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Approximation by polynomials and extension of Parseval's identity for Legendre polynomials to the L^p case

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To Professor Károly Tandori on his 60th birth anniversary

In this note we look at the B-spline polynomial basis in the space $\mathcal{P}_k$ of real algebraic polynomials of order k i.e. of degree not exceeding $k-1$, $k \geqq 1$, over the interval $I=\langle -1, 1\rangle$. For a given sequence $t_0, \ldots, t_k$ with $t_0 \leqq \ldots \leqq t_k$ and $t_0 < t_k$ the corresponding B-spline of order k (cf. [6]) is the function

$$N(t_0, \ldots, t_k;\ t) = (t_k - t_0)[t_0, \ldots, t_k](\cdot - t)_+^{k-1},$$

where the square bracket denotes the devided difference taken at $t_0, \ldots, t_k$ and $s_+ = \max(s, 0)$. In particular, for $i=0, \ldots, k-1$, the spline

$$N_{i,k}(t) = N(\underbrace{-1, \ldots, -1}_{i+1}, \underbrace{1, \ldots, 1}_{k-i};\ t)$$

is a polynomial of degree $k-1$ in I and

(1.1) $$N_{i,k}(t) = \binom{k-1}{i}\left(\frac{1+t}{2}\right)^{k-1-i}\left(\frac{1-t}{2}\right)^{i}.$$

Clearly, we have the following properties:

(1.2) $$N_{i,k} \geqq 0 \quad \text{for} \quad i = 0, \ldots, k-1.$$

(1.3) $$\sum_{i=0}^{k-1} N_{i,k}(t) = 1 \quad \text{for} \quad t \in I.$$

(1.4) $$\mathcal{P}_k = \operatorname{span}[N_{0,k}, \ldots, N_{k-1,k}].$$

(1.5) $$\int_I N_{i,k} = \frac{2}{k}.$$

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For later convenience introduce

$$(f, g) = \int_I fg,$$

$$\|f\|_p = \Big(\int_I |f|^p\Big)^{1/p} \quad \text{if} \quad 1 \leqq p \leqq \infty,$$

$$\|\underline{a}\|_p = \Big(\sum_{i=0}^{k-1} |a_i|^p\Big)^{1/p} \quad \text{for} \quad \underline{a} \in R^k \quad \text{and} \quad 1 \leqq p \leqq \infty,$$

$$N_{i,k,p} = N_{i,k}/(N_{i,k}, 1)^{1/p} = N_{i,k} \cdot (k/2)^{1/p},$$

$$M_{i,k} = N_{i,k,1}.$$

Clearly, $N_{i,k} = N_{i,k,\infty}$. Now, Jensen's inequality and (1.3) imply for $\underline{a} \in R^k$

(1.6) $$\Big\|\sum_{i=0}^{k-1} a_i N_{i,k,p}\Big\|_p \leqq \|\underline{a}\|_p.$$

The kernel for our approximation method is defined as follows

(1.7) $$R_k(s, t) = \sum_{i=0}^{k-1} N_{i,k,p}(s) N_{i,k,q}(t) = \sum_{i=0}^{k-1} M_{i,k}(s) N_{i,k}(t)$$

with $1/p + 1/q = 1$. Clearly, R_k is independent of p and

(1.8) $$R_k(s, t) = R_k(t, s).$$

(1.9) $$R_k(s, t) \geqq 0 \quad \text{for} \quad s, t \in I.$$

For $f \in L^1(I)$ we also define

$$(R_k f)(t) = \int_I f(s) R_k(s, t)\, ds.$$

It now follows by (1.5) and (1.3) that

(1.10) $$R_k 1 = 1.$$

Thus, the standard argument with Hölder's or Jensen's inequality and (1.10) give for the norm of R_k: $L^p(I) \to L^p(I)$

(1.11) $$\|R_k\|_p = 1 \quad \text{for} \quad 1 \leqq p \leqq \infty.$$

Theorem 1. *Let* $f \in L^p(I)$ *if* $1 \leqq p < \infty$ *and let* $f \in C(I)$ *if* $p = \infty$. *Then*

(1.12) $$\|f - R_k f\|_p \to 0 \quad as \quad k \to 0.$$

Proof. Since we have (1.11) it is sufficient to check (1.12) in a dense set. For $f \in \mathcal{P}_n$ we have

(1.13) $$f = \sum_{j=0}^{n-1} c_j P_j$$

with some coefficients c_j, where P_0, P_1, ... are the orthonormal on I Legendre polynomials. In [3] we have proved

$$R_k(s, t) = \sum_{j=0}^{k-1} m_{j,k} P_j(s) P_j(t), \tag{1.14}$$

where

$$m_{j,k} = \frac{(k-1)\dots(k-j)}{(k+1)\dots(k+j)} \quad \text{for} \quad j = 1, \dots, k-1; \quad m_{0,k} = 1. \tag{1.15}$$

Thus for f as given in (1.13) and for $k \geqq n$ we have

$$R_k f = \sum_{j=0}^{n-1} m_{j,k} c_j P_j.$$

However (1.15) implies that for each j, $0 \leqq j \leqq n-1$, $m_{j,k} \to 1$ as $k \to \infty$ and therefore

$$R_k f \to \sum_{j=0}^{n-1} c_j P_j = f \quad \text{as} \quad k \to \infty,$$

and this completes the proof.

Remark. If we start with (1.14) and (1.15) as the definition of R_k, then Theorem 1 can be proved by a different method. Namely, extending the definition (1.15) by letting $m_{j,k}=0$ for $j \geqq k$ we see that R_k: $L^p(I) \to L^p(I)$ is a multiplier operator i.e. for

$$f \sim \sum_{j=0}^{\infty} c_j P_j$$

we have

$$R_k f = \sum_{j=0}^{\infty} m_{j,k} c_j P_j.$$

Now, the theory developed in [7] can be applied to obtain (1.12). The disadvantage of this approach is that it does not seem to imply (1.7).

To state our next result we need some more definitions. In $\mathcal{P}_k$ we introduce the descrete scalar product

$$\langle f, g \rangle = \sum_{i=0}^{k-1} f(i) g(i).$$

With respect to this scalar product the orthogonal Chebyshev polynomials $u_k^{(j)} \in \mathcal{P}_{j+1}$, $j=0, \dots, k-1$, are determined by the condition $u_k^{(j)}(0) = (j+1/2)^{1/2}$ (see e.g. [5]). Here $u_k^{(j)}(i)$ denotes the same value as $u_{i,k}^{(j)}$ in [3].

Theorem 2. *Let* $f\in L^p(I)$ *if* $1\leqq p<\infty$ *and let* $f\in C(I)$ *if* $p=\infty$. *Then for* $f\sim\sum_{j=0}^{\infty} c_j P_j$ *we have*

$$\left(\frac{2}{k}\sum_{i=0}^{k-1}\Big|\sum_{j=0}^{k-1} m_{j,k}u_k^{(j)}(i)c_j\Big|^p\right)^{1/p}\nearrow \|f\|_p \quad as \quad k\nearrow\infty,\quad p<\infty, \tag{1.16}$$

$$\max_{0\leqq i\leqq k-1}\Big|\sum_{j=0}^{k-1} m_{j,k}u_k^{(j)}(i)c_j\Big|\nearrow \|f\|_\infty \quad as \quad k\nearrow\infty.$$

Moreover, for $p=2$ (1.16) *gives*

$$\Big(\sum_{j=0}^{k-1} m_{j,k}c_j^2\Big)^{1/2}\nearrow \|f\|_2 \quad as \quad k\nearrow\infty. \tag{1.17}$$

Remark. The relation (1.17) is equivalent to

$$\sum_{j=0}^{\infty} c_j^2=\|f\|_2^2, \tag{1.18}$$

and therefore (1.16) can be regarded as an extension to the L^p case of (1.18).

Proof. In [2] the following relation is established

$$N_{i,k}=\frac{k-i}{k}N_{i,k+1}+\frac{i+1}{k}N_{i+1,k+1},$$

which rewritten for the $M_{i,k}$'s gives for $i=0,\dots,k-1$

$$M_{i,k}=\frac{k-i}{k+1}M_{i,k+1}+\frac{i+1}{k+1}M_{i+1,k+1}. \tag{1.19}$$

It is important that the right hand side is a convex combination. Introducing

$$\mathfrak{M}_{k,p}(f)=\left(\frac{2}{k}\sum_{i=0}^{k-1}|(f,M_{i,k})|^p\right)^{1/p}$$

we get by (1.19) and Jensen's inequality that

$$\mathfrak{M}_{k,p}(f)\leqq\mathfrak{M}_{k+1,p}(f). \tag{1.20}$$

Moreover Jensen's inequality and (1.3) give

$$\mathfrak{M}_{k,p}(f)\leqq\|f\|_p. \tag{1.21}$$

Now, letting in (1.6) $a_i=(f,M_{i,k})$ we obtain

$$\|R_kf\|_p\leqq\mathfrak{M}_{k,p}(f). \tag{1.22}$$

The combination of the inequalities (1.20)—(1.22) gives

$$\|R_kf\|_p\leqq\mathfrak{M}_{k,p}(f)\leqq\mathfrak{M}_{k+1,p}(f)\leqq\|f\|_p. \tag{1.23}$$

The next step is to identify $\mathfrak{M}_{k,p}(f)$ by means of the Fourier—Legendre coefficients. It is proved in [3] that

$$M_{i,k} = \sum_{j=0}^{k-1} (M_{i,k}, P_j)P_j = \sum_{j=0}^{k-1} m_{j,k} u_k^{(j)}(i) P_j$$

and therefore

$$(f, M_{i,k}) = \sum_{j=0}^{k-1} m_{j,k} u_k^{(j)}(i) c_j, \tag{1.24}$$

whence we infer

$$\mathfrak{M}_{k,p}(f) = \left(\frac{2}{k} \sum_{i=0}^{k-1} \left| \sum_{j=0}^{k-1} m_{j,k} u_k^{(j)}(i) c_j \right|^p \right)^{1/p}. \tag{1.25}$$

To get (1.16) it remains to insert (1.25) into (1.23) and apply Theorem 1. Formula (1.17) we obtain from (1.16) by the following orthogonality relation (cf. [3])

$$\langle u_k^{(j)}, u_k^{(i)} \rangle = \delta_{i,j} \frac{k}{2} m_{i,k}^{-1},$$

and this completes the proof of Theorem 2.

Corollary. *Let* $f \in L^p(I)$ *and* $f \sim \sum_{j=0}^{\infty} c_j P_j$. *Then* $\|f\|_p$ *can be numerically evaluated by means of* (c_j). *Indeed, we at first evaluate* (1.24) *and then* (1.25). *Moreover, inequalities* (1.23) *imply the following error estimate*

$$0 \leqq \|f\|_p - \mathfrak{M}_{k,p}(f) \leqq \|f - R_k f\|_p.$$

In particular, for $f = P_j$ *we get*

$$0 \leqq \|P_j\|_p - \mathfrak{M}_{k,p}(P_j) \leqq (1 - m_{j,k}) \|P_j\|_p.$$

Comments. Inequalities (1.20) and (1.21) are proved already in [4]. Theorem 2 is related to the L^p moment problem on finite interval by the formula

$$(f, N_{i,k}) = (-1)^{k-1-i} \binom{k-1}{i} \Delta^{k-1-i} \mu_i,$$

where

$$\mu_i = \int_{-1}^{1} f(s) \left(\frac{1+s}{2}\right)^i ds.$$

For more details we refer to [4] (see also [1]).

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