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On the stability of the local time of a symmetric random walk

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. rv with $P\{X_1=-1\}=P\{X_1=1\}=1/2$, and consider the symmetric random walk $S_0=0$, $S_n=X_1+\ldots+X_n$ $(n=1, 2, \ldots)$. Define the local time of $\{S_k\}$ by

$$\xi(x, n) := \text{No. } \{k\colon\ 0 < k \leqq n,\ S_k = x\} \quad (n = 1, 2, \ldots;\ x = 0,\ \pm 1,\ \pm 2, \ldots),$$

i.e., $\xi(x, n)$ is the number of visits of $\{S_k\}$ at x up to time n. The properties of $\xi(x, n)$ have been studied by a number of authors for a long time now. Here we present some well known and important results.

Theorem A.

$$P\{\xi(0, 2n) = k\} = 2^{k-2n}\binom{2n-k}{n} (k = 0, 1, 2, \ldots, n;\ n = 1, 2, \ldots),$$

$$\lim_{n\to\infty} P\{n^{-1/2}\xi(x, n) \leqq u\} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^u e^{-t^2/2}\, dt \quad (u > 0;\ x = 0, \pm 1, \pm 2, \ldots).$$

Theorem B (*Kesten,* 1965). *For any* $x=0, \pm 1, \pm 2, \ldots$ *we have*

$$\limsup_{n\to\infty} \frac{\xi(x, n)}{(2n \log\log n)^{1/2}} = \limsup_{n\to\infty} \frac{\sup_{-\infty<x<\infty} \xi(x, n)}{(2n \log\log n)^{1/2}} = 1 \quad a.s.,$$

$$\liminf_{n\to\infty} \left(\frac{\log\log n}{n}\right) \xi(x, n) = \gamma_1 \quad a.s.$$

where γ_1 *is a positive absolute constant.*

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Remark 1. The actual value of γ_1 was not given by Kesten. It was recently evaluated by E. Csáki (oral communication).

Remark 2. Roughly speaking the above two theorems say that $\xi(x, n)$ (for any fixed $x=0, \pm 1, \pm 2, \ldots$) goes to infinity like $n^{1/2}$ does.

Intuitively it is clear that $\xi(x, n)$ is close to $\xi(y, n)$ if x is close to y. This paper is devoted to studying this problem.

Here we present the main results.

Theorem 1. *For any* $k=\pm 1, \pm 2, \ldots$ *we have*

$$\limsup_{N\to\infty} \frac{\xi(k, N)-\xi(0, N)}{(\xi(0, N)\log\log N)^{1/2}} = \limsup_{N\to\infty} \frac{|\xi(k, N)-\xi(0, N)|}{(\xi(0, N)\log\log N)^{1/2}} =$$

$$= \limsup_{N\to\infty} \sup_{n\le N} \frac{\xi(k, n)-\xi(0, n)}{(\xi(0, N)\log\log N)^{1/2}} = \limsup_{N\to\infty} \sup_{n\le N} \frac{|\xi(k, n)-\xi(0, n)|}{(\xi(0, N)\log\log N)^{1/2}} =$$

$$= 2(2k-1)^{1/2} \quad a.s.$$

Theorem 2.

$$\limsup_{N\to\infty} \frac{\xi(1, N)-\xi(0, N)}{N^{1/4}(\log\log N)^{3/4}} = \limsup_{N\to\infty} \frac{|\xi(1, N)-\xi(0, N)|}{N^{1/4}(\log\log N)^{3/4}} =$$

$$= \limsup_{N\to\infty} \sup_{n\le N} \frac{\xi(1, n)-\xi(0, n)}{N^{1/4}(\log\log N)^{3/4}} = \limsup_{N\to\infty} \sup_{n\le N} \frac{|\xi(1, n)-\xi(0, n)|}{N^{1/4}(\log\log N)^{3/4}} = \left(\frac{128}{27}\right)^{1/4} \quad a.s.$$

Theorem 3. *For any* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty} \sup_{|k|\le a_n} \left|\frac{\xi(k, n)}{\xi(0, n)}-1\right| = 0 \quad a.s.$$

where $a_n = n^{1/2}(\log n)^{-(2+\varepsilon)}$.

Remark 3. Theorems 1 and 2 essentially say that for any fixed k the distance between $\xi(k, n)$ and $\xi(0, n)$ for large n behaves like $n^{1/4}$. Since $\xi(0, n)$ is about $n^{1/2}$ asymptotically, this means that $\xi(k, n)$ is relatively close asymptotically to $\xi(0, n)$. The meaning of Theorem 3 is about the same. However in the latter theorem we claim that for large n $\xi(k, n)$ is close to $\xi(0, n)$ whenever $|k|\le a_n$, but the meaning of "close" is not as precise as in Theorems 1 and 2. Theorem 3 is nearly the best possible in the following sense.

Theorem 4.

$$\limsup_{n\to\infty} \sup_{|k|\le b_n} \left|\frac{\xi(k, n)}{\xi(0, n)}-1\right| \ge 1 \quad a.s.,$$

where $b_n = n^{1/2}(\log n)^{-1}$.

2. Proof of Theorem 1.

Among the statements of Theorem 1 we only prove

$$\limsup_{N\to\infty}\frac{\xi(k, N)-\xi(0, N)}{(\xi(0, N)\log\log N)^{1/2}} = 2(2k-1)^{1/2} \quad \text{a.s. for any} \quad k = \pm 1, \pm 2, \ldots.$$

The proofs of its other statements can be obtained without any further difficulty along the same lines.

Let $A_{ij}(m)$ be the event that a symmetric random walk starting form m hits i before j $(i\leqq m\leqq j)$. Then

Lemma 2.1.

$$\mathrm{P}\{A_{ij}(m)\} = (j-m)/(j-i).$$

Proof is trivial.

For any $x=0, \pm 1, \pm 2, \ldots$ define

$$\tau_0(x) := 0,$$

$$\tau_1(x) := \inf\{l:\ l>0,\ S_l = x\},$$

$$\ldots$$

$$\tau_{i+1}(x) := \inf\{l:\ l>\tau_i(x),\ S_l = x\} \quad (i = 0, 1, 2, \ldots),$$

$$\tau_i := \tau_i(0) \quad (i = 0, 1, 2, \ldots),$$

and let

$$\alpha_1(k) := \xi(k, \tau_1)-\xi(0, \tau_1) = \xi(k, \tau_1)-1,$$

$$\ldots$$

$$\begin{aligned}\alpha_i(k) &:= (\xi(k, \tau_i)-\xi(k, \tau_{i-1}))-(\xi(0, \tau_i)-\xi(0, \tau_{i-1}))\\ &= (\xi(k, \tau_i)-\xi(k, \tau_{i-1}))-1 \quad (k = \pm 1,\ \pm 2, \ldots;\ i = 1, 2, \ldots).\end{aligned}$$

Clearly then $\alpha_1(k), \alpha_2(k), \ldots$ is a sequence of i.i.d. rv for any $k=\pm 1, \pm 2, \ldots$. Now we evaluate the distribution of $\alpha_1(k)$. We have

Lemma 2.2.

$$\mathrm{P}\{\alpha_1(k) = -1\} = \mathrm{P}\{\xi(k, \tau_1) = 0\} = \frac{2|k|-1}{2|k|}, \tag{2.1}$$

$$\mathrm{P}\{\alpha_1(k) = l\} = \mathrm{P}\{\xi(k, \tau_1) = l+1\} = \left(\frac{1}{2|k|}\right)^2\left(\frac{2|k|-1}{2|k|}\right)^l \quad (l = 0, 1, 2, \ldots). \tag{2.2}$$

Proof. Without loss of generality we assume that $k>0$. Then

$$\{\xi(k, \tau_1) = 0\} = \{X_1 = -1\}\cup\{X_1 = 1,\ S_2 \neq k,\ S_3 \neq k, \ldots, S_{\tau_1-1} \neq k\}.$$

Hence by Lemma 2.1.

$$P\{\xi(k,\tau_1)=0\}=\frac{1}{2}+\frac{1}{2}\frac{k-1}{k}=\frac{2k-1}{2k},$$

and (2.1) is proven. Similarly, in case of $m>0$ we have

$$\{\xi(k,\tau_1)=m\}=[\{X_1=1\}\cap\{S_2\neq 0, S_3\neq 0,\dots,S_{\tau_1(k)-1}\neq 0, S_{\tau_1(k)}=k\}]\cap$$
$$\cap[\{X_{\tau_1(k)+1}=1\}\cup(\{X_{\tau_1(k)+1}=-1\}\cap\{S_{\tau_1(k)+1}\neq 0, S_{\tau_1(k)+2}\neq 0,\dots,$$
$$\dots, S_{\tau_2(k)-1}\neq 0, S_{\tau_2(k)}=k\})]\cap\dots\cap[\{X_{\tau_{m-1}(k)+1}=1\}\cup(\{X_{\tau_{m-1}(k+1)}=-1\}\cap$$
$$\cap\{S_{\tau_{m-1}(k)+1}\neq 0, S_{\tau_{m-1}(k)+2}\neq 0,\dots, S_{\tau_m(k)-1}\neq 0, S_{\tau_m(k)}=k\})]\cap$$
$$\cap[\{X_{\tau_m(k)+1}=-1\}\cup\{S_{\tau_m(k)+2}\neq k, S_{\tau_m(k)+3}\neq k,\dots, S_{\tau_1(0)-1}\neq k, S_{\tau_1(0)}=0\}].$$

(Note that in case of $\xi(k,\tau_1)=m$ we have $0<\tau_1(k)<\tau_2(k)<\dots<\tau_m(k)<\tau_1(0)<$ $<\tau_{m+1}(k)$). Hence, again by Lemma 2.1

$$P\{\xi(k,\tau_1)=m\}=\frac{1}{2}\frac{1}{k}\left(\sum_{j=0}^{m-1}\binom{m-1}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\frac{k-1}{2}\right)^{m-1-j}\right)\frac{1}{2}\frac{1}{k}=$$
$$=\left(\frac{1}{2k}\right)^2\left(\frac{2k-1}{2k}\right)^{m-1},$$

and (2.2) is also proven. This also completes the proof of Lemma 2.2.

Lemma 2.2 implies

Lemma 2.3.

(2.3) $$E\alpha_1=0,\quad E\alpha_1^2=4k-2,$$

(2.4) $$\lim_{n\to\infty}P\{n^{-1/2}(\alpha_1(k)+\alpha_2(k)+\dots+\alpha_n(k))\leq x(4k-2)^{1/2}\}=$$
$$=(2\pi)^{-1/2}\int_{-\infty}^{x}e^{-u^2/2}\,du,\quad -\infty<x<\infty,$$

(2.5) $$\lim_{n\to\infty}P\{n^{-1/2}\sup_{j\leq n}(\alpha_1(k)+\alpha_2(k)+\dots+\alpha_j(k))\leq x(4k-2)^{1/2}\}=$$
$$=\left(\frac{2}{\pi}\right)^{1/2}\int_0^x e^{-u^2/2}\,du,\quad x>0,$$

and

(2.6) $$\limsup_{n\to\infty}\frac{\alpha_1(k)+\alpha_2(k)+\dots+\alpha_n(k)}{(n\log\log n)^{1/2}}=2(2k-1)^{1/2}\quad a.s.$$

The following two lemmas are simple consequences of (2.6).

Lemma 2.4. *Let $\{\mu_n\}$ be any sequence of positive integer valued rv with $\lim_{n\to\infty} \mu_n=\infty$ a.s. Then*

$$\limsup_{n\to\infty} \frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\mu_n}(k)}{(\mu_n \log\log \mu_n)^{1/2}} \leqq 2\sqrt{2k-1} \quad a.s.$$

Lemma 2.5. *Let $\{v_n\}$ be a sequence of positive integer valued rv with the following properties:*

(i) $\lim_{n\to\infty} v_n=\infty$ *a.s.*

(ii) *there exists a set $\Omega_0\subset\Omega$ such that $\mathrm{P}(\Omega_0)=0$ and for each $\omega\notin\Omega_0$ and $k=1,2,\ldots$ there exists an $n=n(\omega,k)$ for which $v_{n(\omega,k)}=k$.*

Then

$$\limsup_{n\to\infty} \frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{v_n}(k)}{(v_n \log\log v_n)^{1/2}} = 2\sqrt{2k-1} \quad a.s.$$

Utilizing Lemma 2.5. with $v_n=\xi(0,n)$, Theorem 3 and the trivial inequality

$$\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\xi(0,n)}(k) \leqq \xi(k,n)-\xi(0,n) \leqq \alpha_1(k)+\alpha_2(k)+\ldots+\alpha_{\xi(0,n)+1}(k)+1,$$

we obtain Theorem 1.

3. Proof of Theorem 2

Here we only present a proof of the statement

$$\limsup_{N\to\infty} \frac{\xi(1,N)-\xi(0,N)}{N^{1/4}(\log\log N)^{3/4}} = \left(\frac{128}{27}\right)^{1/4} \quad \text{a.s.}$$

The other statements of Theorem 2 are proven along similar lines.

The proof of Theorem 2 is based on the following result of Dobrushin (1955).

Theorem C.

$$\lim_{n\to\infty} \mathrm{P}\{n^{-1/4}(\xi(1,n)-\xi(0,n)) \leqq 2^{1/2}x\} = \frac{2}{\pi} \int_{-\infty}^{x} \int_{0}^{\infty} \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right) dz\, dy.$$

Dobrushin also notes that the density function g of $|N_1|^{1/2}N_2$, where N_1 and N_2 are independent normal (0, 1) rv, is

$$g(y) = \frac{2}{\pi} \int_{0}^{\infty} \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right) dz.$$

Hence Theorem C can be reformulated via saying that

$$2^{-1/2}n^{-1/4}(\xi(1,n)-\xi(0,n)) \xrightarrow{\mathscr{D}} |N_1|^{1/2}N_2 \quad (n\to\infty). \tag{3.1}$$

In fact this statement is not very surprising since on replacing n by $\xi(0, n)$ and k by 1 in (2.4), *intuitively* it is clear that

$$\frac{\alpha_1(1)+\alpha_2(1)+\ldots+\alpha_{\xi(0,n)}(1)}{\sqrt{2\xi(0,n)}} \sim \frac{\xi(1,n)-\xi(0,n)}{\sqrt{2\xi(0,n)}} \xrightarrow{\mathcal{D}} N_2 \quad (n\to\infty). \tag{3.2}$$

(We must emphasize that we do not know any proof of this intuitively clear statement.)

Also, by Theorem A

$$n^{-1/4}(\xi(0,n))^{1/2} \xrightarrow{\mathcal{D}} |N_1|^{1/2} \quad (n\to\infty). \tag{3.3}$$

Intuitively it is again clear (however not yet proved) that

$$\frac{\xi(1,n)-\xi(0,n)}{\sqrt{2\xi(0,n)}} \text{ and } n^{-1/4}(\xi(0,n))^{1/2} \text{ are asymptotically independent rv.} \tag{3.4}$$

"Hence" (3.2), (3.3) and (3.4) together imply (3.1). The proof of Dobrushin is not based on this idea. Following his method however, a slightly stronger version of his Theorem C can be obtained.

Theorem C*. *Let $\{x_n\}$ be any sequence of positive numbers such that $x_n=o(\log n)$. Then*

$$\mathrm{P}\{n^{-1/4}(\xi(1,n)-\xi(0,n))<-2^{1/2}x_n\}\approx\frac{2}{\pi}\int_{-\infty}^{-x_n}\int_0^\infty \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz\,dy$$

and

$$\mathrm{P}\{n^{-1/4}(\xi(1,n)-\xi(0,n))>2^{1/2}x_n\}\approx\frac{2}{\pi}\int_{x_n}^{\infty}\int_0^\infty \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz\,dy.$$

We have also

Lemma 3.1. *There exists a positive constant C such that*

$$g(y)\leqq Cy^{1/3}\exp\left(-(3/2^{5/3})y^{4/3}\right). \tag{3.5}$$

Proof. Substituting $z=xy^{1/3}$ we obtain

$$g(y)=\int_0^\infty \exp\left(-\frac{y^2}{2z^2}-\frac{z^4}{2}\right)dz = y^{1/3}\int_0^\infty \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx.$$

Note that the function

$$f(x)=\frac{1}{x^2}+x^4$$

attains its maximum at $x_0=2^{-1/6}$ and $f(2^{-1/6})=3/2^{2/3}$. Let $x_1=(3/2^{2/3})^{1/4}$. Then

$$g(y) = y^{1/3}\left[\int_0^{x_1} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx+\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx\right] \leqq$$

$$\leqq x_1 y^{1/3}\exp\left(-\frac{y^{4/3}}{2}3\cdot 2^{-2/3}\right)+y^{1/3}\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx.$$

For $y>2^{-3/4}x_1^{-1/4}$ we also have

$$\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx \leqq \int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}x^4\right)dx \leqq 2x_1^3 y^{4/3}\int_{x_1}^{\infty} \exp\left(-\frac{y^{4/3}}{2}x^4\right)dx\leqq$$

$$\leqq 2y^{4/3}\int_{x_1}^{\infty} x^3\exp\left(-\frac{y^{4/3}}{2}x^4\right)dx = \exp\left(-\frac{y^{4/3}}{2}x_1^4\right).$$

Hence we have (3.5).

Lemma 3.2. *For any* $\varepsilon>0$ *there exists a* $C=C(\varepsilon)>0$ *such that*

$$g(y) \geqq C\exp\left(-\frac{y^{4/3}}{2-\varepsilon}3\cdot 2^{-2/3}\right).$$

Proof. With $x_0=2^{-1/6}$ and $\delta>0$ we have

$$g(y) \geqq y^{1/3}\int_{x_0-\delta}^{x_0+\delta} \exp\left(-\frac{y^{4/3}}{2}\left(\frac{1}{x^2}+x^4\right)\right)dx \geqq$$

$$\geqq 2\delta y^{1/3}\exp\left(-\frac{y^{4/3}}{2}3\cdot 2^{-2/3}\frac{1}{1-\varepsilon^*}\right),$$

where ε^* is defined by

$$\max\left(\frac{1}{(x_0-\delta)^2}+(x_0-\delta)^4,\ \frac{1}{(x_0+\delta)^2}+(x_0+\delta)^4\right) = \frac{3/2^{2/3}}{1-\varepsilon^*}.$$

Hence Lemma 3.2 is proved.

Lemmas 3.1, 3.2 and some standard calculus imply

Lemma 3.3. *Let* $\{a_n\}$ *be a sequence of positive numbers with* $a_n\uparrow\infty$. *Then for any* $\varepsilon>0$ *there exist a* $C_1=C_1(\varepsilon)>0$ *and a* $C_2=C_2(\varepsilon)>0$ *such that*

$$C_1\exp\left(-\frac{a_n^{4/3}}{2-\varepsilon}(3/2^{2/3})\right) \leqq \int_{a_n}^{\infty} g(y)\,dy \leqq C_2\exp\left(-\frac{a_n^{4/3}}{2+\varepsilon}(3/2^{2/3})\right).$$

By Theorem C* and Lemma 3.3. we have

Lemma 3.4. *For any* $\varepsilon>0$ *there exist a* $C_1=C_1(\varepsilon)>0$ *and a* $C_2=C_2(\varepsilon)>0$ *such that*

$$\mathrm{P}\left\{n^{-1/4}(\xi(1,n)-\xi(0,n))\geqq(1+2\varepsilon)\left(\frac{128}{27}\right)^{1/4}(\log\log n)^{3/4}\right\}\leqq C_2(\log n)^{-(1+\varepsilon)}$$

and

$$\mathrm{P}\left\{n^{-1/4}(\xi(1,n)-\xi(0,n))\geqq(1-2\varepsilon)\left(\frac{128}{27}\right)^{1/4}(\log\log n)^{3/4}\right\}\geqq C_1(\log n)^{-(1-\varepsilon)}.$$

Next we prove

Lemma 3.5.

$$\limsup_{n\to\infty}\frac{\xi(1,n)-\xi(0,n)}{n^{1/4}(\log\log n)^{3/4}}\geqq\left(\frac{128}{27}\right)^{1/4}\qquad a.s.$$

Proof. Let

$$n_k:=[\exp(k\log k)],\quad b_k:=\left(\frac{128}{27}\right)^{1/4}n_k^{1/4}(\log\log n_k)^{3/4},$$

$$\zeta(n):=\xi(1,n)-\xi(0,n),\quad \xi(x,(m,n)):=\xi(x,n)-\xi(x,m)\quad(m<n),$$
$$\zeta(m,n):=\xi(1,(m,n))-\xi(0,(m,n)),\ A_k:=\{\zeta(n_k)\geqq(1-2\varepsilon)b_k\}.$$

By Lemma 3.4

$$\mathrm{P}\{A_k\}\geqq C(k\log k)^{-(1-\varepsilon)}. \tag{3.6}$$

Let $j<k$ and consider

$$\mathrm{P}\{A_kA_j\}=\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{A_k,\zeta(n_j)=l\}=$$
$$=\sum_x\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{A_k,\zeta(n_j)=l,\ S_{n_j}=x\}=$$
$$=\sum_x\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{A_k|\zeta(n_j)=l,\ S_{n_j}=x\}\mathrm{P}\{\zeta(n_j)=l,\ S_{n_j}=x\}=$$
$$=\sum_x\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{\zeta(n_j,n_k)\geqq(1-2\varepsilon)b_k-l\,|\,S_{n_j}=x\}\mathrm{P}\{\zeta(n_j)=l,\ S_{n_j}=x\}\leqq$$
$$\leqq\sum_{l=(1-2\varepsilon)b_j}^{\infty}\sup_x\mathrm{P}\{\zeta(n_j,n_k)\geqq(1-2\varepsilon)b_k-l\,|\,S_{n_j}=x\}\sum_x\mathrm{P}\{\zeta(n_j)=l,\ S_{n_j}=x\}=$$
$$=\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{\zeta(n_k-n_j)\geqq(1-2\varepsilon)b_k-l\}\mathrm{P}\{\zeta(n_j)=l\}\leqq$$
$$\leqq\sum_{l=(1-2\varepsilon)b_j}^{\infty}\mathrm{P}\{\zeta(n_k)\geqq(1-2\varepsilon)b_k-l\}\mathrm{P}\{\zeta(n_j)=l\}\approx$$
$$\approx\int_{(1-2\varepsilon)2^{-1/2}n_j^{-1/4}b_j}\mathrm{P}\{\zeta(n_k)\geqq(1-2\varepsilon)b_k-2^{1/2}n_j^{1/4}y\}\mathrm{P}\{\zeta(n_j)=2^{1/2}n_j^{1/4}y\}\,dy=$$
$$=\int_A^{\infty}g(y)\int_{B(y)}^{\infty}g(z)\,dz\,dy,$$

where

$$A := (1-2\varepsilon)2^{-1/2}n_j^{-1/4}b_j = (1-2\varepsilon)2^{-1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_j)^{1/4}$$

and

$$B(y) := (1-2\varepsilon)b_k 2^{-1/2}n_j^{-1/4}b_j - 2^{1/2}n_j^{1/4}y2^{-1/2}n_k^{-1/4} =$$

$$= (1-2\varepsilon)2^{-1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_k)^{1/4} - y\left(\frac{n_j}{n_k}\right)^{1/4}.$$

Now a simple but tedious calculation yields that for any $\varepsilon>0$ there exists a j_0 such that if $j_0<j<k$ then

$$\mathrm{P}\{A_j A_k\} \leqq (1+\varepsilon)\mathrm{P}\{A_j\}\mathrm{P}\{A_k\}. \tag{3.7}$$

Here we omit the details of the proof of this fact, and sketch only the main idea behind it. Since $(n_j/n_k)^{1/4} \leqq k^{-1/4}$ $(j=1, 2, ..., k-1)$, the lower limit of integration $B(y)$ above is nearly equal to

$$(1-2\varepsilon)2^{1/2}\left(\frac{128}{27}\right)^{1/4}(\log\log n_k)^{1/4} \quad \text{if} \quad y \leqq k^{1/4}, \quad \text{say.}$$

Hence for the latter y values the integral $\int_{B(y)}^{\infty} g(z)\,dz$ is nearly equal to $P\{A_k\}$. Similarly, the integral $\int_{A}^{\infty} g(y)\,dy$ gives $P\{A_j\}$, and our claim (3.7) follows, for in the case of $y>k^{1/4}$ the value of $g(y)$ is very small.

Now (3.6), (3.7) and the Borel—Cantelli lemma combined give Lemma 3.5. We have also

Lemma 3.6. *Let* $m_k := [\exp(k/\log^2 k)]$ *and*

$$B_k :=$$

$$= \left\{\xi(0, (m_k, m_{k+1})) \geqq (1+\varepsilon)\left[(m_{k+1}-m_k)\left(\log\frac{m_{k+1}}{m_{k+1}-m_k} + 2\log\log m_{k+1}\right)\right]^{1/2}\right\}.$$

Then of the events B_k only finitely many occur with probability one.

Proof. This lemma is an immediate consequence of Theorem 1 of Csáki—Csörgő—Földes—Révész (1983), where the corresponding statement is formulated in terms of Wiener process instead of symmetric random walk. The analogue statement is easily obtained.

Lemma 3.7. *Let*

$$M_{k+1} := ((2+\varepsilon)m_{k+1}\log\log m_{k+1})^{1/2},$$

$$a_{k+1} := (1+\varepsilon)\left[(m_{k+1}-m_k)\left(\log\frac{m_{k+1}}{m_{k+1}-m_k}+2\log\log m_{k+1}\right)\right]^{1/2}$$

and

$$D_k := \left\{\sup_{l\le M_{k+1}-a_{k+1}}\ \sup_{j\le a_{k+1}} |\alpha_l+\alpha_{l+1}+\dots+\alpha_{l+j}| \ge \right.$$

$$\left.\ge \left[(2+\varepsilon)a_{k+1}\left(\log\frac{M_{k+1}}{a_{k+1}}+\log\log M_{k+1}\right)\right]^{1/2}\right\}.$$

Then of the events D_k only finitely many occur with probability one.

Proof. Cf. Theorem 3.11 of Csörgő—Révész (1981).

A simple consequence of Lemmas 3.6, 3.7 and Theorem B is

Lemma 3.8. *Let*

$$E_k := \left\{\sup_{m_k\le n\le m_{k+1}} |\zeta(m_k, n)| \ge \left[(2+\varepsilon)a_{k+1}\left(\log\frac{M_{k+1}}{a_{k+1}}+\log\log M_{k+1}\right)\right]^{1/2}\right\}.$$

Then of the events E_k only finitely many occur with probability one.

Lemma 3.9.

$$\limsup_{n\to\infty}\frac{\xi(1, n)-\xi(0, n)}{n^{1/4}(\log\log n)^{3/4}} \le \left(\frac{128}{27}\right)^{1/4} \quad a.s. \tag{3.8}$$

Proof. Let

$$c_k := \left(\frac{128}{27}\right)^{1/4} m_k^{1/4}(\log\log m_k)^{3/4}, \quad E_k := \{\zeta(m_k) \ge (1+2\varepsilon)c_k\}.$$

Then by Lemma 3.4 only finitely many of the events E_k occur with probability one. Now observing that

$$\left[(2+\varepsilon)a_{k+1}\left(\log\frac{M_{k+1}}{a_{k+1}}+\log\log M_{k+1}\right)\right]^{1/2} = o(c_k),$$

we have (3.8) by Lemma 3.8, and Lemma 3.9 is proved.

Also Lemmas 3.5 and 3.9 combined give Theorem 2.

4. Proof of Theorem 3.

A simple calculation and Lemma 2.2 imply

Lemma 4.1. *For any* $k=1,2,\ldots,n$; $n=1,2,\ldots$ *we have*

$$E\exp\left(-\frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_n(k)}{((4k-2))n^{1/2}}\right)\leqq C,$$

where C is an absolute positive constant.

The above lemma together with the Chebishev inequality and the Borel—Cantelli lemma imply

Lemma 4.2. *For any* $\varepsilon>0$

$$\lim_{n\to\infty}\sup_{|k|\leqq n}\frac{\alpha_1(k)+\alpha_2(k)+\ldots+\alpha_n(k)}{(kn)^{1/2}(\log n)^{1+\varepsilon}}=0\quad a.s.$$

Consequently, on replacing n by $\xi(0,n)$, we get

$$\lim_{n\to\infty}\sup_{|k|\leqq\xi(0,n)}\frac{\xi(k,n)-\xi(0,n)}{(k\xi(0,n))^{1/2}(\log n)^{1+\varepsilon}}=0\quad a.s.$$

and

$$\lim_{n\to\infty}\sup_{|k|<\xi(0,n)(\log n)^{-(2+3\varepsilon)}}\frac{\xi(k,n)-\xi(0,n)}{\xi(0,n)(\log n)^{-\varepsilon/2}}=0\quad a.s. \tag{4.1}$$

By (4.1) we have also Theorem 3.

5. Proof of Theorem 5.

A theorem of Hirsch (1965) (cf. p. 124 in Csörgő—Révész (1981)) says:

$$\max_{1\leqq k\leqq n}S_k\leqq n^{1/2}(\log n)^{-1}\quad i.o.$$

with probability one. This, in turn, implies Theorem 4.

6. A problem

To fill in the gap between Theorems 3 and 4 is an interesting enough problem. The following conjecture, however, is even more challenging.

Conjecture.

$$\lim_{n\to\infty}\sup_{m_n\leqq k\leqq M_n}\left|\frac{\xi(k,n)}{\xi(0,n)}-1\right|=0\quad a.s.,$$

where

$$m_n := \frac{\inf\limits_{1 \le k \le n} S_k}{\log\log n}, \quad M_n := \frac{\sup\limits_{1 \le k \le n} S_k}{\log\log n}.$$

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