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# Completeness in coalgebras

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*To Professor Károly Tandori on his sixtieth birthday*

**1. Preliminaries.** For a set $A$ and $n$ positive integer, denote by $A^{(n)}$ the $n$'th copower (i.e. the union of $n$ disjoint copies) of A. Dualizing the notion of an $n$-ary operation we obtain that of an *$n$-ary co-operation on* A: this is a mapping $f\colon A\to A^{(n)}$. The corresponding notion may be introduced in any well-copowered category, cf. [4], [6], [10]. For $A$ non-empty and $F$ a set of co-operations on $A$, the pair $\langle A;\ F\rangle$ is called a *coalgebra.* Coalgebras were considered by DRBOHLAV [2]; he introduced the common algebraic notions and proved the Birkhoff variety theorem for them. Here we shall study completeness of sets of co-operations on finite sets.

Let $\mathbf{n}$ stand for $\{0, \ldots, n-1\}$. One can introduce $A^{(n)}$ as $\mathbf{n}\times A$, and so each co-operation $f\colon A\to A^{(n)}$ is uniquely determined by a pair of mappings $\langle f_0, f_1\rangle$ where $f_0\colon A\to\mathbf{n}$ and $f_1\colon A\to A$. We call $f_0$ and $f_1$ *the labelling* and *the mapping of $f$*, respectively. We can imagine co-operations — as well as other mappings — by means of graphs, e.g. Fig. 1 displays the ternary co-operation on $\mathbf{3}$ having the cycle (012) as labelling and the transposition (01) as mapping.

The *$n$-ary coprojections* may be defined by dualizing the notion of the $n$-ary projection. We write $p^{n,i}$ for the $i$'th $n$-ary coprojection $(i=0, \ldots, n-1)$; then $p_0^{n,i}(a)=i$ and $p_1^{n,i}(a)=a$ for each $a\in A$.

The superposition $f(g_0, \ldots, g_{n-1})$ of an operation $f\colon A^n\to A$ and $n$ operations $g_i\colon A^k\to A$ $(i=0, \ldots, n-1)$ may be considered as follows. There exists a (unique) $g\colon A^n\to A^n$ such that $g_i=ge_i^n$ for each $i\in\mathbf{n}$. Then $f(g_0, \ldots, g_{n-1})=gf$. Dually, for arbitrary co-operations $f\colon A\to A^{(n)}$, $g^{(i)}\colon A\to A^{(k)}$ $(i=0, \ldots, n-1)$ there exists a (unique) mapping $g\colon A^{(n)}\to A^{(k)}$ such that $g^{(i)}=p^{n,i}g$ for each $i\in\mathbf{n}$. The co-operation $fg\colon A\to A^{(k)}$ is called the *superposition* of $f$ and $g^{(i)}$; we denote it by $f(g^{(0)}, \ldots, g^{(n-1)})$. Fig. 2 and 3 display $f(g^{(0)}, g^{(1)}, g^{(2)})$ with $f, g^{(0)}, g^{(1)}$ the co-operation on Fig. 1, and $g^{(2)}=p^{3,2}$. For the labelling and mapping of a superposition $s=f(g^{(0)}, \ldots, g^{(n-1)})$

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we have

$$(1)\qquad \begin{aligned} s_0(a) &= g_0^{(f_0(a))}(f_1(a)),\\ s_1(a) &= g_1^{(f_0(a))}(f_1(a)). \end{aligned}$$

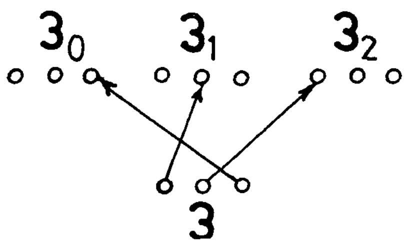

Fig. 1

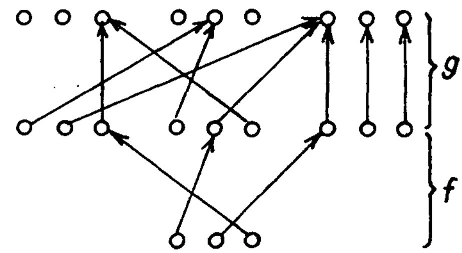

Fig. 2

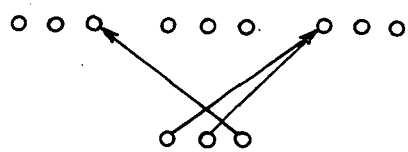
Fig. 3

Analogously to the case of operations, a set of co-operations on a set $A$ is called a *clone* if it is closed under superpositions and contains all coprojections. A clone of co-operations is also an abstract clone, i.e. it is a heterogeneous clone in the sense of TAYLOR [13]. Indeed, it satisfies the identities (2.8.1)—(2.8.3) in the definition of heterogeneous clone in [13]; they may be written in the form

$$(2.1)\qquad f(g^{(0)}(h^{(0)}, \dots, h^{(k-1)}), \dots, g^{(n-1)}(h^{(0)}, \dots, h^{(k-1)})) = \\ = (f(g^{(0)}, \dots, g^{(n-1)}))(h^{(0)}, \dots, h^{(k-1)})$$

for arbitrary $f, g^{(i)}, h^{(j)}$ of appropriate arities;

$$(2.2)\qquad f(p^{n,0}, \dots, p^{n,n-1}) = f$$

for $f$ $n$-ary; and

$$(2.3)\qquad p^{n,i}(f^{(0)}, \dots, f^{(n-1)}) = f^{(i)}$$

for $f^{(0)}, \dots, f^{(n-1)}$ of the same arity. Denote, e.g., the left and right side of (2.1) by $p$ and $q$, and let $\bar f$ and $\bar g^{(i)}$ stand for $f(g^{(0)}, \dots, g^{(n-1)})$ and $g^{(i)}(h^{(0)}, \dots, h^{(k-1)})$, respectively. Then, for every $a \in A$, the equations (1) give

$$p_0(a) = \bar g_0^{(f_0(a))}(f_1(a)) = h_0^{(g_0^{(f_0(a))}(f_1(a)))}(g_1^{(f_0(a))}(f_1(a))) = \\ = h_0^{(\bar f_0(a))}(\bar f_1(a)) = q_0(a),$$

and similarly we obtain $p_1(a) = q_1(a)$. One can verify also (2.2) and (2.3).

We shall denote the clone of all co-operations on $A$ by $\mathscr{C}_A$, and the set of all $n$-ary co-operations of $A$ by $\mathscr{C}_A^n$.

An $n$-ary operation $f$ on $A$ depends on its $i$'th variable iff there is an $n$-ary $g$ on $A$ such that $f(e_1^n, \dots, e_{i-1}^n, g, e_{i+1}^n, \dots, e_n^n) \neq f$. Accordingly, an $f \in \mathscr{C}_A^n$ *depends on its $i$'th variable* if there exists a $g \in \mathscr{C}_A^n$ with $f(p^{n,0}, \dots, p^{n,i-1}, g, p^{n,i+1}, \dots, p^{n,n-1}) \neq f$. It is easy to verify that $f$ depends on its $i$'th variable iff $f_0^{-1}(i)$ is not void. We say that $f(\in \mathscr{C}_A^n)$ is *essentially $k$-ary* if there exist exactly $k$ elements $i \in \mathbf{n}$ such that $f$ depends on its $i$'th variable, i.e. if $f_0$ has $k$-element range.

**2. Complete sets of co-operations.** We shall study co-operations on finite sets $\mathbf{n}$ $(n > 1)$. For $C \subseteq \mathscr{C}_\mathbf{n}$, the least clone in $\mathscr{C}_\mathbf{n}$ containing $C$ will be denoted by $[C]$ and called *the clone generated by $C$*. If $[C] = \mathscr{C}_\mathbf{n}$ (i.e. every co-operation on $n$ may be obtained from those in $C$ and coprojections using superposition) then $C$ is said to be *complete*. In this case we call also the coalgebra $\langle \mathbf{n}; C \rangle$ *primal.*

We shall need terms and notations for special co-operations. The *diagonal co-operation $d$* on $\mathbf{n}$ is $n$-ary with $d_0$, $d_1$ identical. The $n$-ary *$(i, j)$-constant co-operation* $i^{n,j}$ is determined by $i_0^{n,j}(k) = j$, $i_1^{n,j}(k) = i$ for each $i, j \in \mathbf{n}$ and for each $k \in \mathbf{n}$. An *$(i, j)$-translation* is a co-operation $t$ with $t_1(i) = j$. If such a $t$ is $m$-ary then $t(p^{n,l_0}, \dots, \dots, p^{n,l_{m-1}})$ is an $n$-ary $(i, j)$-translation (which is essentially $|\{l_0, \dots, l_{m-1}\}|$-ary). Similarly, from an $(i, j)$-constant we can get an $(i, j)$-constant of arbitrary arity. We call a co-operation $g$ *$(i, j)$-gluing* if $g_k(i) = g_k(j)$ for all $k \in \mathbf{2}$; $g$ is *gluing* if it is $(i, j)$-gluing for some $i, j \in \mathbf{n}$. Thus, $g$ is not gluing iff the mapping $i \mapsto \langle g_0(i), g_1(i) \rangle$ is 1—1 on $\mathbf{n}$.

The following observations are trivial:

Proposition 0. *An essentially $k$-ary co-operation is a superposition of a $k$-ary co-operation and some coprojections. If a co-operation on $\mathbf{n}$ is essentially $k$-ary then $k \leqq n$.*

This implies

Proposition 1. *The set of all at most $n$-ary co-operations on $\mathbf{n}$ is complete.*

Thus, studying completeness on $\mathbf{n}$, we can restrict ourselves to co-operations with arity $\leqq n$.

The mappings of a set $C$ of co-operations on $n$ generate a semigroup $\mathscr{S}(C)$ of self-mappings of $\mathbf{n}$, called *the semigroup of $C$*. We call $C$ *transitive* if $\mathscr{S}(C)$ is transitive. Note that *each self-mapping in $\mathscr{S}(C)$ is the mapping of some (unary) co-operation in* $[C]$, *i.e.*, $\mathscr{S}(C) \subseteq \mathscr{S}[C]$. Indeed, for co-operations $f$ and $g$ of arbitrary arities, let $h = f(g(p^{1,0}, \dots, p^{1,0}), \dots, g(p^{1,0}, \dots, p^{1,0}))$. Then for each $i \in \mathbf{n}$, $h_1(i) = g_1(f_1(i))$, proving that $\mathscr{S}[C]$ is closed under products of mappings, whence the assertion follows.

Proposition 2. *A transitive set of co-operations on* $\mathbf{n}$ *is complete provided it contains an essentially n-ary co-operation.*

Proof. By Proposition 1, we have to prove that, for a set of co-operations $C$ satisfying the conditions of Proposition 2, every at most $n$-ary co-operation $g$ on $\mathbf{n}$ is a composition of some co-operations in $C$. Let $f\in C$ be essentially $n$-ary. Then the labelling of $f$ is onto, hence it is a permutation of $\mathbf{n}$. Form $f(p^{k,g_0(f_0^{-1}(0))}, \ldots$ $\ldots, p^{k,g_0(f_0^{-1}(n-1))})=f'$; then, for each $i\in\mathbf{n}$, we have $f_0'(i)=p_0^{k,g_0(i)}(f_1(i))=g_0(i)$, i.e., the arity and labelling of $f$ are the same as those of $g$, while its mapping is the mapping of $f$: for $i\in\mathbf{n}$, $f_1'(i)=p_1^{k,g_0(i)}(f_1(i))=f_1(i)$.

On the other hand, as $C$ is transitive, for every $k, l\in\mathbf{n}$ there exists a $(k, l)$-translation $t^{k,l}$; we can assume that $t^{k,l}$ is unary. Then $t^{k,l}(p^{n,j})$ is an $n$-ary $(k, l)$-translation whose labelling is the constant function with value $j$. Now form

$$f\big(t^{f_1(f_0^{-1}(0)),g_1(f_0^{-1}(0))}(p^{n,0}), \ldots, t^{f_1(f_0^{-1}(n-1)),g_1(f_0^{-1}(n-1))}(p^{n,n-1})\big) = f^*.$$

Then, for each $i\in\mathbf{n}$, $f_0(i) = \big(t^{f_1(i),g_1(i)}(p^{n,f_0(i)})\big)_0(f_1(i)) = f_0(i)$, and $f_1(l) = $ $=\big(t^{f_1(i),g_1(i)}(p^{n,f_0(i)})\big)=g_1(i)$, i.e., we have an essentially $n$-ary $f^*$ whose labelling coincides with that of $f$, while its mapping is the mapping of $g$.

Finally, $g=(f^*)'\in[C]$.

Corollary 2.1. *If $f$ is diagonal and the mapping of $g$ is a cycle on* $\mathbf{n}$ *then* $\{f, g\}$ *is complete.*

Indeed, the diagonal co-operation is essentially $n$-ary, and a cycle on $\mathbf{n}$ generates a transitive group on $\mathbf{n}$.

Corollary 2.2. *The set* $\mathscr{C}_{\mathbf{n}}^2$ *of all binary co-operations is complete.*

This is the coalgebraic version of Sierpiński's completeness theorem [11]. As clearly there are binary co-operations whose mapping is cyclic (hence generates a transitive semigroup), we have to show only that theie is an essentially $n$-ary co-operation in the clone generated by the superposition of binary co-operations on $\mathbf{n}$. Define $b^{n,i}\in\mathscr{C}_{\mathbf{n}}^2$ $(i\in\mathbf{n}-1)$ by $b_0^{n,i}(k)=0$ if $k\leqq i$ and $b_0^{n,i}(k)=1$ otherwise, while $b_1^{n,i}$ is identical. Then $b^{n,0}(p^{n,0}, n^{n,1}(p^{n,1}, \ldots, b^{n,n-2}(p^{n,n-2}, p^{n,n-1})\ldots))=d$, the diagonal (i.e., any essentially $n$-ary) co-operation on $\mathbf{n}$.

Next we determine the Sheffer c operations: a co-operation on $\mathbf{n}$ is *Sheffer* if it generates the clone of all co-operations on $\mathbf{n}$ (cf. [7]). Consider a partition $\pi$ of $\mathbf{n}$. We say that a co-operation $f$ on $\mathbf{n}$ *preserves* $\pi$ if $\pi$ is a refinement of the partition induced on $\mathbf{n}$ by $f_0$ (i.e. $f_0$ is constant on each block of $\pi$), and is compatible with $f_1$ (i.e. on each block of $\pi$ all the values are in the same block of $\pi$). A set $C$ of co-operations preserves $\pi$ if each $f\in C$ preserves $\pi$. Every co-operation preserves the least partition (the one with 1-element blocks) and exactly the essentially unary co-operations preserve the greatest partition (with one block). Further, let $S$ be a non-empty subset of $\mathbf{n}$. We say that a set $C$ of co-operations on $\mathbf{n}$ *preserves* $S$ if $S$ is closed under $f_1$ for every $f\in C$.

Proposition 3. *A co-operation $f$ on $\mathbf{n}$ is Sheffer if and only if it preserves neither non-least partitions nor non-empty proper subsets of $\mathbf{n}$.*

Proof. Sufficiency. The second condition means that $[\{f\}]$ is transitive. By Proposition 2, it is enough to prove that $f$ contains an essentially $n$-ary co-operation.

Suppose that $f$ is $m$-ary. Then $m\geqq 2$, and $f$ is essentially at least binary, since it does not preserve the partition of $n$ consisting of one block. Further, $f_1$ is cyclic, since $f$ is transitive; hence $f$ is not gluing.

We show that, for each pair $i, j$ of different elements from $\mathbf{n}$, there exists a non-negative integer $k$ such that $f_0(f_1^k(i))\neq f_0(f_1^k(j))$. Write $i^0$ for $i$, and $i^k$ for $f_1(i^{k-1})$. Suppose that $f_0(i^k)=f_0(j^k)$ for every integer $k\geqq 0$, contrary to the claim; in particular, $f_0(i)=f_0(j)$. As $f_1$ is cyclic, there is a least natural number $t(<n)$ such that $j=i^t$, and hence $j^k=i^{t+k}$. It follows $j^{(r-1)t}=i^{rt}$, and thus $f_0(i^{rt})=f_0(i)$ for every non-negative integer $r$. If $(t, n)=1$ then $\{i^{rt}\colon r\geqq 0\}=\mathbf{n}$, so $f_0$ is constant, a contradiction, because $f$ is at least binary. Hence $1<(t, n)<n$. Now we see that $f_0(i^u)=f_0(i^v)$ whenever $u\equiv v \pmod{(t, n)}$. Define an equivalence $\sim$ on $\mathbf{n}$ by $i^u\sim i^v$ iff $u\equiv v$ $\pmod{(t, n)}$; this is a refinement of the equivalence induced by $f_0$. Also, clearly, $\sim$ is preserved by $f_1$. Hence $f$ preserves the (non-trivial) partition of this equivalence, a contradiction again.

Given an integer $k\geqq 0$, there exists a unary co-operation $h$ in $[f]$ such that, for each $i\in\mathbf{n}$, $h_1(i)=f_1^k(i)$. Hence for the $m$-ary co-operation $s^{i,j}=h(f)$ we have $s_0^{i,j}(i)=f_0(h_1(i))=f_0(f_1^k(i))\neq f_0(f_1^k(j))=s_0^{i,j}(j)$.

Now, if $2\leqq k<n$, for every non-gluing essentially $k$-ary co-operation $c\in[f]$ we construct a non-gluing essentially at least $(k+1)$-ary co-operation $c'\in[f]$ as follows:

Since $k<n$, and $c$ is not gluing, there exist $i, j\in\mathbf{n}$ such that $c_0(i)=c_0(j)$, and $c_1(i)\neq c_1(j)$. Let $c$ be (formally) $l$-ary. Put

$$c' = c\big(p^{l+1,0}, \dots, p^{l+1,c_0(i)-1}, s^{c_1(i),c_1(j)}(\underbrace{p^{l+1,l}}_{0}, \dots, p^{l+1,l}, \underbrace{p^{l+1,c_0(i)}}_{s_0^{c_1(i),c_1(j)}(c_1(i))}, \quad p^{l+1,l}, \dots, \underbrace{p^{l+1,l}}_{m-1}), p^{l+1,c_0(i)+1}, \dots, p^{l+1,l-1}\big).$$

Assume that $c$ depends on its $q$'th variable. Then there is an $r\in\mathbf{n}$ such that $c_0(r)=q$. If $q\neq c_0(i)$ then $c_0'(r)=p_0^{l+1,q}(c_1(r))=q$, and if $q=c_0(i)$ then $c_0'(i)=$ $=p_0^{l+1,c_0(i)}(s_1^{c_1(i),c_1(j)}(c_1(i)))=c_0(i)=q$, i.e., $c'$ also depends on its $q$'th variable. In addition, $c'$ depends on its $l$'th variable, too: $c_0(j)={}_0^{l+1,l}(s_1^{c_1(i),c_1(j)}(c_1(j)))=l$.

We have shown that $c'$ is essentially at least $(k+1)$-ary. It remains to show that $c'$ is not gluing. Observe that, for $a\in\mathbf{n}$, $c'_0(a)=l$ if $a\neq i$ and $c_0(a)=c_0(i)$, while $c'_0(a)=c_0(a)$ otherwise; further $c'_1(a)=s_1^{c_1(i),c_1(j)}(c_1(a))$ if $c_0(a)=c_0(i)$, and $c'_1(a)=c_1(a)$ otherwise. Since $s_1^{c_1(i),c_1(j)}$ is a permutation of $\mathbf{n}$, we obtain that, for $a,b\in\mathbf{n}$ with $c'_0(a)=c'_0(b)$, $c'_1(a)\neq c'_1(b)$ whenever $c_1(a)\neq c_1(b)$. This means that $c'$ is $(a, b)$-gluing only if $c$ is $(a, b)$-gluing. Thus, $c'$ is not gluing, as required.

Using this construction, from $f$ we get an essentially $n$-ary co-peiation in $f$ in a finite number of steps, proving the sufficiency.

Necessity. We have to show that if a co-operation $f$ preserves a non-trivial partition $\pi$ of $\mathbf{n}$ then every co-operation in $[f]$ also preserves $\pi$, and the same holds for non-empty subsets instead of non-trivial partitions. As the coprojections preserve everything, it is enough to show that any composition $f(g^0, \ldots, g^{k-1})$ preserves the partition $\pi$ provided $f, g^0, \ldots, g^{k-1}$ preserve it.

Put $h=f(g^0, \ldots, g^{k-1})$, and let $a\equiv b(\pi)$. Then $h_0(a)=g_0^{f_0(a)}(f_1(a))$, $h_0(b)=$ $=g_0^{f_0(b)}(f_1(b))$. Here $f_1(a)\equiv f_1(b)(\pi)$ and $f_0(a)=f_0(b)$, hence $g_0^{f_0(a)}(f_1(a))=$ $=g_0^{f_0(b)}(f_1(b))$, as needed. Also we have $h_1(a)=h_1(b)$, again by (1) and the definition of preservation. The case of subsets is even simpler. Thus, Proposition 3 is proved.

Consider the case when $n$ is a prime number. Then the non-preserving of non-empty proper subsets by $f$ means that $f_1$ is a prime-order cycle, hence it preserves no non-trivial partition with more than one blocks. Thus we have to exclude the preservation of the one-block partition only. This can be done by requiring that $f$ is essentially at least binary. Hence it follows:

Corollary 3.1. *Let $n$ be a prime number. A co-operation $f$ on $\mathbf{n}$ is Sheffer if and only if it is essentially at least binary and $f_1$ is a cyclic permutation of $\mathbf{n}$.*

Introducing some natural algebraic notions for coalgebras, we can given a more familiar form to Proposition 3. Let $\mathbf{A}=\langle A;\ F\rangle$ be a coalgebra. If the subset $B$ of $A$ is preserved by $F$, we can obtain a subcoalgebra $\mathbf{B}=\langle B;\ F'\rangle$ of $\mathbf{A}$ by putting $F'=\{f'\colon f\in F\}$ where $f'_i$ $(i\in\mathbf{2})$ are the restrictions of $f_i$ to $B$. A subcoalgebra $\mathbf{B}$ of $\mathbf{A}$ is *proper* if $B$ is a proper subset of $A$.

Furthermore, if the partition $\pi$ of $A$ is preserved by $F$, we can obtain a coalgebra $\bar{\mathbf{A}}=\langle\bar{A};\ \bar{F}\rangle$, where $\bar{A}=\{\bar{a}\colon a\in A\}$ is the set of blocks of $\pi$, while $\bar{F}=\{\bar{f}\colon f\in F\}$ and $\bar{f}$ is defined by $\bar{f}_0(\bar{a})=f_0(a)$, $\bar{f}_1(\bar{a})=\overline{f_1(a)}$ for each $a\in A$. Coalgebras $\bar{\mathbf{A}}$ arising in such a way are called *factorcoalgebras* of $\langle A;\ F\rangle$; $\bar{\mathbf{A}}$ is *proper* if it is induced by a partition with at least one non-trivial block. As it is usual for algebras, a coalgebra $\mathbf{B}$ which may be obtained from another coalgebra $\mathbf{A}$ by forming a subcoalgebra of a factorcoalgebra is called a *factor* of $A$. A factor of $\mathbf{A}$ is *proper* if in the process of its formation we take a proper sub- or factoralgebra. Using the just introduced notions, Proposition 3 states:

Proposition 3. *A finite coalgebra with one co-operation is primal if and only if it has no proper factors.*

This is the coalgebraic version of Rousseau's theorem (a finite algebra with one operation is primal iff it has no proper factors and is rigid [8], [7]).

The following proposition corresponds to Słupecki's completeness criterion for operations [12], [7]. Call a co-operation *essential* if it is essentially at least binary and non-gluing.

Proposition 4. *The set consisting of all unary co-operations and an arbitrary essential co-operation is complete on any* $\mathbf{n}$.

Proof. Denote the set and the essential co-operation in the proposition by $S$ and $f$, respectively. We show that there is a Sheffer co-operation in $[S]$. For this aim we prove the following two claims:

($\alpha$) There exists a Sheffer co-operation $g$ on $\mathbf{n}$ such that $g_0=f_0$.

($\beta$) If $g$ is a co-operation on $\mathbf{n}$ such that $g_0=f_0$, then $g\in[S]$.

*Proof of* ($\alpha$). A co-operation $g$ on $\mathbf{n}$ with $g_0=f_0$ is fully determined by its mapping $g_1$. We have to define a $g_1$ such that neither non-empty proper subsets nor non-least partitions would be preserved by $g$. Concerning the subsets, it is sufficient to choose $g_1$ a cyclic permutation of $\mathbf{n}$. As for the partitions, the co-operation $g$ may preserve only refinements of the partition $\lambda$ induced by its labelling. Thus, we have to show that under appropriate choice of the cycle $g_1$, no non-trivial refinement of $\lambda$ will be preserved by (the unary operation) $g_1$. We can suppose that $\lambda$ itself is not least, else we are done.

Given a cyclic permutation $g_1$ of $\mathbf{n}$ and an element $i\in\mathbf{n}$, each element of $\mathbf{n}$ may be written in the form $g_1^r(i)$; for this element, we write shortly $i^r$. Partitions preserved by $g_1$ are the same as congruences of the algebra $\langle\mathbf{n};\, g_1\rangle$. Each such non-trivial and proper congruence is uniquely determined by a divisor $d(1<d<n)$ of $n$ (and hence it may be denoted by $\pi_d$) in the following way: $i^r\equiv i^s(\pi_d)$ if and only if $r\equiv s \pmod d$.

Let $\bar{i}$ be a block of $\lambda$ with minimal number of elements, and $i\in\bar{i}$. Then $|\bar{i}|\leqq n/2$. On the other hand, the number of non-trivial proper divisors of $n$ is less than $n/2$; hence we can define $g_1$ so that for each non-trivial proper divisor $d$ of $n$ $i^d\notin\bar{i}$. Now if, for some $s$, $\pi_d\leqq\lambda$ then from $i^d\equiv i^0=i\ (\pi_d)$ it follows $i^d\equiv i(\lambda)$, i.e., $i^d\in\bar{i}$; a contradiction.

*Proof of* ($\beta$). Let $f$ be $l$-ary, $k\in\mathbf{l}$, $i\in\mathbf{n}$. As $f$ is not gluing, the system of equations

$$f_0(x)=k,\quad f_1(x)=i$$

has at most one solution $x^{k,i}$ in $\mathbf{n}$. Clearly, each element of $\mathbf{n}$ may be written in form

$x^{k,i}$ with uniquely determined $k$ and $i$. Define the unary co-operation $t^k$ by

$$t_1^k(i)=\begin{cases} g_1(x^{k,i}) & \text{if} \quad x^{k,i} \quad \text{exists,}\\ 0 & \text{otherwise.}\end{cases}$$

Define $f'\in[S]$ by $f'=f\big(t^0(p^{l,0}), \dots, t^{l-1}(p^{l,l-1})\big)$. Then

$$f_0'(x^{k,i}) = p_0^{l,f_0(x^{k,i})}\big(t_1^{f_0(x^{k,i})}(f_1(x^{k,i}))\big) = p_0^{l,k}\big(t_1^k(i)\big) = k = f_0(x^{k,i}) =$$

$$= g_0(x^{k,i}), \quad \text{and} \quad f_1'(x^{k,i}) = p_1^{l,k}\big(t_1^k(i)\big) = g_1(x^{k,i}).$$

Thus, $g=f'\in[S]$, as required, and the proposition is proved.

Call a co-operation $f$ *sharp* if it is $k$-ary and essentially $k$-ary for some $k$. From Proposition 0 if follows that the number of sharp co-operations is finite on every $\mathbf{n}$, and a clone of co-operations is uniquely determined by the sharp co-operations it contains. Hence we infer that the number of clones of co-operations $i$ finite for each $\mathbf{n}$, i.e. the clones of co-operations on $\mathbf{n}$ form a finite lattice. For $n=2$, there is as few as 12 sharp co-operations, and even this number decreases to 8 if we do not distinguish between $f=f(p^{2,0}, p^{2,1})$ and $f(p^{2,1}, p^{2,0})$ (as they are the same „up to a permutation of variables”). Fig. 4 shows the

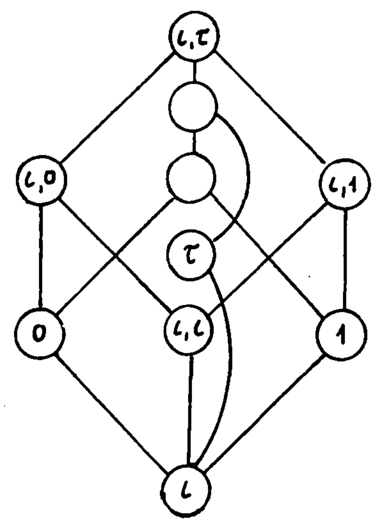


Fig. 4

lattice of clones of co-operations on **2** (the coalgebraic version of the Post diagram; cf. [5]). Circles standing for clones contain pairs or single signs; the denote the labelling-mapping pair or the mapping of the co-operation generating the given clone (if it is generated by one co-operation). We write $\iota$ and $\tau$ for the identical and non-identical permutation of **2**, and $i(\in\mathbf{2})$ for the constant mapping with value $i$.

**3. Co-operations and selective operations.** Given arbitrary non-empty sets $P$ and $M$, a natural number $k$, and mappings $f_0$: $P \to \mathbf{k}$, $f_1$: $P \to P$, we define a $k$-ary operation $f$ on $M^P$ by agreeing that, for every $p \in P$, the $p$-component of the result of $f$ is the $f_1$-component of the $f_0$'th operand. Operations obtained in this way are called regular selective operations (see [1]). The mappings $f_0$ and $f_1$ are referred to as the first and second selectors of $f$. Observe that they can be considered as the labelling and the mapping of a co-operation (of the same arity as $f$) on P. Moreover, for any nontrivial $M$ and nonempty $P$, there is a bijection between the regular selective operations on $M^P$ and the co-operations on $P$ assigning to a selective operation $f$ a co-operation whose labelling and mapping are the first and second selectors of $f$, respectively. This bijection is a clone isomorphism, i.e. it sends a projection into the coprojection with appropriate indices, and a superposition of operations into the superposition of co-operations being the images thereof. This follows immediately from (2) in [1] and (1) in this paper. Hence the study of clones (including lattices of clones) of regular selective operations on a finite power of a set reduces to the study of clones of co-operations on a finite set.

E.g., Corollary 2.1. implies that the basic operations of a $k$-dimensional die D (see [3]) generate the clone of all selective operations on the base set $M^k$ of D. Hence it follows that *the variety of k-dimensional dice is equivalent to the* $k$'th *power-variety of sets,* an observation due to TAYLOR [15] (see also [14]).

Further, we can reformulate Corollary 3.1., using the following consequence of Corollary 2.2.: a co-operation $f$ on $\mathbf{n}$ is Sheffer iff $\mathscr{C}_{\mathbf{n}}^2 \subseteq [f]$, and translating it into the language of selective operations, we obtain the following fact: *For p prime, all binary selective operations on* $M^p$ $(|M|>1)$ *are term functions of the given binary selective operation f if and only if f is essentially binary and the second selector of f is a cyclic permutation of* $\mathbf{p}$. Formulated in different terms, this is the main result in [9].

Finally, Fig. 4 may be considered as the lattice of clones of selective operations on $M^2$ $(|M|>1)$.

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