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Contractions as restricted shifts

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Dedicated to Professzor K. Tandori on the occasion of his 60th birthday

In what follows T is a contraction (i.e. linear operator with $\|T\| \leqq 1$) on a Hilbert space $\mathcal{H}$ and "c.n.u." stands for "completely nonunitary".

A familiar result (cf. [2], [1]) guarantees that, if $T^n \to 0$ strongly as $n \to \infty$, then T is unitarily equivalent to a restriction of a backward shift. If T is c.n.u., on a complex separable space, then its functional model [3, Ch. VI, Sec. 2] shows that T is unitarily equivalent to a restriction of the orthogonal sum of a backward shift and a bilateral shift.

The purpose of this note is to generalize Rota's construction [2] to the case of an arbitrary c.n.u. contraction. We shall give the shift operators in question, the space of the restriction as well as the operator which provides the unitary equivalence to a certain extent explicitely in terms of T. Maybe Lemma 1 is of own independent interest. Our method is elementary and self-containing in the sense that it uses only very standard facts of operator theory.

Since $\{T^{*n}T^n\}_{n=0}^{\infty}$ is a decreasing sequence of selfadjoint operators, its strong limit exists, is a positive contraction and therefore

$$A = \left(\lim_{n\to\infty} T^{*n}T^n\right)^{1/2}$$

exists, A is selfadjoint and $0 \leqq A \leqq I$. Similarily, the selfadjoint operator

$$\hat{A} = \left(\lim_{n\to\infty} T^n T^{*n}\right)^{1/2}$$

exists and $0 \leqq \hat{A} \leqq I$. We define an operator V on $\overline{A\mathcal{H}}$ by

$$VAh = ATh \quad (h \in \mathcal{H})$$

and then by taking closure. The definition of A shows that $T^*A^2T = A^2$ and thus

$$\|ATh\| = \|Ah\| \quad (h \in \mathcal{H}),$$

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consequently V is an isometry on $\overline{A\mathcal{H}}$.

Lemma 1. *$\|AV^n g\|\uparrow\|g\|$ and $\|AV^{*n}g\|\downarrow\|\hat{A}Ag\|$ for $g\in\overline{A\mathcal{H}}$ as $n\to\infty$.*

Proof. For $h\in\mathcal{H}$ we have

$$\|AVAh\| = \|A^2Th\| \geqq \|T^*A^2Th\| = \|A^2h\|$$

and this implies that

$$\|AVf\| \geqq \|Af\| \quad \text{for} \quad f\in\overline{A\mathcal{H}}.$$

Substituting $f=V^n g$, we obtain

$$\|AV^{n+1}g\| \geqq \|AV^n g\| \quad (g\in\overline{A\mathcal{H}}, \quad n=0,1,\ldots).$$

This shows that $\{\|AV^n g\|\}_{n=0}^{\infty}$ is an increasing sequence.

Now let P_λ denote the spectral projection of A belonging to $[0,\lambda]$. If $0\leqq\lambda<1$ and $h\in\mathcal{H}$, then

$$\|Ah\|^2 = \|AT^n h\|^2 = \|P_\lambda AT^n h\|^2 + \|(I-P_\lambda)AT^n h\|^2 =$$
$$= \|AP_\lambda T^n h\|^2 + \|A(I-P_\lambda)T^n h\|^2 \leqq \lambda^2\|P_\lambda T^n h\|^2 + \|(I-P_\lambda)T^n h\|^2 =$$
$$= (\lambda^2-1)\|P_\lambda T^n h\|^2 + \|T^n h\|^2.$$

So we get

$$0 \leqq (1-\lambda^2)\|P_\lambda T^n h\|^2 \leqq \|T^n h\|^2 - \|Ah\|^2 \to 0 \quad \text{as} \quad n\to\infty.$$

This implies that for each $h\in\mathcal{H}$ and $0\leqq\lambda<1$,

$$\|P_\lambda T^n h\| \to 0 \quad \text{as} \quad n\to\infty.$$

Thus

$$\|P_\lambda V^n Ah\| = \|P_\lambda AT^n h\| \leqq \|A\|\,\|P_\lambda T^n h\| \to 0 \quad \text{as} \quad n\to\infty.$$

So we have

$$\|AV^n Ah\|^2 \geqq \|(I-P_\lambda)AV^n Ah\|^2 = \|A(I-P_\lambda)V^n Ah\|^2 \geqq \lambda^2\|(I-P_\lambda)V^n Ah\|^2 =$$
$$= \lambda^2\|V^n Ah\|^2 - \lambda^2\|P_\lambda V^n Ah\|^2 \to \lambda^2\|Ah\|^2 \quad (n\to\infty).$$

This means that, for each $h\in\mathcal{H}$ and $\varepsilon>0$,

$$\|Ah\| \geqq \|AV^n Ah\| \geqq (1-\varepsilon)\|Ah\|$$

if n is sufficiently large, i.e.:

$$\|AV^n Ah\| \to \|Ah\| \quad (h\in\mathcal{H}, \quad n\to\infty).$$

Now for $\varepsilon>0$ and $g\in\overline{A\mathcal{H}}$ there exists $h\in\mathcal{H}$ such that $\|g-Ah\|<\varepsilon$ and for sufficiently large n we have

$$0 \leqq \|g\| - \|AV^n g\| \leqq \big|\|g\| - \|Ah\|\big| + \big|\|Ah\| - \|AV^n Ah\|\big| + \big|\|AV^n Ah\| - \|AV^n g\|\big| <$$
$$< \|g - Ah\| + \varepsilon + \|AV^n\| \, \|Ah - g\| < 3\varepsilon.$$

So the first statement of the lemma is proved. In order to prove the second one, let $g \in \overline{A\mathscr{H}}$ and $h \in \mathscr{H}$. Then we have

$$(AV^{*n}g, h) = (g, V^n Ah) = (g, AT^n h) = (T^{*n}Ag, h)$$

and consequently

$$AV^{*n}g = T^{*n}Ag \quad (g \in \overline{A\mathscr{H}}, \quad n = 1, 2, \ldots). \tag{1}$$

Thus, using the definition of $\hat{A}$ we obtain

$$\|AV^{*n}g\| = \|T^{*n}Ag\| \downarrow \|\hat{A}Ag\| \quad (g \in \overline{A\mathscr{H}}, \quad n \to \infty),$$

and so the lemma is proved.

Let us introduce an operator $B\colon \overline{A\mathscr{H}} \to \overline{A\mathscr{H}}$ by

$$B = (I - A\hat{A}^2 A)^{1/2} | \overline{A\mathscr{H}}$$

and a linear manifold $\mathscr{H}_0$ by

$$\mathscr{H}_0 = \{h \in \mathscr{H}\colon\ Ah \in \operatorname{Ran} B\}.$$

Lemma 2. *B commutes with V and V^*, and $T\mathscr{H}_0 \subset \mathscr{H}_0$. If T is c.n.u., then* $\operatorname{Ker} B = \{0\}$ *and $\mathscr{H}_0$ is dense in $\mathscr{H}$.*

Proof. Using the fact that $\hat{A}^2 = T\hat{A}^2 T^*$ (which is an easy consequence of the definition of $\hat{A}$) and (1), for $g \in \overline{A\mathscr{H}}$ we obtain

$$V^*A\hat{A}^2Ag = V^*AT\hat{A}^2T^*Ag = V^*VA\hat{A}^2AV^*g = A\hat{A}^2AV^*g.$$

This shows that $A\hat{A}^2A|\overline{A\mathscr{H}}$ commutes with V^*. Therefore B, being the limit of a sequence of polynomials of $A\hat{A}^2A|\overline{A\mathscr{H}}$, also commutes with V^*. Since B is self-adjoint, it commutes with V, too.

If $h \in \mathscr{H}_0$, then

$$ATh = VAh \in V(\operatorname{Ran} B) = \operatorname{Ran}(BV) \subset \operatorname{Ran} B$$

and therefore $T\mathscr{H}_0 \subset \mathscr{H}_0$.

Suppose now that T is c.n.u. and $f \in \operatorname{Ker} B$. Then

$$0 = \|Bf\|^2 = \big((I - A\hat{A}^2A)f, f\big) = \|f\|^2 - \|\hat{A}Af\|^2,$$

i.e.:

$$\|\hat{A}Af\| = \|f\|. \tag{2}$$

Since A and $\hat{A}$ are contractions, this implies that

$$\|Af\| = \|f\| \tag{3}$$

and, since $0 \leqq A \leqq I$, $Af=f$ follows. Thus (2) implies

$$\|\hat{A}f\| = \|f\|. \tag{4}$$

Using the definitions of A and $\hat{A}$, (3) and (4) imply

$$\|T^n f\| = \|f\| = \|T^{*n} f\| \quad (n = 1, 2, \ldots).$$

Since by assumption T is c.n.u., $f=0$ follows [3, Ch. I, Th. 3.2].

In order to prove the last statement of the lemma, suppose that T is c.n.u. and $h \in \operatorname{Ker}(I-A^2\hat{A}^2)$. In this case

$$0 = ((I-A^2\hat{A}^2)h, h) = \|h\|^2-(\hat{A}^2 h, A^2 h)$$

and thus

$$\|h\|^2 = |(\hat{A}^2 h, A^2 h)| \leqq \|\hat{A}^2 h\| \|A^2 h\| \leqq \|\hat{A} h\| \|A h\| \leqq \|h\|^2.$$

This shows that $\|\hat{A}h\| \|Ah\| = \|h\|^2$ and therefore

$$\|Ah\| = \|h\| = \|\hat{A}h\|$$

which implies

$$\|T^n h\| = \|h\| = \|T^{*n} h\| \quad (n = 1, 2, \ldots)$$

and consequently $h=0$.

So we have proved that, if T is c.n.u., then $\operatorname{Ker}(I-A^2\hat{A}^2)=\{0\}$ and consequently $\operatorname{Ran}(I-\hat{A}^2A^2)$ is dense in $\mathscr{H}$. Since

$$A(I-\hat{A}^2A^2) = (I-A\hat{A}^2A)A = B^2A,$$

we obtain

$$A \operatorname{Ran}(I-\hat{A}^2A^2) = B^2A\mathscr{H},$$

i.e.: $\mathscr{H}_0 \supset \operatorname{Ran}(I-\hat{A}^2A^2)$ and so $\mathscr{H}_0$ is also dense in $\mathscr{H}$. This completes the proof of the lemma.

Let us introduce the following notations:

$$D = (I-T^*T)^{1/2}, \quad \hat{D} = (I-TT^*)^{1/2},$$

$$V_n = \begin{cases} V^n & \text{if } \; n \geqq 0 \\ V^{*-n} & \text{if } \; n < 0. \end{cases}$$

Clearly $V_n^* = V_{-n}$, and (1) implies that

$$T^*AV_n g = AV_{n-1} g \quad (g \in \overline{A\mathscr{H}}, \quad n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Lemma 3. *If T is c.n.u., then*

$$\sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n f\|^2 = \|f\|^2 \quad \textit{for} \quad f \in \operatorname{Ran} B.$$

Proof. If $f \in \operatorname{Ran} B$, i.e. $f = Bg$ with a $g \in \overline{A\mathscr{H}}$ then, by Lemma 2 and the definition of V_n, we have

$$V_n f = V_n Bg = BV_n g \quad (n = 0, \pm 1, \ldots).$$

Since by assumption T is c.n.u., Lemma 2 implies the existence of B^{-1} on Ran B. Let $N, M \geqq 0$. Using (5) we obtain

$$\sum_{n=-M}^{N} \|\hat{D}AB^{-1}V_n f\|^2 = \sum_{n=-M}^{N} \|\hat{D}AV_n g\|^2 =$$

$$= \sum_{n=-M}^{N} [(V_{-n}A^2V_n g, g) - (V_{-n}ATT^*AV_n g, g)] =$$

$$= \sum_{n=-M}^{N} [\|AV_n g\|^2 - \|AV_{n-1} g\|^2] = \|AV^N g\|^2 - \|AV^{*M+1} g\|^2.$$

Now Lemma 1 implies that

$$\sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n f\|^2 = \|g\|^2 - \|\hat{A}Ag\|^2 = \|Bg\|^2 = \|f\|^2,$$

and so the lemma is proved.

Now define $\mathscr{K}$ by

$$\mathscr{K} = [\bigoplus_{m=0}^{\infty} \overline{D\mathscr{H}}] \oplus [\bigoplus_{n=-\infty}^{\infty} \overline{\hat{D}A\mathscr{H}}].$$

Let S denote the operator on $\mathscr{K}$ defined by

$$S\{[\bigoplus_{m=0}^{\infty} h_m] \oplus [\bigoplus_{n=-\infty}^{\infty} h'_n]\} = [\bigoplus_{m=0}^{\infty} h_{m+1}] \oplus [\bigoplus_{n=-\infty}^{\infty} h'_{n+1}].$$

This S is the orthogonal sum of a backward shift and a bilateral shift. An easy computation shows that

$$\sum_{m=0}^{\infty} \|DT^m h\|^2 = \sum_{m=0}^{\infty} [\|T^m h\|^2 - \|T^{m+1} h\|^2] = \|h\|^2 - \|Ah\|^2 \quad (h \in \mathscr{H}).$$

In what follows suppose that T is c.n.u. Then the above formula and Lemma 3 imply that

(6) $$\sum_{m=0}^{\infty} \|DT^m h\|^2 + \sum_{n=-\infty}^{\infty} \|\hat{D}AB^{-1}V_n Ah\|^2 = \|h\|^2 \quad \text{for} \quad h \in \mathscr{H}_0.$$

Thus the linear manifold $\mathscr{H}_0'$, defined by

$$\mathscr{H}_0' = \{[\bigoplus_{m=0}^{\infty} DT^m h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n Ah]:\ h \in \mathscr{H}_0\},$$

exists and is contained in $\mathscr{K}$. Let $\mathscr{H}'$ denote the closure of $\mathscr{H}_0'$ in $\mathscr{K}$. Define a mapping U_0: $\mathscr{H}_0 \to \mathscr{H}_0'$ by

$$U_0 h = [\bigoplus_{m=0}^{\infty} DT^m h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n Ah] \quad (h \in \mathscr{H}_0).$$

Our main result is the following

Theorem. *If T is c.n.u. then, using the above notations, U_0 extends to a unitary operator U: $\mathscr{H} \to \mathscr{H}'$, $\mathscr{H}'$ is an invariant subpace for S and $UT = (S|\mathscr{H}')U$.*

Proof. The definition of U_0 and (6) show that U_0 is linear and isometric. Since, by Lemma 2, Dom U_0 is dense in $\mathscr{H}$ and, by the definition of $\mathscr{H}'$, Ran U_0 is dense in $\mathscr{H}'$, U_0 extends by continuity to a unitary operator U: $\mathscr{H} \to \mathscr{H}'$.

If $h \in \mathscr{H}_0$ then, by Lemma 2, $Th \in \mathscr{H}_0$ and so we have

$$SUh = SU_0 h = [\bigoplus_{m=0}^{\infty} DT^{m+1} h] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_{n+1}Ah] =$$

$$= [\bigoplus_{m=0}^{\infty} DT^m (Th)] \oplus [\bigoplus_{n=-\infty}^{\infty} \hat{D}AB^{-1}V_n A(Th)] = U_0 Th = UTh.$$

Therefore, by continuity, for every element h of $\mathscr{H}$ we have $SUh = UTh$ and, since $U\mathscr{H} = \mathscr{H}'$, we can conclude that $\mathscr{H}'$ is invariant for S and $UTh = (S|\mathscr{H}')Uh$ for $h \in \mathscr{H}$. So the theorem is proved.

References

[1] C. Foiaş, O observaţie asupra modelului universal de contracţie al lui G. C. Rota, *Com. Acad. R. P. Romîne*, 13 (1963), 349—352.

[2] G. C. Rota, On models for linear operators, *Comm. Pure Appl. Math.*, 13 (1960), 468—472.

[3] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbers Space*, North Holland (Amsterdam, 1970).

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