Acta Sci. Math., 48 (1985), 71—74

On the product of certain permutable subgroups

KERESZTÉLY CORRÁDI and PÉTER Z. HERMANN

Dedicated to Professor K. Tandori on his 60th birthday

It is well-known that the finite p-nilpotent groups form a Fitting class; in particular, for $N_1, N_2 \lhd G$ and N_1, N_2 p-nilpotent, $\langle N_1, N_2\rangle$ is also p-nilpotent. In [1] there was defined $\mathcal{N}(p, q)$ (generalizing the concept of p-nilpotence) as the class of finite groups, in which for every p-subgroup P, $|N(P)/C(P)|$ is not divisible by the prime q. By the theorem in [1], $\mathcal{N}(p, q)$ is a Fitting class for any primes $p\neq q$. In this paper we prove a stronger result:

Theorem. *Let G be a finite group and $H_1, H_2 \leqq G$. Assume that $H_1H_2 \leqq G$ (i.e. $H_1H_2=H_2H_1$) and $H_tM\leqq G(t=1, 2)$ for every q-subgroup M. Then $H_1, H_2 \in \mathcal{N}(p, q)$ implies $H_1H_2 \in \mathcal{N}(p, q)$.*

For the proof we need the following lemmas, dealing with the permutability of subgroups of a group G. (Throughout in the text, p and q are distinct fixed primes.)

Lemma 1. *Suppose that $H\leqq G$ and $HM=MH$ for any q-subgroup M. Let S be a subgroup of G then $(H\cap S)D=D(H\cap S)$ for any q-subgroup D in S.*

Proof. $(H\cap S)D=HD\cap S=S\cap DH=D(S\cap H)$.

Lemma 2. *Assume $H, K, L, T\leqq G$ and $L\leqq H\cap K$. If $G=HK=LT$ then $T=(T\cap H)(T\cap K)$.*

Proof. $H=G\cap H=LT\cap H=L(T\cap H)$, similarly $K=L(T\cap K)$, hence $G=HK=L(T\cap H)L(T\cap K)=L(T\cap H)(T\cap K)$, thus $T=T\cap L(T\cap H)(T\cap K)= =(T\cap L)(T\cap H)(T\cap K)=(T\cap H)(T\cap K)$.

Lemma 3. *If $R<G$, $|G:R|=q$ then $RD=DR$ for any q-subgroup D.*

Proof. It can be assumed that $D\nleqq R$. Let z be an element in $D\setminus R$ then $R\langle z\rangle=G=\langle z\rangle R$, hence $RD=G=DR$.

Received May 12, 1983.

Lemma 4. (see KEGEL [4] or [2, p. 677]). *Let A and B be subgroups of the finite group G. Suppose that for all* $x\in G$, $AB^x=B^xA$; *if* $AB\neq G$, *then at least one of A and B is contained in a normal subgroup of G, different from G.*

Proof of the theorem. By induction; suppose it were false and let G be a counterexample for which $f(G, H_1, H_2):=|G|+|H_1: H_1\cap H_2|_q+|H_2: H_1\cap H_2|_q+$ $+|G: H_1|+|G: H_2|$ is minimal. So $G=H_1H_2$ and there is a subgroup $U=U_pU_q$ in G with a normal Sylow p-subgroup U_p and cyclic Sylow q-subgroups like U_q such that all subgroups of U except U are (p-)nilpotent. (For the standard properties of such U-s we will use see [2, chapter IV.] or [3]). As $U=\langle U_r^q\colon r\in U\rangle$, each $H_tU=UH_t$.

(∗) For any subgroup $X\leqq G$ and $H_t\leqq X$ (for at least one t) $X=X\cap G=$ $=X\cap H_tH_{t'}=H_t(X\cap H_{t'})$, hence by Lemma 1, $X=G$ or $X\in\mathcal{N}(p, q)$. In particular, $H_tU=G$ $(t=1, 2)$.

Suppose $U_q\leqq H_t$, then $U_q^{H_t\cap U}:=\langle U_q^s\colon s\in H_t\cap U\rangle\leqq H_t\cap U$, so $H_t\cap U\leqq$ $\leqq Z(U)U_q$; thus for a suitable $u\in U$ we get $U_q^u\not\leqq H_t$, hence $H_t<H_tU_q^u$, yielding $1\neq|H_tU_q^u\colon H_t|\,|\,(|U_q|, |G\colon H_t|)\,|\,(|U_q|, |U_p|)=1$, a contradiction, which gives

$$(1)\qquad U_q\not\leqq H_t\quad (t=1, 2).$$

(2) Either (i) $q=|G\colon H_t|$ and $U_p\leqq\bigcap_{y\in G}H_t^y$ (for at least one t), or

(ii) $T:=\langle Q\colon Q\in \mathrm{Syl}_q(G)\rangle\neq G$ and $(H_1\cap H_2)T\neq G$:

If $T<G$ then $(H_1\cap H_2)T=G$ would yield by Lemma 2 that

$$T=(T\cap H_1)(T\cap H_2)\in\mathcal{N}(p, q)$$

by the minimality of G and Lemma 1, contrary to $U\leqq T$; so $(H_1\cap H_2)T<G$ in this case.

Now assume $T=G$. Suppose $H_tQ<G$ for both t and all $Q\in\mathrm{Syl}_q(G)$, then $H_t^G<G$ for each t by Lemma 4, hence $H_1^G, H_2^G\in\mathcal{N}(p, q)$ by (∗); so

$$G=H_1^GH_2^G\in\mathcal{N}(p, q)$$

by [1], a contradiction. Thus we can assume $H_1Q=G$ (with a $Q\in\mathrm{Syl}_q(G)$). Then with a suitable $Q_1<Q$ we get $|G\colon H_1Q_1|=q$. By Lemma 3 and $f(G, H_1Q_1, H_2)\leqq$ $\leqq f(G, H_1, H_2)-(|G\colon H_1|-q)$ we see that $|G\colon H_1|=q$. Let $x\in G$, then $G=H_1U^x$ by (1), thus $|U^x\colon H_1\cap U^x|=|G\colon H_1|=q$, so $U_p^x\leqq H_1$, as required.

(3) If $H_tU_q<G$ then $H_t\cap U=1$ and $|U_q|=q$: $H_tU_q<G$ implies $H_tU_q\in$ $\in\mathcal{N}(p, q)$, thus $H_t\cap U\leqq Z(U)$ by (1). $D:=(H_t\cap U)^G=(H_t\cap U)^{UH_t}=(H_t\cap U)^{H_t}\leqq$ $\leqq H_t$, hence $D\cap U=H_t\cap U\leqq Z(U)$, thus $U/D\cap U\notin\mathcal{N}(p, q)$. If $D>1$, then — all conditions of the theorem remaining valid for G/D, H_1D/D, H_2D/D — $G/D\in\mathcal{N}(p, q)$, contrary to $UD/D\leqq G/D$; so $D=1$. Let $D_1=(\Phi(U_q))^G$, then $D_1=$

$=(\Phi(U_q))^{UH_t}=(\Phi(U_q))^{H_t}\leqq H_t\Phi(U_q)$, hence $D_1\cap U\leqq H_t\Phi(U_q)\cap U=(H_t\cap U)\Phi(U_q)=$ $=\Phi(U_q)\leqq Z(U)$. Thus we get (factorizing by D_1) $D_1=1$.

Now, by (2), we separate two cases.

Case 1: $U_p\leqq N:=\bigcap_{x\in G} H_1^x$, $|G:H_1|=q$.

Case 1/a: $NH_2<G$. As $G=H_2U=NH_2U_q$, $1\neq|G:NH_2|$ is a power of q and $NH_2\in\mathcal{N}(p,q)$ by Lemma 1. Then $f(G,H_1,NH_2)\leqq f(G,H_1,H_2)$ with equality iff $N\leqq H_2$. Thus $|G:H_2|$ is a power of q, consequently $H_2\leqq\tilde{H}_2<G$ with $|G:H_2|=q$. As $f(G,H_1,\tilde{H}_2)\leqq f(G,H_1,H_2)$, $H_2=\tilde{H}_2$ is of index q. So $U_p\leqq M:=N\cap\bigcap_{x\in G} H_2^x$. Let $U_p\leqq R\in\mathrm{Syl}_p(M)$, then $G=MN_G(R)$, so by Lemma 2, $N_G(R)=N_{H_1}(R)N_{H_2}(R)$. Thus $N_G(R)\in\mathcal{N}(p,q)$ by Lemma 1, if $N_G(R)<G$. If so, then for $Q_1\in\mathrm{Syl}_q(C_G(R))$ there exists a Sylow q-subgroup Q_2 of M such that (Q_1 normalizes Q_2, hence) $Q_1Q_2\in\mathrm{Syl}_q(G)$. Let $U_q=\langle b\rangle$, then $b\in T=\langle(Q_1Q_2)^x\colon x\in G\rangle\leqq(C_G(R)M)^G=C_G(R)M$, thus $b=b_Cb_M$ with $b_C\in C_G(R)\leqq C_G(U_p)$ and $b_M\in M$. So $b_M\in N_M(U_p)\setminus C_G(U)_p$ and for any u in U_p, $u^b=u^{b_M}$. Hence $1=u^{b^q}=u^{b_M^q}$, yielding with a suitable power b_M^k a q-subgroup $\langle b_M^k\rangle$, that normalizes but does not centralize the p-subgroup U_p, contrary to $M\leqq H_1\in\mathcal{N}(p,q)$; thus $R\lhd G$. For $t=1,2$ let $S_t\in\mathrm{Syl}_q(H_t)$, then $S_1^G, S_2^G\leqq C_G(R)$. Let $S\in\mathrm{Syl}_q(G)$; there exist elements e,f in G with $S_1^e, S_1^f\leqq S$. $S\nleqq C_G(R)$ and $|S:S_1^e|=q=|S:S_2^f|$ (because of $|G:H_1|=q=|G:H_2|$), so $S_1^e=S_2^f$. $ef^{-1}=g_1g_2$ (with $g_t\in H_t$) and $S_1^{g_1g_2}\leqq H_1^{g_2}\cap H_2$; as $f(G,H_1^{g_2};H_2^{g_2}=H_2)\leqq$ $\leqq f(G,H_1,H_2)-\sum_{t=1,2}|H_t\colon H_1\cap H_2|_q$, we get that $|H_1\cap H_2|_q=|H_1|_q=|H_2|_q$, contrary to $|H_1\cap H_2|=|H_1||H_2||G|^{-1}|=q^{-1}|H_1|$.

Case 1/b: $NH_2=G$. As $NU=NU\cap NH_2=N(NU\cap H_2)$, $NU=G$ by Lemma 1. Thus $NU_q=G$, G/N is cyclic, so $H_1\lhd G$. Suppose $H_2U_q=G$, then $H_2\leqq\hat{H}_2<G$ with a $|G:\hat{H}_2|=q$, so by induction, $H_2=\hat{H}_2$. Let $E=\bigcap_{x\in G} H_2^x$, then $G\neq H_1E$ by [1] and $U_p\leqq E$, producing Case 1/a with (H_2,E,H_1) instead of (H_1,N,H_2). Thus $H_2U_q\neq G$. $H_2\lhd\!\!\!|\lhd G$ by [1], so $L:=U_q^G<G$ by Lemma 4.

$H_1\cap L\lhd G$, $|L:H_1\cap L|=q$, $L\notin\mathcal{N}(p,q)$, hence $L\neq(H_1\cap L)(H_2\cap L)$ by Lemma 1, which yields $H_2\cap L\leqq H_1\cap L$. Suppose $(L\cap H_1)H_2<G$, then (as $G=H_2U=$ $=H_2L$), $|G:(L\cap H_1)H_2|=q$, $f(G,H_1,(L\cap H_1)H_2)\leqq f(G,H_1,H_2)$. Thus H_2 is of index q in G, $G=H_2U_q$, which is not the case; so $G=(L\cap H_1)H_2$. We get $G/L\cap H_1\simeq H_2/L\cap H_1\cap H_2=H_2/L\cap H_2\simeq G/L$, $L\leqq H_1$, a contradiction.

Case 2: $T=\langle Q\colon Q\in\mathrm{Syl}_q(G)\rangle\neq G$. Having eliminated Case 1 we may assume by (2) and (3) that $H_t\cap\hat{U}=1$ $(t=1,2)$ and $|\hat{U}_q|=q$ for *any* $\hat{U}$, being of the same type as U. Also by (2), $(H_1\cap H_2)T<G$.

As $\hat{U}\cap H_t=1$, $|T:T\cap H_t|=|\hat{U}|$; let $Q\in\mathrm{Syl}_q(G)$, then by Lemma 1, $(T\cap H_t)Q^x=Q^x(T\cap H_t)$ for any $x\in G$. $T\neq(T\cap H_t)Q$, hence by Lemma 4, there exist $W_t \lneqq T$ $(t=1,2)$ with $T \cap H_t \leqq W_t$. $\hat{U}_q \nleqq W_t$ yields $W_t \in \mathcal{N}(p,q)$ and the existence of $V_t \lhd T$ with $W_t \leqq V_t$ and $|T: V_t|=q$ $(t=1,2)$. Still $\hat{U}_q \nleqq V_t$, so $V_1, V_2 \in \mathcal{N}(p,q)$. By $|T: V_t|=q$, $T \notin \mathcal{N}(p,q)$ and [1], $V_1=V_2$.

On the other hand, $V_t=V_t \cap T=V_t \cap (T \cap H_t)\hat{U}=(T \cap H_t)(V_t \cap \hat{U})=(T \cap H_t)\hat{U}_p$; thus

(4) $(T \cap H_1)\hat{U}_p=(T \cap H_2)\hat{U}_p$, consequently $|T \cap H_1|=|T \cap H_2|$.

(5) $|G: T|$ is a power of p, hence for any $A \leqq B \leqq G$, $|B: A|_q=|B \cap T: A \cap T|_q$:

Let P be a Sylow p-subgroup of G then $(|G: TP|, |G: H_1 \cap H_2|)=(|G: TP|, |\hat{U}|^2)=1$, thus $G=(H_1 \cap H_2)TP$, so $TP=(TP \cap H_1)(TP \cap H_2)$ by Lemma 2. By $U \leqq TP$, $TP=G$.

Let $Q_t \in \mathrm{Syl}_q(H_t \cap T)$ for $t=1,2$, then by (4), $Q_2=Q_1^g$ for some g. Let $g=g_1g_2$ with $g_t \in H_t$; as $Q_2 \leqq (T \cap H_1^{g_2}) \cap (T \cap H_2)$, $f(G, H_1^{g_2}, H_2^{g_2}=H_2)=|G|+|G: H_1|+|G: H_2|+|H_1^{g_2}: H_1^{g_2} \cap H_2|_q+|H_2: H_1^{g_2} \cap H_2|_q = |G|+|G: H_1|+|G: H_2|+|T \cap H_1^{g_2}: T \cap H_1^{g_2} \cap H_2|_q+|T \cap H_2: T \cap H_1^{g_2} \cap H_2|_q$ by (5). But $|T \cap H_1^{g_2}: T \cap H_1^{g_2} \cap H_2|_q = 1 = |T \cap H_2: T \cap H_1^{g_2} \cap H_2|_q$ as $Q_2 \leqq T \cap H_1^{g_2} \cap H_2$, so by the minimality of $f(G, H_1, H_2)$, we have $|H_1: H_1 \cap H_2|_q=|H_2: H_1 \cap H_2|_q=1$.

On the other hand, $|H_1: H_1 \cap H_2|_q=|G: H_2|_q=|U|_q=q=|G: H_1|_q=|H_2: H_1 \cap H_2|_q$, the final contradiction.

References

[1] P. Hermann, Groups without certain subgroups form a Fitting class (to appear).
[2] B. Huppert, *Endliche Gruppen. I*, Springer-Verlag (Berlin, 1967).
[3] N. Ito, Note on (LM)-groups of finite order, Kodai Math. Seminar Report (1951), 1—6.
[4] O. H. Kegel, Produkte nilpotenter Gruppen, *Arch. Math.*, **12** (1961), 90—93.

(K. C.)
DEPT. NUMERICAL METHODS
AND COMPUTATION,
EÖTVÖS UNIVERSITY,
BUDAPEST, 1088, HUNGARY

(P. H.)
DEPT. ALGEBRA AND
NUMBER THEORY,
EÖTVÖS UNIVERSITY,
BUDAPEST, 1088, HUNGARY