overset{\text{def}}{=} \bar{L}_0$, *further for* $0<\tau<1/2$ *we have*

$$\|E_\lambda f - f\|_{C^\tau} \to 0 \quad (\lambda \to \infty, f \in H^{3/2+\tau}(\mathbf{R}^3)). \tag{15}$$

Here E_λ is the spectral family of L and C^τ denotes the Hölder class of functions (TRIEBEL [9], 2.7.1(2)), *i.e.*

$$\|f\|_{C^\tau} \overset{\text{def}}{=} \sup_{x\in\mathbf{R}^3} |f(x)| + \sup_{x,y\in\mathbf{R}^3} \frac{|f(x)-f(y)|}{|x-y|^\tau}.$$

Proof of Theorem 2. Using the imbedding $H^{3/2+\tau}\subset C^\tau$ (cf. [9], 2.8.1(16), $p=2$, $n=3$) (15) follows by the method of the proof of Theorem 1.

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