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An extension of the Lindeberg—Trotter operator-theoretic approach to limit theorems for dependent random variables I. General convergence theorems; approximation theorems with o-rates

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Dedicated to Professor Károly Tandori on the occasion of his sixtieth birthday, in high esteem

1. Introduction. This paper is concerned with a generalization of the classical Lindeberg—Trotter operator-theoretic approach to the central limit theorem (=CLT) (see e.g. [24], [15, p. 113], [10, p. 248], [20, p. 223], [18, p. 207]), so far restricted to independent random variables (=r.v.'s), to the general case of arbitrary *dependent* r.v.'s. One of the great advantages of the classical Trotter approach is that it can also cover the weak law of large numbers (=WLLN), indeed any limit theorem dealing with convergence in distribution of r.v.'s and, above all, it can even cover limit theorems equipped with O-rates or o-rates of convergence, all in the case that the r.v.'s are independent (see e.g. [6], [11, p. 157], [19], [22], [5], [21]). A further advantage of the method is that it is elementary in the sense that it does not use Fourier analytic machinery at all.

Any attempt to generalize the Trotter approach to the situation of dependent r.v.'s leads to principal difficulties. Already in the „resctrictedly" dependent case of martingale difference sequences (MDS) and arrays (MDA) did the Trotter approach have to be modified considerably in order to cover the particular type of dependency in question (see e.g. [1], [23], [9], [2], [7], [8]). In order to comprehend these difficulties let us ıecall the basic principles of the Trotter approach.

If $(X_k)_{k\in \mathbf{N}}$ is a sequence of independent r.v.'s and f any function belonging to the space C_B (see Section 2 for definition), the Trotter operator $V_{X_k}\colon C_B\to C_B$ associated with X_k is defined (cf. (3.1)) for each $y\in \mathbf{R}$ as the expectation of the r.v.

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$f(X_k+y)$,

$$V_{X_k}f(y) = E[f(X_k+y)] \quad (k\in\mathbf{N}).$$

One of the basic properties of this operator, the proof of which rests upon the relation $P_{\sum_{k=1}^n X_k} = \mathop{*}\limits_{k=1}^{n} P_{X_k}$, valid for the distributions P_{X_k} of independent r.v.'s X_k, is

$$V_{\sum_{k=1}^n X_k} f = V_{X_1}V_{X_2}\dots V_{X_n} f \quad (f\in C_B;\ n\in\mathbf{N}). \tag{1.1}$$

If $(Z_k)_{k\in\mathbf{N}}$ is a further sequence of r.v.'s which are independent not only amongst themselves but also of the X_k, then (1.1) leads to the basic inequality

$$\|V_{\sum_{k=1}^n X_k} f - V_{\sum_{k=1}^n Z_k} f\|_{C_B} \leqq \sum_{n=1}^{n} \|V_{X_k} f - V_{Z_k} f\|_{C_B} \tag{1.2}$$

valid for any $f\in C_B$, where $\|g\|_{C_B} = \sup_{y\in\mathbf{R}} |g(y)|$.

Now an equivalent formulation of the CLT for independent, identically distributed r.v.'s states that

$$|E[f(n^{-1/2}S_n)] - E[f(X^*)]| = o(1) \quad (n\to\infty) \tag{1.3}$$

for any $f\in C_B$, where $S_n = \sum_{k=1}^n X_k$, and X^* is a standard normally distributed r.v. This is a particular case ($y=0$!) of

$$\|V_{n^{-1/2}S_n} f - V_{X^*} f\|_{C_B} = o(1) \quad (n\to\infty)$$

for any $f\in C_B$. In order to be able to include such limit theorems under (1.2), X^* must be $(n^{-1/2})$-decomposable in the form

$$P_{X^*} = P_{n^{-1/2}\sum_{k=1}^n Z_k}, \tag{1.4}$$

where the decomposition components Z_k are $\mathcal{N}(0, \sigma_k^2)$-distributed r.v.'s with $\sigma_k^2 = \mathrm{Var}\,[X_k]$ which may, without loss of generality (see [3, p. 164]), be chosen to be independent amongst themselves as well as of the r.v.'s X_k. In that case one has by (1.4), noting that the Trotter operator just involves the distribution of the associated r.v.,

$$V_{X^*} f = V_{n^{-1/2}\sum_{k=1}^n Z_k} f \quad (f\in C_B).$$

So establishing (1.3) just amounts to showing, noting (1.2) with all r.v.'s multiplied by the factor $n^{-1/2}$, that

$$\|V_{n^{-1/2}X_k} f - V_{n^{-1/2}Z_k} f\|_{C_B} = o\left(\frac{1}{n}\right) \quad (f\in C_B;\ n\to\infty). \tag{1.5}$$

Assertions for single differences of type (1.5) can easily be estimated by assuming that the moments of the r.v.'s X_k and Z_k coincide up to the order 2.

If one would wish to equip the CLT in the form (1.3) or other weak limit theorems with rates, it would suffice to supply (1.5) with rates better than $o(1/n)$, which is possible if the corresponding moments of higher orders are equal to another. However, the whole procedure is only applicable to *independent* r.v.'s since the basic properties used, namely (1.1) and (1.2), are only valid for such r.v.'s.

A first indication that cognate methods of proof could possibly be applicable to *dependent* r.v.'s is the paper [12] by Z. Govindarajulu who established the WLLN for triangular arrays of dependent r.v.'s. Foı this purpose he used a property corresponding to (1.1), one tailored to the situation of dependent r.v.'s; but he had to replace inequality (1.2) by estimates of a different type.

The chief aim of this paper, however, is to present an operator-theoretic approach that allows one to generalize the Trotter operator-technique, one that has stood the test, to dependent r.v.'s. The development of the present approach may in some sense be compared with that of Trotter's: similarly as did Lindeberg's proof of the CLT of 1922 (cf. [17]) serve Trotter as the basis for his operator approach, so did Govindarajulu's paper give the impulse to our definition of a „conditional Trotter operator" (cf. Def. 1 in Section 3). However, its applicability is not only confined to a proof of the CLT or WLLN. These theorems will, much more, be deduced as particular cases of a very general limit theorem, which can even be supplied with o-rates or O-rates of convergence, as will be shown in Section 5 and 6.

For the sake of clarity let us present a particular case of the "conditional Trotter operator" tailored to MDS. If $(X_k)_{k\in\mathbf{N}}$ is a MDS, i.e., $E[X_k|\mathfrak{F}_{k-1}]=0$ a.s., $k\in\mathbf{N}$, where $\mathfrak{F}_{k-1}=\mathfrak{A}(X_1, \ldots, X_{k-1})$ is the σ-algebra generated by the r.v.'s $X_1, \ldots, X_{k-1}$, then the conditional Trotter operator $V_{X_k}^{\mathfrak{F}_{k-1}}$ is defined for each $f\in C_B$ and $y\in\mathbf{R}$ as the conditional expectation of the r.v. $f(X_k+y)$ relative to $\mathfrak{F}_{k-1}$, i.e.,

$$(V_{X_k}^{\mathfrak{F}_{k-1}}f)(y) := E[f(X_k+y)|\mathfrak{F}_{k-1}] \quad (k\in\mathbf{N}).$$

If the r.v.'s X_k are independent, then the properties associated with conditional expectation yield that

$$V_{X_k}^{\mathfrak{F}_{k-1}}f = E[f(X_k+\cdot)|\mathfrak{F}_{k-1}] = E[f(X_k+\cdot)] = V_{X_k}f,$$

so that the conditional Trotter operator coincides with the classical Trotter operator. Furthermore, the operator $V_{X_k}^{\mathfrak{F}_{k-1}}$ has all of the basic characteristics of V_{X_k}, so that it is possible to establish with its help the counterparts of the properties (1.1) and (1. 2) for dependent r.v.'s (see (3.6) and (3.7)). For this reason it is not only possible to extend all of the limit theorems established by means of Trotter operators for independent r.v.'s to the case of arbitrary dependent r.v.'s — whereby the dependency structure just depends upon moment conditions of type (4.1) — but also to extend them to particular types of restrictedly dependent r.v.'s, namely to MDS and MDA, without having to modify the proofs as has been necessary so far (see e.g. [7, 8]).

Concerning a comparison with the literature existing in the field, let us first note that apart from the paper [12] cited for the WLLN as well as another [13] by P. Gudynas, no further papers are known to the authors that deal with assertions on convergence in distribution without restricting the dependency structure in some way or other. The r.v.'s are either assumed to be independent or dependent in the sense of MDS, MDA, or inverse martingales. Whereas the WLLN without rates is also a particular case of our results (see Theorem 3), direct comparisons with the results of Gudynas are hardly possible since he is concerned with inequalities for metrics of vector-valued r.v.'s. Points of comparison with other papers devoted to independent r.v.'s or to MDS or MDA will be gone into in the course of the paper.

Part I of this paper consists of five sections, the second of which is concerned with the preliminary results needed from approximation and probability theory. Section 3 deals with the definition of the conditional Trotter operator and its basic properties, while Section 4 is devoted to the general limit theorem, namely Theorem 1, which is then applied to give the CLT and WLLN. Section 5 contains the general approximation with o-rates, Theorem 4, together with applications. The second and last part of the paper, covering Sections 6 to 8, begins with two general approximation theorems with O-rates for convergence in distribution (Theorems 7 and 8) which are applied to yield to O-error estimates for assertions of Berry—Esséen-type, i.e. for the uniform convergence of distribution functions (Theorem 11 and 12), dealt with in Section 7. Section 8 is concerned with the particular case of MDA as well as with the existing literature in the matter.

2. Notations and preliminaries. In the following, $C_B=C_B(\mathbf{R})$ will denote the vector space of all real-valued, bounded, uniformly continuous functions defined on the reals $\mathbf{R}$, endowed with norm $\|f\|_{C_B}\colon \sup_{x\in\mathbf{R}}|f(x)|$. For $r\in\mathbf{P}:=\mathbf{N}\cup\{0\}$ we set

$$C_B^0:=C_B,\quad C_B^r:=\{g\in C_B;\ g^{(j)}\in C_B,\ 1\leqq j\leqq r\},$$

the seminorm on C_B^r being given by $|g|_{C_B^r}:=\|g^{(r)}\|_{C_B}$. For any $f\in C_B$ and $t\geqq 0$ the K-functional, needed in Part II, is defined by

$$K(t;\ f;\ C_B,\ C_B^r):=\inf_{g\in C_B^r}\{\|f-g\|_{C_B}+t|g|_{C_B^r}\}.$$

This functional is equivalent to the rth modulus of continuity, defined for $f\in C_B$ by

$$\omega_r(t;\ f;\ C_B):=\sup_{|h|\leqq t}\Big\|\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}f(u+kh)\Big\|_{C_B},$$

in the sense that there are constants $c_{1,r};c_{2,r}>0$, independent of f and $t\geqq 0$, such that (see [4, pp. 192, 258])

$$c_{1,r}\omega_r(t^{1/r};\ f;\ C_B)\leqq K(t;\ f;\ C_B,\ C_B^r)\leqq c_{2,r}\omega_r(t^{1/r};\ f;\ C_B). \tag{2.1}$$

Lipschitz classes of index $r \in \mathbf{N}$ and order $\alpha, 0<\alpha \leqq r$ will be needed in Part I. They are defined for $f \in C_B$ by

$$\text{(2.2)} \qquad \mathrm{Lip}\,(\alpha; r; C_B) := \{\omega_r(t; f; C_B) \leqq L_f t^\alpha\},$$

L_f being the so-called Lipschitz constant. Note that for $\alpha = r' + \beta$, $r' \leqq r-1$, $0<\beta\leqq 1$ (see [14])

$$\text{(2.3)} \qquad f^{(r')} \in \mathrm{Lip}\,(\beta;\ r-r';\ C_B) \Rightarrow f \in \mathrm{Lip}\,(r'+\beta;\ r;\ C_B).$$

Several preliminaries from probability theory will be noted. Let $(\Omega, \mathfrak{A}, P)$ denote a probability space with set Ω, σ-algebra $\mathfrak{A}$ and probability measure P, $\mathfrak{B}$ the σ-algebra of Borel sets in $\mathbf{R}$, $\mathfrak{Z}(\Omega, \mathfrak{A}) := \{X\colon\ \Omega \to \mathbf{R},\ X \text{ is } \mathfrak{A}, \mathfrak{B}\text{-measurable}\}$ the set of all real r.v.'s on Ω, and $\mathfrak{L}(\Omega, \mathfrak{A}, P) := \{X \in \mathfrak{Z}(\Omega, \mathfrak{A});\ X \text{ is } P\text{-integrable}\}$ the set of all real P-integrable r.v.'s on Ω.

The general convergence theorems of this paper will be formulated, as indicated in the introduction, for φ-decomposable r.v.'s. If $\varphi\colon \mathbf{N} \to \mathbf{R}^+$ is a positive normalizing function, then $Z \in \mathfrak{Z}(\Omega, \mathfrak{A})$ is called φ-decomposable, if for each $n \in \mathbf{N}$ there exist n independent r.v.'s $Z_k = Z_{k,n}$, $1 \leqq k \leqq n$, such that the distributions of the r.v. Z and the normalized sums $\varphi(n) \sum_{k=1}^{n} Z_k$ coincide, i.e., if

$$\text{(2.4)} \qquad P_Z = P_{\varphi(n) \sum_{k=1}^{n} Z_k}.$$

An important concept needed for the proofs will be the conditional expectation (see e.g. [3, p. 292]), to be denoted for $X \in \mathfrak{L}(\Omega, \mathfrak{A}, P)$ and each sub-σ-algebra $\mathfrak{G} \subset \mathfrak{A}$ by $E[X|\mathfrak{G}]$. If Y also belongs to $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, and $\mathfrak{G}'$ is a further sub-σ-algebra of $\mathfrak{A}$, then there hold the properties (see e.g. [3, p. 293f.])

$$\text{(2.5)} \qquad E[E[X|\mathfrak{G}]] = E[X];$$

$$\text{(2.6)} \qquad E[X|\mathfrak{G}_0] = E[X] \quad \text{a.s. for} \quad \mathfrak{G}_0 = \{\Phi, \Omega\};$$

$$\text{(2.7)} \qquad X \leqq Y \quad \text{a.s. implies} \quad E[X|\mathfrak{G}] \leqq E[Y|\mathfrak{G}] \quad \text{a.s.};$$

$$\text{(2.8)} \qquad X = c \quad \text{a.s., some} \quad c \in \mathbf{R}, \quad \text{implies} \quad E[X|\mathfrak{G}] = c \quad \text{a.s.};$$

$$\text{(2.9)} \qquad E[\alpha X + \beta Y|\mathfrak{G}] = \alpha E[X|\mathfrak{G}] + \beta E[Y|\mathfrak{G}] \quad \text{a.s.} \quad (\alpha, \beta \in \mathbf{R});$$

(2.10) $E[X|\mathfrak{G}] = E[X]$ a.s. provided the σ-algebra $\mathfrak{A}(X)$, generated by X, is independent of $\mathfrak{G}$;

$$\text{(2.11)} \qquad E[E[X|\mathfrak{G}]|\mathfrak{G}'] = E[E[X|\mathfrak{G}']|\mathfrak{G}] = E[X|\mathfrak{G}] \quad \text{a.s.}$$

The aim now is to represent the conditional expectation as an integral. For this purpose two concepts need be recapitulated. If $\mathfrak{G} \subset \mathfrak{A}$ is a σ-algebra and $X \in \mathfrak{Z}(\Omega, \mathfrak{A})$, a function $\hat{P_X}\colon\ \Omega \times \mathfrak{A} \to \mathbf{R}$ is said to be a *regular conditional probability distribution* of X relative to $\mathfrak{G}$, if it satisfies the conditions (see e.g. [16, p. 372ff.]): (i) For every fixed $\omega\in\Omega$, the set function $P_X^{\hat{}}(\omega,\cdot)$, defined on $\mathfrak{A}$, is a probability measure; (ii) for every fixed $A\in\mathfrak{A}$, $P_X^{\hat{}}(\cdot, A)\in\mathfrak{Z}(\Omega,\mathfrak{G})$; (iii) for every $A\in\mathfrak{A}$ and $G\subset\mathfrak{G}$, there holds

$$\int_G P_X^{\hat{}}(\omega, X^{-1}(A))\,dP = P(G\cap X^{-1}(A)).$$

The function F_Z: $\mathbf{R}\times\Omega\to\mathbf{R}$, defined by

$$F_X(x|\mathfrak{G}) = F_X(x|\mathfrak{G})(\omega) = P_{\hat{X}}(\omega, (-\infty, x]) \quad \text{a.s.} \quad (x\in\mathbf{R});$$

is called a *conditional distribution function* of X with respect to $\mathfrak{G}$. [Note that if $(\Omega, \mathfrak{A}, P)$ is an arbitrary probability space, and $\mathfrak{G}$ an arbitrary sub-σ-algebra of $\mathfrak{A}$, then for each $X\in\mathfrak{Z}(\Omega,\mathfrak{A})$ there always exists a regular conditional distribution (and so also a conditional distribution function) of X with respect to $\mathfrak{G}$ (see e.g. [16, p. 373])].

Now to the integral representation. Let $X\in\mathfrak{L}(\Omega,\mathfrak{A},P)$, $\mathfrak{G}$ be a sub-σ-algebra of $\mathfrak{A}$, g: $\mathbf{R}\to\mathbf{R}$ a Borel-measurable function with $E[g(X)]<\infty$, and $F_X(X|\mathfrak{G})$ be a conditional distribution function of X relative to $\mathfrak{G}$. Then there exists a $G\in\mathfrak{G}$ with $P(G)=0$ such that for all $\omega\in\Omega\setminus G$ (sdee [16, p. 375])

$$E[g(X)|\mathfrak{G}](\omega) = \int g(x)\,d\big(F_X(x|\mathfrak{G})(\omega)\big). \tag{2.12}$$

For the proofs an (ordinary) Lindeberg condition of order s, $s>0$ — generalized to the situation of a φ-decomposable limiting r.v. (cf. [5]) — and sometimes the usual Feller condition will be needed. Both will be formulated for $X_k\in\mathfrak{Z}(\Omega,\mathfrak{A})$. If $X_k^s\in\mathfrak{L}(\Omega,\mathfrak{A},P)$ for some $s\in(0,\infty)$ and all $k\in\mathbf{N}$, then the sequence $(X_k)_{k\in\mathbf{N}}$ satisfies a *Lindeberg condition of order* s, if for every $\delta>0$

$$\Big(\sum_{k=1}^{n}\int_{|x|\geqq\delta/\varphi(n)}|x|^s\,dF_{X_k}(x)\Big)\Big/\Big(\sum_{k=1}^{n}E[|X_k|^s]\to 0 \quad (n\to\infty). \tag{2.13}$$

If $0<\sigma_k^2<\infty$, where $\sigma_k^2:=E[X_k^2]$, $k\in\mathbf{N}$, and $s_n=\big(\sum_{k=1}^{n}\sigma_k^2\big)^{1/2}$, then $(X_k)_{k\in\mathbf{N}}$ satisfies a *Feller-condition,* if

$$\lim_{n\to\infty}\max_{1\leqq k\leqq n}\frac{\sigma_k^2}{s_n} = 0. \tag{2.14}$$

3. A generalization of the Trotter-operator for dependent r.v.'s. As already mentioned in the introduction, the Trotter-operator plays an important role in establishing rates of convergence for independent r.v.'s. For the development of corresponding assertions in the instance of dependent r.v.'s a new operator concept — closely related to the usual Trotter-operator — will be introduced in this paper. To elucidate the connections, let us first recall the definition of the Trotter-operator and its most important properties.

For any $X\in\mathfrak{Z}(\Omega, \mathfrak{A})$ having distribution function F_X the associated Trotter-operator V_X: $C_B\to C_B$ is defined for $f\in C_B$ by

(3.1) $$V_X f(y) := \int_{\mathbf{R}} f(x+y)\,dF_X(x) = E[f(X+y)], \quad (y\in\mathbf{R}).$$

Lemma 1. *Let* $X, Y\in\mathfrak{Z}(\Omega, \mathfrak{A})$. *Let* $X_1, \ldots, X_n$, $Z_1, \ldots, Z_n$, $n\in\mathbf{N}$, *be independent r.v.'s belonging to* $\mathfrak{Z}(\Omega, \mathfrak{A})$. *Then*

a) V_X *is a positive, linear operator satisfying inequality*

(3.2) $$\|V_X f\|_{C_B} \leqq \|f\|_{C_B} \quad (f\in C_B);$$

b) $V_X=V_Y$ *provided* X *and* Y *are identically distributed;*

c) V_X *and* V_Y *are commutative provided* X *and* Y *are independent;*

(3.3) d) $$V_{S_n} f = V_{X_1}V_{X_2}\ldots V_{X_n} f \quad (f\in C_B);$$

(3.4) e) $$\|V_{S_n} f - V_{\sum_{k=1}^n Z_k} f\|_{C_B} \leqq \sum_{k=1}^n \|V_{X_k} f - V_{Z_k} f\|_{C_B} \quad (f\in C_B).$$

The Trotter operator may be generalized as follows by using the concept of conditional expectation.

Definition 1. Let $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$ and $\mathfrak{G}$ be an arbitrary sub-σ-algebra of $\mathfrak{A}$. The conditional Trotter operator $V_X^{\mathfrak{G}}$: $C_B\to C_B\times(\mathfrak{Z}(\Omega, \mathfrak{G}))$ of X relative to $\mathfrak{G}$ is defined for $f\in C_B$ by

$$V_X^{\mathfrak{G}} f(y) := E[f(X+y)\,|\,\mathfrak{G}] \quad (y\in\mathbf{R}).$$

The most important properties of this operator, which is uniquely determined up to a set of measure zero by definition, are collected in the following lemma; below one has set $((V_X^{\mathfrak{G}} f)(y))(\omega)=(V_X^{\mathfrak{G}} f)(y, \omega)$.

Lemma 2. *Let* $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$ *be an arbitrary sub-σ-algebra of* $\mathfrak{A}$, *and* f *and* g *belong to* C_B. *Then*

a) $(V_X^{\mathfrak{G}} f)(y, \cdot)\in\mathfrak{Z}(\Omega, \mathfrak{G})$ $(y\in\mathbf{R})$;

b) *there exists a set* $G_1\in\mathfrak{G}$ *with* $P(G_1)=0$ *such that*

$$\sup_{y\in\mathbf{R}} |(V_X^{\mathfrak{G}} f)(y, \omega)| \leqq \|f\|_{C_B} \quad (\omega\in\Omega\setminus G_1;\ f\in C_B);$$

c) *there exists a set* $G_2\in\mathfrak{G}$ *with* $P(G_2)=0$ *such that* $(V_X^{\mathfrak{G}} f)(\cdot, \omega)\in C_B$ *for all* $\omega\in\Omega\setminus G_2$;

d) *there exists a set* $G_3\in\mathfrak{G}$ *with* $P(G_3)=0$ *such that* $(V_X^{\mathfrak{G}}(\alpha f+\beta g))(\cdot, \omega)=$ $=\alpha(V_X^{\mathfrak{G}} f)(\cdot, \omega)+\beta(V_X^{\mathfrak{G}} g)(\cdot, \omega)$ *for all* $\omega\in\Omega\setminus G_3$ *and* $\alpha, \beta\in\mathbf{R}$;

e) $(V_X^{\mathfrak{G}} f)(y)=E[f(X+y)|\mathfrak{G}]=(V_X f)(y)$ *a.s. provided* $\mathfrak{A}(X)$ *is independent of* $\mathfrak{G}$.

Proof a). An immediate consequence of Definition 1.

b) In view of (2.7), (2.8) one has

$$\sup_{y\in\mathbf{R}} |V_X^{\mathfrak{G}} f(y,\omega)| \leqq \sup_{y\in\mathbf{R}} |E[f|(X+y)||\mathfrak{G}](\omega)| \leqq E[\|f\|_{C_B}|\mathfrak{G}] = \|f\|_{C_B} \quad \text{a.s.}$$

c) Since $V_X^{\mathfrak{G}} f(y)$ is bounded a.s. by part b), it remains to show that $V_X^{\mathfrak{G}} f$ is uniformly continuous a.s. Let $\varepsilon>0$ be arbitrary. Since $f\in C_B(\mathbf{R})$, there exists a $\delta>0$ such that $|f(y_1)-f(y_2)|<\varepsilon$ for all $y_1, y_2\in\mathbf{R}$ with $|y_1-y_2|<\delta$, so that $\sup_{x\in\mathbf{R}} |f(x+y_1)-f(x+y_2)|<\varepsilon$. But (2.9) and (2.7) yield

$$|V_X^{\mathfrak{G}} f(y_1,\omega)-V_X^{\mathfrak{G}} f(y_2,\omega)| = |E[f(X+y_1)|\mathfrak{G}](\omega)-E[f(X+y_2)|\mathfrak{G}](\omega)| \leqq$$

$$\leqq E[|f(X+y_1)-f(X+y_2)||\mathfrak{G}](\omega) \leqq \sup_{x\in\mathbf{R}} |f(x+y_1)-f(x+y_2)| < \varepsilon \quad \text{a.s.}$$

establishing c).

d) and e) follow directly from (2.9), (2.10), respectively.

From Lemma 2 b)—d) one obtains

Corollary 1. *Let $(\Omega, \mathfrak{A}, P)$, $\mathfrak{G}$, X and f be given as in Lemma 2. There exists a set $G\in\mathfrak{G}$ with $P(G)=0$ such that $(V_X^{\mathfrak{G}} f)(\cdot,\omega)$ is a linear operator of C_B into itself for all $\omega\in\Omega\setminus G$ satisfying $\|(V_X f)(\cdot,\omega)\|_{C_B}\leqq\|f\|_{C_B}$.*

Proof. With G_1, G_2, G_3 given as in Lemma 2 b)—d), then $(V_X^{\mathfrak{G}} f)(\cdot,\omega)$ is a contraction endomorphism on C_B for each $\omega\in\Omega\setminus G$, where $G:=G_1\cup G_2\cup G_3$ with $P(G)=0$.

Basic for the main convergence theorem of this paper is the counterpart of inequality (3.4) for the operator $V_{S_n}^{\mathfrak{G}}$ for partial sums S_n of not necessarily independent r.v.'s. For this purpose two lemmas will be needed.

Lemma 3. *Given $(\Omega, \mathfrak{A}, P)$ and any $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, there exists a set $A=A(X)\in\mathfrak{A}$ with $P(A)>0$ such that*

$$|E[X]| \leqq |X(\omega)| \quad (\omega\in A).$$

Take $A:=\{\omega\in\Omega;\ |E[X]|\leqq|X(\omega)|\}$ and show that assumption $P(A)=0$ leads to a contradiction.

Lemma 4. *Let $X, Y\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, $f\in C_B$ and $\mathfrak{G}$ be any sub-σ-algebra of $\mathfrak{A}$. To each $y\in\mathbf{R}$ there exists a set $G^y=G^y(f, X, Y)\in\mathfrak{G}$ with $P(G^y)>0$ such that*

$$|V_{X+Y} f(y)| \leqq |V_X(V_Y^{\mathfrak{G}} f)(y,\omega)| \quad (\omega\in G^y). \tag{3.5}$$

Proof. According to Lemma 2 c) there is a set $G_1^y\in\mathfrak{G}$ with $P(G_1^y)=0$ such that $(V_Y^{\mathfrak{G}} f)(\cdot,\omega)\in C_B$ for all $\omega\in\Omega\setminus G_1^y$. Since $E[f(X+Y+y)|\mathfrak{G}]\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$, on account of Lemma 3 to each $y\in\mathbf{R}$ there exists a set $G_2^y=G_2^y(f, X, Y)\in\mathfrak{G}$ with

$P(G_2^y)>0$ such that, noting (3.1), Definition 1 and (2.5),

$$\left|V_X((V_Y^{\mathfrak{G}}f)(y,\omega))(y) = \left|\int_{\mathbf{R}}(V_Y^{\mathfrak{G}}f)(x+y,\omega)\,dF_X(x)\right| =\right.$$

$$= \left|E[E[f(X+Y+y)|\mathfrak{G}](\omega)]\right| =$$

$$= \left|[E[f(X+Y+y)]|\mathfrak{G}](\omega)\right| \geqq \left|E[E[f(X+Y+y)|\mathfrak{G}]]\right| =$$

$$= |E[f(X+Y+y)]| = |V_{X+Y}f(y)|$$

for all $\omega\in G^y:=\Omega\setminus(G_1^y\cap G_2^y)$. Since $G^y\in\mathfrak{G}$ and $P(G^y)=0$ by definition of G^y, the proof is complete.

Now to the fundamental lemma of the paper, namely the counterpart of assertions (3.3) and (3.4) for the operator $V_X^{\mathfrak{G}}$.

Lemma 5. *Given* $\mathfrak{L}(\Omega,\mathfrak{A},P)$, *let* $(X_n)_{n\in\mathbf{N}}$ *be a sequence of r.v.'s from* $\mathfrak{L}(\Omega,\mathfrak{A},P)$, $(\mathfrak{G}_n)_{n\in\mathbf{N}}$ *a sequence of sub-σ-algebras from* $\mathfrak{A}$, $\mathfrak{G}_0=\{\Phi,\Omega\}$.

a) *For each* $f\in C_B$ *one has*

$$(V_{X_1}^{\mathfrak{G}_0}V_{X_2}^{\mathfrak{G}_1}\dots V_{X_n}^{\mathfrak{G}_{n-1}}f)(y) = V_{S_n}f(y) \quad a.s. \quad (y\in\mathbf{R};\ n\in\mathbf{N}). \tag{3.6}$$

b) *If* $(Z_n)_{n\in\mathbf{N}}$ *is a further sequence from* $\mathfrak{L}(\Omega,\mathfrak{A},P)$ *it being assumed that the* Z_n, *for each* $n\in\mathbf{N}$, *are independent amongst themselves as well as of the* X_n, *then there exist for each* $y\in\mathbf{R}$, $n\in\mathbf{N}$ *and* $1\leqq k\leqq n$ *sets*

$$G_{n,k-1}^y\in\mathfrak{G}_{k-1} \quad \textit{with} \quad P(G_{n,k-1}^y)>0$$

such that for each $\omega=\omega(n,k,y)\in G_{n,k-1}^y$

$$\|V_{S_n}f-V_{\sum_{k=1}^n Z_k}f\| \leqq \sum_{k=1}^n \sup_{y\in\mathbf{R}}|(V_{X_k}^{\mathfrak{G}_{k-1}}f)(y,\omega)-V_{Z_k}f(y)| \quad (n\in\mathbf{N}). \tag{3.7}$$

Proof. Now $E[X_k|\mathfrak{G}_0]=E[X_n]$ a.s., all $k\in\mathbf{N}$ by (2.6). So a repeated application of (2.11) as well as (3.3) yield for $n\in\mathbf{N}$ and $y\in\mathbf{R}$

$$(V_{X_1}^{\mathfrak{G}_0}V_{X_2}^{\mathfrak{G}_1}\dots V_{X_n}^{\mathfrak{G}_{n-1}}f)(y) =$$

$$= E[E\dots E[f(X_1+\dots+X_n+y)|\mathfrak{G}_{n-1}]\dots|\mathfrak{G}_1]|\mathfrak{G}_0] =$$

$$= E[f(X_1+\dots+X_n+y)|\mathfrak{G}_0] = E[f(X_1+\dots+X_n+y)] =$$

$$= (V_{X_1}\dots V_{X_n}f)(y) = (V_{S_n}f)(y) \quad \text{a.s.,}$$

establishing (3.6). Concerning part b), one has by (3.6), (3.2) and Lemma 1 c),

$$\|V_{S_n}f - V_{\sum_{k=1}^n Z_k} f\|_{C_B} = \tag{3.8}$$

$$= \sup_{y\in \mathbf{R}} \Big|\sum_{k=1}^n (V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \dots V_{X_{k-1}}^{\mathfrak{G}_{k-2}} [V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] V_{\sum_{j=k+1}^n Z_j} f)(y)\Big| \leqq$$

$\leqq \sum_{k=1}^n \sup_{y\in} |(V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \dots V$ [illegible] $V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] f)(y)|.$

According to Lemma 4 applied to the r.v.'s S_{k-1} and X_k, there exists to each $y\in\mathbf{R}$ a set $G_{k-1}^y \in \mathfrak{G}_{k-1}$ with $P(G_{k-1}^y)>0$. Associating to each $y\in\mathbf{R}$ a fixed $\omega_{n,k}\in G_{k-1}^y$ for which inequality (3.5) holds, one deduces by applying (3.6) and (3.2) the estimate

$$|(V_{X_1}^{\mathfrak{G}_0} V_{X_2}^{\mathfrak{G}_1} \dots V_{X_{k-1}}^{\mathfrak{G}_{k-2}} [V_{X_k}^{\mathfrak{G}_{k-1}} - V_{Z_k}] f)(y)| \leqq$$

$$\leqq |V_{S_{k-1}} (V_{X_k}^{\mathfrak{G}_{k-1}} f)(y, \omega_{n,k}) - (V_{S_{k-1}} V_{Z_k} f)(y)| \leqq$$

$$\leqq \sup_{y\in\mathbf{R}} |(V_{X_k}^{\mathfrak{G}_{k-1}} f)(y, \omega_{n,k}) - V_{Z_k} f(y)|.$$

If one now takes the supremum over all $y\in\mathbf{R}$ on the left side of this inequality and then sums over k, the proof of (3.7) follows in conjunction with (3.8).

4. Convergence theorems for dependent random variables. This section is concerned with weak convergence theorems in the case of arbitrary dependent r.v.'s. The basis is a general limit theorem which yields both the CLT and WLLN by specializing the limit r.v. Since the results of this section deal with convergence without rates, it is possible to formulate them also for uniform convergence of distribution functions or for stochatic convergence. The hypotheses are, apart from the usual Lindeberg conditions for the sequences of r.v.'s $(X_k)_{k\in\mathbf{N}}$ and the decomposition components $(Z_k)_{k\in\mathbf{N}}$, the positivity and the uniform boundedness of the second moments of the X_k, as well as the moment condition (4.1). The latter reduces to the coincidence of the first and second moments of X_k and Z_k provided the r.v.'s are independent. Since the σ-algebras $\mathfrak{G}_k$, $k\in\mathbf{P}$, occurring in (4.1) may, apart from $\mathfrak{G}_0$, be chosen freely, distinct forms of dependency are admitted.

4.1. General limit theorem.

Theorem 1. *Let $(X_k)_{k\in\mathbf{N}}$ be a sequence of dependent r.v.'s such that $0<m\leqq E[X_k^2]\leqq M<\infty$ for $k\in\mathbf{N}$, and some constants m, $M>0$. Let $(\mathfrak{G}_k)_{k\in\mathbf{N}}$ be a sequence of sub-σ-algebras of $\mathfrak{A}$, $\mathfrak{G}_0=\{\Phi, \Omega\}$, and Z a φ-decomposable r.v. with decomposition components Z_k, $k\in\mathbf{N}$. If*

$$E[X_k^j|\mathfrak{G}_{k-1}] = E[Z_k^j] \quad (k\in\mathbf{N};\ j\in\{1, 2\}) \tag{4.1}$$

and the sequences $(X_k)_{k\in\mathbf{N}}$, $(Z_k)_{k\in\mathbf{N}}$ both satisfy Lindeberg conditions of order 2 (cf.

(2.13)), *then there holds for each* $f \in C_B^2$ *in case*

(4.2) $$\varphi(n) = O(n^{-1/2}) \quad (n \to \infty)$$

(4.3) $$\|V_{\varphi(n)S_n} f - V_Z f\|_{C_B} = o_f(1) \quad (n \to \infty).$$

If the distribution function F_Z *of* Z *is continuous, one has in addition*

(4.4) $$\sup_{x \in \mathbf{R}} |F_{\varphi(n)S_n}(x) - F_Z(x)| = o(1) \quad (n \to \infty).$$

Proof. Firstly, one can ensure that the r.v.'s Z_k, $k \in \mathbf{N}$, are independent of the X_k as well as of the sub-σ-algebras $\mathfrak{G}_k$, $k \in \mathbf{N}$ by means of an appropriate choice of the probability space. According to Lemma 5 b) to each $y \in \mathbf{R}$, $n \in \mathbf{N}$ and $1 \leqq k \leqq n$ there exists sets $G_{n,k-1}^y \in \mathfrak{G}_{k-1}$ with $P(G_{n,k-1}^y) > 0$ such that for each $\omega = \omega(n, k, y) \in G_{n,k-1}$ by (2.4)

(4.5) $$\|V_{\varphi(n)S_n} f - V_Z f\|_{C_B} = \|V_{\varphi(n)S_n} f - V_{\varphi(n)\sum_{k=1}^n Z_k} f\|_{C_B} \leqq$$
$$\leqq \sum_{k=1}^n \sup_{y \in \mathbf{R}} |(V_{\varphi(n)X_k}^{\mathfrak{G}_{k-1}} f)(y, \omega) - (V_{\varphi(n)Z_k} f)(y)|.$$

Now choose $\bar{\omega} \in G_{n,k-1}^y$ such that condition (4.1) is satisfied for it. An application of (2.12) plus Taylor's formula to both $f(\varphi(n)X_k + y)$ and $f(\varphi(n)Z_k + y)$ then gives

$$|V_{\varphi(n)X_k}^{\mathfrak{G}_{k-1}} f(y, \bar{\omega}) - V_{\varphi(n)Z_k} f(y)| =$$

(4.6) $$= \Big| \int_{\mathbf{R}} f(x+y)\, d\big(F_{\varphi(n)X_k}(x|\mathfrak{G}_{k-1})\big)(\bar{\omega}) - \int_{\mathbf{R}} f(x+y)\, dF_{\varphi(n)X_k}(x) \Big| =$$

$$= \Big| \int_{\mathbf{R}} \Big\{ \sum_{j=0}^{2} \frac{\varphi(n)^j x^j}{j!} f^{(j)}(y) + \frac{1}{2} \varphi(n)^2 x^2 [f^2(\eta) - f^2(y)] \Big\} d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar{\omega})\big) -$$
$$- \int_{\mathbf{R}} \Big\{ \sum_{j=0}^{2} \frac{\varphi(n)^j x^j}{j!} f^{(j)}(y) + \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)] \Big\} dF_{Z_k}(x) \Big|$$

where $|\eta - y| \leqq \varphi(n)|x|$. Since $f^{(2)} \in C_B$, to any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that $|f^{(2)}(\eta) - f^{(2)}(y)| < \varepsilon$ for $|\eta - y| < \delta$. But by (4.2) to each $\delta > 0$ and $x \in \mathbf{R}$ there is an $n \in \mathbf{N}$ with $|\eta - y| \leqq \varphi(n)|x| < \delta$. So splitting up the range of integration in (4.6) into $\{x \in \mathbf{R};\ |x| < \delta/\varphi(n)\}$ and its complementary set, one obtains by (2.12) and (4.1) the expression

$$\Big| \Big(\int_{|x| < \delta/\varphi(n)} + \int_{|x| \geqq \delta/\varphi(n)} \Big) \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)]\, d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar{\omega})\big) -$$
$$- \Big(\int_{|x| < \delta/\varphi(n)} + \int_{|x| \geqq \delta/\varphi(n)} \Big) \frac{1}{2} \varphi(n)^2 x^2 [f^{(2)}(\eta) - f^{(2)}(y)]\, dF_{Z_k}(x) \Big|$$

which can in turn be estimated by

$$(4.7)\qquad \frac{\varphi(n)^2}{2}\Big\{\varepsilon\big(E[|X_k|^2\,|\,\mathfrak{G}_{k-1}](\bar\omega)+E[|Z_k|^2]\big)+ + \|f^{(2)}\|_{C_B}\Big(\int\limits_{|x|\ge\delta/\varphi(n)} x^2\,d\big(F_{X_k}(x\,|\,\mathfrak{G}_{k-1})(\bar\omega)\big)+\int\limits_{|x|\ge\delta/\varphi(n)} x^2\,dF_{Z_k}(x)\Big)\Big\}.$$

Since $E[X_k^2]\le M$, (2.5) and (4.1) yield that $E[Z_k^2]\le M$ as well as $E[X_k^2|\mathfrak{G}_{k-1}]\le M$ a.s. for all $k\in\mathbf{N}$. Since further $E[X_k^2]\ge m>0$, there are constants $M_1, M_2>0$ such that $E[X_k^2|\mathfrak{G}_{k-1}]\le M_1E[X_k^2]$ a.s., and

$$E[X_k^2\mathbf{1}_{\{|X_k|\ge\delta/\varphi(n)\}}\,|\,\mathfrak{G}_{k-1}]\le M_2E[X_k^2\mathbf{1}_{\{|X_k|\ge\delta/\varphi(n)\}}]\quad\text{a.s.,}$$

$\mathbf{1}_A$, $A\in\mathfrak{A}$ being the indicator function. Then one deduces from (4.6) and (4.7) that for each $y\in\mathbf{R}$, $n\in\mathbf{N}$ and $1\le k\le n$

$$(4.8)\qquad |(V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k}f)(y,\bar\omega)-V_{\varphi(n)Z_k}f(y)|\le \le\frac{\varphi(n)^2}{2}\Big[\varepsilon\big(M_1E[X_k^2]+E[Z_k^2]\big)+ +\|f^{(2)}\|_{C_B}\Big(M_2\int\limits_{|x|\ge\delta/\varphi(n)} x^2\,dF_{X_k}(x)+\int\limits_{|x|\ge\delta/\varphi(n)} x^2\,dF_{Z_k}(x)\Big)\Big].$$

Taking now the supremum on the left side of this inequality for all $y\in\mathbf{R}$, summing up over k from 1 to n, and finally dividing the result by the strictly positive expression $(\varphi(n)^2/2)\sum_{k=1}^{n}(M_1E[X_k^2]+E[Z_k^2])$, one obtains from (4.5)

$$(4.9)\qquad 2\|V_{\varphi(n)S_n}f-V_Zf\|_{C_B}\Big/\Big(\varphi(n)^2\sum_{k=1}^{n}(M_1E[X_k^2]+E[Z_k^2])\Big)\le \le\varepsilon+\|f^{(2)}\|_{C_B}\left\{\frac{M_2\sum\limits_{k=1}^{n}\int\limits_{|x|\ge\delta/\varphi(n)}x^2\,dF_{X_k}(x)}{M_1\sum\limits_{k=1}^{n}E[X_k^2]}+\frac{\sum\limits_{k=1}^{n}\int\limits_{|x|\ge\delta/\varphi(n)}x^2\,dF_{Z_k}(x)}{\sum\limits_{k=1}^{n}E[Z_k^2]}\right\}.$$

Since the sequences $(X_k)_{k\in\mathbf{N}}$, $(Z_k)_{k\in\mathbf{N}}$ are assumed to satisfy Lindeberg conditions of order 2, the term in square brackets converges to zero for $n\to\infty$ by (4.2). Since $\varepsilon>0$ was arbitrary, assertion (4.3) follows by noting that the denominator on the left side of (4.9) is uniformly bounded in n because of (4.2) and the uniform boundedness of $E[X_k^2]$ and $E[Z_k^2]$. This in turn yields (4.4) since F_Z is continuous (cf. [11, p. 140]).

4.2. The central limit theorem. A particular case of Theorem 1 is the following version of the CLT.

Theorem 2. *Let* $(X_k)_{k\in N}$ *and* $(\mathfrak{G}_k)_{k\in P}$ *be given as in Theorem* 1, *denote* $\sigma_k:=E[X_k^2]$, $k\in N$, *and* $s_n:=(\sum_{k=1}^{n}\sigma_k^2)^{1/2}$.

a) *If there holds*

(4.10) $$E[X_k^j|\mathfrak{G}_{k-1}]=\sigma_k^j E[X^{*j}] \quad \text{a.s.} \quad (k\in N;\ j\in\{1,2\}),$$

and if the sequences $(X_k)_{k\in N}$ *and* $(\sigma_k)_{k\in N}$ *satisfy a Lindeberg condition of order 2 and the Feller condition* (2.14), *respectively, then one has*

(4.11) $$\|V_{s_n^{-1}S_n}f-V_{X^*}f\|_{C_B}=o_f(1) \quad (f\in C_B^2;\ n\to\infty)$$

or, equivalently,

(4.12) $$\sup_{x\in R}|F_{s_n^{-1}S_n}(x)-F_{X^*}(x)|=o(1) \quad (n\to\infty).$$

b) *If the r.v.'s* X_k *are in addition idetically distributed,* (4.10) *is valid with* $\sigma_k=1$, *then one has for* $f\in C_B^2$

(4.13) $$\|V_{n^{-1/2}S_n}f-V_{X^*}f\|_{C_B}=o_f(1) \quad (n\to\infty)$$

or, equivalently,

(4.14) $$\sup_{x\in R}|F_{n^{-1/2}S_n}(x)-F_{X^*}(x)|=o(1) \quad (n\to\infty).$$

Proof. a) Choosing for the decomposition components Z_k of Theorem 1 the independent r.v.'s $\sigma_k X^*$, then condition (2.4) is satisfied with $\varphi(n):=s_n^{-1}$. Further, $Z=\sigma_k X^*$ implies that hypothesis (4.10) corresponds to (4.1). Since also $E[X_k^2]\le M<\infty$ for all $k\in N$, $\varphi(n)=s_n^{-1}\le M_n^{-1/2}$, and so (4.2) is satisfied. It can be shown (cf. [3, p. 268]) that the Lindeberg condition for $(X_k)_{k\in N}$ plus the Feller condition for $(\sigma_k)_{k\in N}$ yields the Lindeberg condition for $(Z_k)_{k\in N}$. So assertion (4.11) follows from (4.3). Finally, (4.12) is a derivation of (4.4) in view of the continuity of F_{X^*}.

b) Assertions (4.13) and (4.14) are immediate consequences of (4.11) and (4.12), noting that conditions (2.13) and (2.14) are always automatically satisfied for identically distributed r.v.'s.

4.3. The weak law of large numbers. Since the partial sums in the WLLN are normalized by n^{-1} and not $n^{-1/2}$ as for the CLT, just a Lindeberg condition of order one need be assumed for $(X_k)_{k\in N}$ while the moment condition (4.1) reduces to the condition that the conditional moments of the X_k with respect to $\mathfrak{G}_{k-1}$ be zero.

Theorem 3. *Let* $(X_k)_{k\in N}$ *and* $(\mathfrak{G}_k)_{k\in P}$ *be defined as in Theorem* 1, *let* X_0 *be a r.v. taking on the value zero with probability* 1, *and let*

$$E[X_k|\mathfrak{G}_{k-1}]=0 \quad a.s. \quad (k\in N).$$

a) *If the sequence* $(X_k)_{k\in N}$ *satisfies a Lindeberg condition of order* 1 *with* $\varphi(n)=$

$=n^{-1}$, *then* $(X_k)_{k\in\mathbf{N}}$ *satisfies the weak law of large numbers, i.e., for each* $\varepsilon>0$

(4.15) $$\lim_{n\to\infty} P(\{|n^{-1}S_n|\geqq\varepsilon\})=0.$$

b) *If the r.v.'s* X_k *are just identically distributed, then* (4.15) *again holds.*

Proof. a) If one chooses the decomposition components Z_k such that $P_{Z_k}=P_{X_0}$ for all $k\in\mathbf{N}$, then (2.4) is satisfied with $\varphi(n)=n^{-1}$. An application of the Taylor expansion of $f\in C_B^1$ up to the order 1 yields, just as in the proof of Theorem 1,

(4.16) $$\|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B}=o_f(1)\quad(n\to\infty).$$

Since convergence in distribution is equivalent to stochastic convergence for the limit r.v. X_0 (cf. e.g. [3, 220]), (4.16) implies assertion (4.15). Part b) is a particular case of a), Lindeberg's condition being satisfied automatically.

5. Convergence theorems for dependent random variables with o-rates. It is possible to equip the limit theorems of Section 4 with rates without any larger modifications of the proofs; just stronger assumptions upon the moments and higher-order Lindeberg conditions will be needed. However, the assertions will now be restricted to the convergence in distribution of the normalized partial sums, since the equivalence of convergence in distribution with uniform convergence of the distribution functions in case of the CLT and with stochastic convergence in the case of the WLLN is only valid for convergence *without* rates.

5.1. A general approximation theorem.

Theorem 4. *Let* $(X_k)_{k\in\mathbf{N}}$ *be a sequence of r.v.'s,* $r\in\mathbf{N}\setminus\{1\}$, *and* m, M *be two positive constants with* $0<m\leqq E[|X_k|^r]\leqq M<\infty$ *for* $k\in\mathbf{N}$. *Let* $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ *be a sequence of sub-σ-algebras of* $\mathfrak{A}$ *with* $\mathfrak{G}_0=\{\Phi,\Omega\}$. *If* Z *is a φ-decomposable r.v. with decomposition components* Z_k, $k\in\mathbf{N}$, *condition* (4.1) *is fulfilled for* $j\in\mathbf{N}$, $1\leqq j\leqq r$, *and the sequence* $(X_k)_{k\in\mathbf{N}}$ *as well as* $(Z_k)_{k\in\mathbf{N}}$ *satisfy Lindeberg conditions of order* r, *then for* $f\in C_B^r$

(5.1) $$\|V_{\varphi(n)S_n}f-V_Zf\|_{C_B}=o_f(n[\varphi(n)]^r)\quad(n\to\infty).$$

Proof. The proof of this theorem is based upon that of Theorem 1. Just as there one has inequality (4.5). For a suitable $\bar\omega$ (cf. the proof of Theorem 1), an application of Taylor's expansion, this time up to the order r for $f\in C_B^r$, yields

(5.2) $$|(V^{\mathfrak{G}_{k-1}}_{\varphi(n)X_k}f)(y,\bar\omega)-(V_{\varphi(n)Z_k}f)(y)|=$$

$$=\Bigg|\int_{\mathbf{R}}\Big\{\sum_{j=0}^{r}\frac{\varphi(n)^jx^j}{j!}f^{(j)}(y)+\frac{\varphi(n)^rx^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\Big\}\,d(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar\omega))-$$

$$-\int_{\mathbf{R}}\Big\{\sum_{j=0}^{r}\frac{\varphi(n)^jx^j}{j!}f^{(j)}(y)+\frac{\varphi(n)^rx^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\Big\}\,dF_{Z_k}(x)\Bigg|.$$

Following the arguments in the proof of Theorem 1 with $f^{(2)}$ replaced by $f^{(r)}$, one obtains after the range of integration has been split up and estimates analogous to (4.7) and (4.8) have been carried out, that for $1\leqq k\leqq n$, $y\in\mathbf{R}$, the right side of (5.2) is bounded from above by

$$\frac{\varphi(n)^r}{r!}\Big\{\varepsilon\big(M_1^* E[|X_k|^r\,|\,\mathfrak{G}_{k-1}](\bar\omega)+E[|Z_k|^r]\big)+$$

$$+\|f^{(r)}\|_{C_B}\Big(M_2^*\int\limits_{|x|\geqq\delta/\varphi(n)}|x|^r\,d\big(F_{X_k}(x\,|\,\mathfrak{G}_{k-1})(\bar\omega)\big)+\int\limits_{|x|\geqq\delta/\varphi(n)}|x|^r\,dF_{Z_k}(x)\Big)\Big\},$$

where M_1^* and M_2^* are the constants corresponding to M_1 and M_2 in inequality (4.8), noting that the remaining terms of the Taylor expansion up to the order r vanish on account of (4.1). The Lindeberg conditions of order r for the sequences $(X_k)_{k\in\mathbf{N}}$ and $(Z_k)_{k\in\mathbf{N}}$ then yield, as in Theorem 1,

$$(5.3)\qquad r!\,|V_{\varphi(n)S_n}f-V_Zf|/\Big(\varphi(n)^r\sum_{k=1}^{n}(M_1^*E[|X_k|^r]+E[|Z_k|^r])\Big)=o_f(1)\quad(n\to\infty).$$

Since $E[|X_k|^r]$ is uniformly bounded by hypothesis, and so also $E[|Z_k|^r]$ by (4.1), there exists a constant $M_3>0$ such that $\Big(\sum_{k=1}^{n}(M_1^*E[|X_k|^r]+E[Z_k]^r]\Big)\leqq nM_3$. Inserting this estimate into (5.3) gives statement (5.1).

5.2. Applications to the CLT and WLLN with o-rates. By specializing the limit r.v. in Theorem 4 one obtains

Theorem 5. *Let $(X_k)_{k\in\mathbf{N}}$ and $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ be given as in Theorem* 1, σ_k, s_n *as defined in Theorem* 2, *and let* $r\in\mathbf{N}$.

a) *If* (4.10) *is satisfied for* $1\leqq j\leqq r$, *and the sequences $(X_k)_{k\in\mathbf{N}}$ and $(\sigma_k)_{k\in\mathbf{N}}$ satisfy a Lindeberg condition of order r and the Feller condition* (2.14), *respectively, then* $f\in C_B^r$ *implies*

$$(5.4)\qquad \|V_{s_n^{-1}S_n}f-V_{X^*}f\|_{C_B}=o_f(ns_n^{-r})\quad(n\to\infty).$$

b) *If the r.v.'s X_k are identically distributed,* (4.10) *holds for* $\sigma_k=1$ *and* $1\leqq j\leqq r$, *then for* $f\in C_B^r$

$$(5.5)\qquad \|V_{n^{-r/2}S_n}f-V_{X^*}f\|_{C_B}=o_f(n^{(2-r)/2})\quad(n\to\infty).$$

Concerning the proof, assertion (5.4) follows from (5.1) just as does (4.13) from (4.3); (5.5) is immediate by (5.4).

If one compares the rate in (5.5) with that known for independent r.v.'s and MDS (cf. [5] or [7]), it will be seen that the same approximation order could be achieved even though the X_k are now dependent.

Now to the WLLN. Since a moment condition corresponding to (4.1) for $r\geqq 2$, i.e., a condition of form $E[X_k^j|\mathfrak{G}_{k-1}]=E[X_0^j]$ a.s., $1\leqq j\leqq r$ would now mean that only those r.v.'s X_k can be admitted that take on the value zero with probability 1 just as does X_0, since $E[E[X_k^2|\mathfrak{G}_{k-1}]]=E[X_k^2]=E[X_0^2]=0$, such a condition will now be replaced by the weaker (5.6).

Theorem 6. *Let $(X_k)_{k\in\mathbf{N}}$ and $(\mathfrak{G}_k)_{k\in\mathbf{P}}$ be defined as in Theorem 4, X_0 as in Theorem 3. If $(X_k)_{k\in\mathbf{N}}$ satisfies condition*

$$\text{(5.6)} \qquad n^{r-j}\sum_{k=1}^{n}|E[X_k^j|\mathfrak{G}_{k-1}]| = o\Big(\sum_{k=1}^{n}E[|X_k|^r]\Big) \quad a.s. \quad (1\le j\le r;\ n\to\infty),$$

for some $r\in\mathbf{N}$ as well as a Lindeberg condition of order r, then for $f\in C_B^2$, $n\to\infty$

$$\|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B} = o_f\Big(n^{-r}\sum_{k=1}^{n}E[|X_k|^r]\Big).$$

Proof. Choosing the decompostion components Z_k such that $P_{Z_k}=P_{X_0}$ and sets $\varphi(n)=n^{-1}$ as in the proof of Theorem 3, a Taylor expansion up to the order r yields, by taking into account that $E[|Z_k|^j]=0$, $1\le j\le r$, that for $y\in\mathbf{R}$ and suitable $\bar\omega$ (see (4.6) and (4.8))

$$|(V_{n^{-1}X_k}^{\mathfrak{G}_{k-1}}f)(y,\bar\omega)-V_{n^{-1}Z_k}f(y)| =$$

$$= \left|\int_{\mathbf{R}}\sum_{j=0}^{r}\frac{n^{-j}x^j}{j!}f^{(j)}(y)+\frac{n^{-r}x^r}{r!}[f^{(r)}(\eta)-f^{(r)}(y)]\,d\big(F_{X_k}(x|\mathfrak{G}_{k-1})(\bar\omega)\big)-f(0)\right| \le$$

$$\le \sum_{j=1}^{r}\frac{n^{-j}}{j!}\|f^{(j)}\|_{C_B}E[|X_k|^j|\mathfrak{G}_{k-1}](\bar\omega)+$$

$$+\frac{n^{-r}}{r!}\Big\{\varepsilon M_1^{**}E[|X_k|^r]+\|f^{(r)}\|_{C_B}\Big(M_2^{**}\int_{|x|\ge\delta n}|x|^r\,dF_{X_k}(x)\Big)\Big\},$$

M_1^{**} and M_2^{**} being the constants corresponding to M_1 and M_2 from (4.8). As in the proof of Theorem 1 one has in view of (5.6)

$$\text{(5.7)} \qquad \|V_{n^{-1}S_n}f-V_{X_0}f\|_{C_B}\Big/\Big(n^{-r}\sum_{k=1}^{n}E[|X_k|^r]\Big) = o_f(1) \quad (n\to\infty).$$

Note that the rate of approximation in (5.7) is a good as that given in [5] and [7] for the WLLN for independent r.v.'s and MDS, respectively. For $r=3$ the rate is $o(n^{-2})$, provided the r.v.'s are identically distributed. Though the r.v.'s X_k are now arbitrarily dependent, no additional assumption was needed to obtain this rate of convergence. So Theorem 6 can be regarded as a true generalization of the corresponding assertions in [5] and [7].

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