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Rates of uniform convergence for the empirical characteristic function

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In honour of Professor Károly Tandori on his sixtieth birthday

Introduction, results, and discussion

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed d-dimensional random vectors, $d \geqq 1$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with common distribution function $F(x)$, $x \in \mathbf{R}^d$, and characteristic function

$$C(t) = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbf{R}^d$. The nth, empirical characteristic function of the sequence is

$$C_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{i\langle t, X_j\rangle} = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF_n(x), \quad t = (t_1, \ldots, t_d) \in \mathbf{R}^d,$$

where $F_n(x)$, $x \in \mathbf{R}^d$, denotes the empirical distribution function of $X_1, \ldots, X_n$. For any extended number $0 < T \leqq \infty$ consider the random variable

$$\Delta_n(T) = \sup_{|t| < T} |C_n(t) - C(t)|.$$

It is trivial that $\Delta_n(T) \to 0$ almost surely as $n \to \infty$ for any fixed $T < \infty$, but CSÖRGŐ and TOTIK [2] have pointed out that $\Delta_n(\infty) \to 0$ almost surely if and only if F is purely discrete. These two facts lead to considering the quantities $\Delta_n(T_n)$ for some sequences $\{T_n\}$ of finite positive numbers converging to infinity at an intermediate rate. In Theorem 1 of [2] we have shown in a simple elementary fashion that $\Delta_n(\exp\{n/G_n\}) \to 0$ almost surely for any sequence $\{G_n\}$ such that $G_n \to \infty$. More interesting is the fact that this result is optimal in general. We proved in Theorem 2 of [2] that for any characteristic function C which vanishes at infinity along at least one

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path, $\Delta_n(\exp\{n/G_n\})$ does not converge to zero even in probability if G_n does not converge to infinity. It is natural to expect that if we specify the rate at which G_n goes to infinity then we should be able to derive rates at which $\Delta_n(\exp\{n/G_n\})$ converges to zero almost surely. The present note adresses exactly this problem.

It should be pointed out that later but independently YUKICH [4] also proved, at least in the univariate special case $d=1$, that $\Delta_n(\exp\{n/G_n\})\to 0$ almost surely whenever $G_n\to\infty$. He derives this result, necessarily in a more complicated way, from a general theorem of his concerning the law of large numbers for empirical measures on general measurable state spaces and indexed by classes of functions. He does not attempt the optimality of the result. However, the fact that he obtained an apparently unimprovable corollary shows the strength of his general theorem. It was YUKICH [4] who first obtained a special rate result on moving intervals. Again in the univariate case $d=1$ he deduced from his general theorem what is Example (liii) below, with a larger constant on the right side.

Our approach is direct and elementary. The proof of the following result, presented in the next section, appears as a straightforward extension of the proof of Theorem 1 in [2]. In order to avoid trivialities we assume that $X=X_1$ is nondegenerate.

Theorem. *Let* $K_n=\inf\{x>0\colon P\{|X|>x\}\leqq R_n\}$ *where* $\{R_n\}$ *is a nonincreasing sequence of positive numbers. If*

$$\sum_{n=n_0}^{\infty} e^{-M_1R_n^2 n}+\sum_{n=n_0}^{\infty}(K_nT_n/R_n)^d e^{-M_2R_n^2 n}<\infty$$

for some $M_1, M_2>0$ *such that*

$$n_0=n_0(M_1,M_2)=\inf\left\{n\colon R_n\leqq 1/\left(4\sqrt{\max(M_1,M_2)}\right)\right\}<\infty,$$

then

$$\limsup_{n\to\infty}\frac{1}{R_n}\Delta_n(T_n)\leqq 2+2\sqrt{M_1}+4\sqrt{M_2}$$

almost surely.

Setting $R_n\equiv\varepsilon$ and $K_n\equiv K(\varepsilon)$ with any small $\varepsilon>0$ and $T_n=\exp\{o(n)\}$, we see that this theorem gives Theorem 1 of CSÖRGŐ and TOTIK [2]. On the other hand, it is not surprising that on smaller balls the rates depend on the tail behaviour of the underlying distribution. Even in the case of a fixed ball when $T_n\equiv T<\infty$, the ideal almost sure rate result $\Delta_n(T)=O\left(\sqrt{(\log\log n)/n}\right)$ can be achieved only under some tail condition. Improving an earlier result of CSÖRGŐ [1], who required $\varepsilon>1$, the presently available weakest such condition of LEDOUX [3] is satisfied if $Eg_\varepsilon(|X|)<\infty$ for any $\varepsilon>0$, where

$$g_\varepsilon(u)=\begin{cases}(\log u)(\log_2 u)^2(\log_3 u)^{1+\varepsilon}, & \text{if } u\geqq\exp\{\exp\{e\}\},\\ 1, & \text{if } 0\leqq u<\exp\{\exp\{e\}\}.\end{cases}$$

Here and in what follows, for any integer $k \geqq 1$, $\log_k$ denotes the k times iterated logarithm.

In view of this circumstance, it is perhaps tolerable that the Theorem above, designed for growing balls, can at best give the rate $R_n = \sqrt{(\log n)/n}$ even when $T_n \equiv T < \infty$ and X is a bounded random variable. The following version of the Theorem is more restrictive because it can be applied only for balls with radius $T_n \geqq n^A$, where $A > 0$. Nevertheless, in this domain it is more suggestive.

Corollary. *Let* $K_n = \inf\{x > 0\colon P\{|X| > x\} \leqq 1/\sqrt{G_n}\}$, *where* $\{G_n\}$ *is a sequence of positive numbers such that* $G_n \to \infty$ *as* $n \to \infty$. *If*

$$\sum_{n=1}^{\infty} e^{-M_1 n/G_n} + \sum_{n=1}^{\infty} G_n^{d/2} K_n^d e^{-M_3 n/G_n} < \infty$$

for some $M_1, M_3 > 0$, *then*

$$\limsup_{n\to\infty} \sqrt{G_n}\, \Delta_n(\exp\{n/G_n\}) \leqq 2 + 2\sqrt{M_1} + 4\sqrt{d + M_3}$$

almost surely.

We illustrate our result by way of natural examples. Note that if $P\{|X| > x\} \leqq h(x)$ for large enough $x > 0$, where h is a nonincreasing function and if $V_n = \inf\{x > 0\colon h(x) \leqq R_n\}$ then $K_n \leqq V_n$ for large enough n. Therefore, if the condition of the Theorem or the Corollary is satisfied with V_n replacing K_n, then the corresponding conclusion is applicable. The following examples follow by elementary calculation. All the limsup statements are meant to hold almost surely.

Example (1). Suppose that $P\{|X| > x\} \leqq Lx^{-\alpha}$ for all large enough x, where L and α are arbitrary positive constants. Then

(i) for any $A > 0$ and integer $k \geqq 1$,

$$S_1(A, k) = \limsup_{n\to\infty} (\log_k n)^{A/2} \Delta_n(\exp\{n/(\log_k n)^A\}) \leqq 2 + 4\sqrt{d},$$

(ii) for any $0 < A < 1$,

$$S_2(A) = \limsup_{n\to\infty} n^{(1-A)/2} \Delta_n(\exp\{n^A\}) \leqq 2 + 4\sqrt{d},$$

(iii) for any $A > 0$

$$S_3(A) = \limsup_{n\to\infty} \sqrt{\frac{n}{\log n}}\, \Delta_n(n^A) \leqq 2 + 2\sqrt{\min(A, 1)} + 4\sqrt{1 + \left(A + \frac{1+\alpha}{2\alpha}\right)d},$$

(iv) for any $A > 0$ and integer $k \geqq 1$,

$$S_4(A, k) = \limsup_{n\to\infty} \sqrt{\frac{n}{\log n}}\, \Delta_n((\log_k n)^A) \leqq 4 + 4\sqrt{1 + d\frac{1+\alpha}{2\alpha}}.$$

Note that in case (iii) the right-side constant is obtained by applying both the Theorem and the Corollary. When $d=1$, the result in (iii) was obtained by YUKICH [4], as mentioned above, with the greater upper bound $12\{1+A+(1+\alpha)/(2\alpha)\}$.

Example (2). Suppose that $P\{|X|>x\}\leqq L_1 \exp\{-L_2 x^\alpha\}$ for all large enough x and positive constants L_1, L_2 and α. Then we obtain $S_2(A,k)\leqq 2+4\sqrt{d}$, $S_2(A)\leqq \leqq 2+4\sqrt{d}$ and

$$S_3(A) \leqq 2+2\sqrt{\min(A,1)}+4\sqrt{1+\left(A+\frac{1}{2}\right)d}, \quad S_4(A,k) \leqq 4+4\sqrt{1+\frac{d}{2}}.$$

This means that the Theorem cannot distinguish the exponential decrease of a tail from power decrease. Of course, the latter inequalities follow directly from those in Example (1) by taking the limit as $\alpha\to\infty$. If we assume that X is a bounded random variable, the Theorem and Corollary give nothing better than the latter four inequalities.

Example (3). Suppose that $P\{|X|>x\}\leqq 1/(\log_k x)^\alpha$ for all large enough x, a positive constant α and an integer $k\geqq 1$. Then

$$\limsup_{n\to\infty} (\log_k n)^\alpha \Delta_n\left(\exp\{n/(\log_k n)^{2\alpha}\}\right) \leqq 2+4\sqrt{d},$$

and even on a fixed ball of radius $T<\infty$,

$$\limsup_{n\to\infty} (\log_k n)^\alpha \Delta_n(T) \leqq 2.$$

We know from the mentioned results in [1] and [3] that the Theorem is not optimal when $T_n\equiv T<\infty$. It is tempting to beleive that the above rates cannot essentially be improved upon on large balls with $T_n\geqq n^A$. However, this remains an open question at this writing.

Proof of Theorem

Introducing the truncated integrals

$$B_n(t) = \int_{|x|\leqq K_n} e^{i\langle t,x\rangle} dF_n(x) = \frac{1}{n}\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j|\leqq K_n),$$

$$\tilde{B}_n(t) = \int_{|x|\leqq K_n} e^{i\langle t,x\rangle}\, dF(x),$$

where $\chi(A)$ denotes the indicator of the event A, and writing $D_n(t)=B_n(t)-\tilde{B}_n(t)$, we have just as in [2] that

$$\Delta_n(T_n) \leqq \sup_{|t|\leqq T_n} |D_n(t)| + \sup_{|t|\leqq T_n} |B_n(t)-C_n(t)| + \sup_{|t|\leqq T_n} |\tilde{B}_n(t)-C(t)|.$$

The third term is obviously less than or equal to R_n. The second term is

$$\frac{1}{n}\sup_{|t|\leqq T_n}\Big|\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j|>K_n)\Big| \leqq \frac{1}{n}\sum_{j=1}^{n}\chi(|X_j|>K_n),$$

and hence

$$q_n = P\Big\{\sup_{|t|\leqq T_n}|B_n(t)-C_n(t)| > (1+2\sqrt{M_1})R_n\Big\} \leqq$$

$$\leqq P\Big\{\frac{1}{n}\sum_{j=1}^{n}\big(\chi(|X_j|>K_n)-P\{|X|>K_n\}\big) > 2\sqrt{M_1}\,R_n\Big\} \leqq$$

$$\leqq \begin{cases} 2\exp\{-M_1R_n^2 n/\sigma_n^2\}, & \text{if } \sqrt{M_1}<\sigma_n^2/4,\\ 2\exp\{-\sqrt{M_1}\,R_n n/4\}, & \text{if } \sqrt{M_1}\geqq\sigma_n^2/4,\end{cases}$$

by Bernstein's inequality, where $\sigma_n^2=\mathrm{Var}\,\chi(|X|>K_n)\leqq 1$. Thus, if $n\geqq n_0$, then $q_n\leqq 2\exp\{-M_1R_n^2 n\}$.

Let $\varepsilon>0$ be arbitrarily small, $\varepsilon\leqq 4d^{3/2}$, and let us cover the cube $[-T_n, T_n]^d$, and hence our ball $\{t\colon |t|\leqq T_n\}$ by $N_n=\big([(4d^{3/2}K_nT_n)/(\varepsilon R_n)]+1\big)^d$ disjoint small cubes $\Delta_1, \dots, \Delta_{N_n}$, the edges of each of which are of length $(\varepsilon R_n)/(2d^{3/2}K_n)$. If $t_1, \dots, t_{N_n}$ denote the centres of these cubes then

$$\sup_{|t|\leqq T_n}|D_n(t)| \leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)| + \max_{1\leqq k\leqq N_n}\sup_{t\in\Delta_k}|D_n(t)-D_n(t_k)| \leqq$$

$$\leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)|+\varepsilon R_n,$$

for, exactly as in [2], $|D_n(s)-D_n(t)|\leqq 2d^{3/2}K_n|s-t|$. On the other hand, proceeding exactly as in [2] again, we obtain via another application of the Bernstein inequality that

$$P\Big\{\max_{1\leqq k\leqq N_n}|D_n(t_k)| > 4\sqrt{M_2}\,R_n\Big\} \leqq N_n\max_{1\leqq k\leqq N_n}P\big\{|D_n(t_k)| > 4\sqrt{M_2}\,R_n\big\} \leqq$$

$$\leqq Q(K_nT_n/R_n)^d\sup_{t\in\mathbf{R}^d}P\big\{|D_n(t)| > 4\sqrt{M_2}\,R_n\big\} \leqq$$

$$\leqq 4Q(K_nT_n/R_n)^d\exp\{-M_2R_n^2 n\}$$

whenever $n\geqq n_0$, where $Q=(4d^{3/2}/\varepsilon)^d$. Summing up,

$$\sum_{n=n_0}^{\infty}P\big\{\Delta_n(T_n) > (2+2\sqrt{M_1}+4\sqrt{M_2}+\varepsilon)R_n\big\} \leqq$$

$$\leqq 4Q\Big(\sum_{n=n_0}^{\infty}\exp\{-M_1R_n^2 n\}+\sum_{n=n_0}^{\infty}(K_nT_n/R_n)^d\exp\{-M_2R_n^2 n\}\Big),$$

and the Borel—Cantelli lemma yields the desired result.

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