Acta Sci. Math., 48 (1985), 163—171

Metric equivalence of tree automata

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To Professor K. Tandori on his 60th birthday

In [2] and [3] it has been shown that for both frontier-to-root and root-to-frontier tree automata the general product and the α_0-product have the same power from the point of view of metric completeness. In this paper we strengthen these results by showing that for both classes of tree automata mentioned above the α_0-product is metrically equivalent to the general product.

For all the notions and notations not defined in this paper we refer the reader to [2], [3] and [4].

1. Frontier-to-root tree automata

Throughout this section we use a fixed rank type R. To exclude trivial cases, it will be supposed that for an $m>0$, $m\in R$.

Let Σ and Σ^i $(i\in I)$ be ranked alphabets of rank type R, and consider the algebars $\mathcal{A}_i=(A_i, \Sigma^i)$ $(i\in I)$. Furthermore, let

$$\varphi = \{\varphi^m\colon (\prod(A_i|i\in I))^m\times\Sigma_m \to \prod(\Sigma^i_m|i\in I)|m\in R\}$$

be a family of mappings. Then by the *general product* of $\mathcal{A}_i$ $(i\in I)$ with respect to Σ and φ we mean the Σ-algebra

$$\mathcal{A} = (A, \Sigma) = \prod(\mathcal{A}_i|i\in I)[\Sigma, \varphi]$$

with $A=\prod(A_i|i\in I)$, and for arbitrary m, $\sigma\in\Sigma_m$ and $a_1, \dots, a_m\in A$

$$\mathrm{pr}_i(\sigma^{\mathcal{A}}(a_1, \dots, a_m)) = \sigma_i(\mathrm{pr}_i(a_1), \dots, \mathrm{pr}_i(a_m)),$$

where pr_i is the ith projection operator and $\sigma_i=pr_i(\varphi^m(a_1, \dots, a_m, \sigma))$. In the sequel we assume that I is given together with a linear ordering $\leqq$.

Received May 30, 1984.

We now define a special type of the general product. To this take the mappings φ_i^m $(m\in R, i\in I)$ given by $\varphi_i^m(a, \sigma)=\mathrm{pr}_i(\varphi^m(a, \sigma))$ $(a\in A, \sigma\in\Sigma_m)$. We say the product $\mathscr{A}$ above is an *α_0-product* if for every $i\in I$ and $m\in R$, $\varphi_i^m(a_1, \ldots, a_m, \sigma)$ $(a_1, \ldots, \ldots, a_m\in A, \sigma\in\Sigma_m)$ is independent of $\mathrm{pr}_j(a_1), \ldots, \mathrm{pr}_j(a_m)$ $(j\in I)$ whenever $i\leqq j$.

Let K be a class of algebras of rank type R. Then the operators $\mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{P}_g$, $\mathbf{P}_{gf}$, $\mathbf{P}_{\alpha_0}$ and $\mathbf{P}_{\alpha_0 f}$ are defined in the following way.

$\mathbf{H}(K)$: homomorphic images of algebras from K.

$\mathbf{S}(K)$: subalgebras of algebras from K.

$\mathbf{P}(K)$: direct products of algebras from K.

$\mathbf{P}_g(K)$: general products of algebras from K.

$\mathbf{P}_{gf}(K)$: products from $\mathbf{P}_g(K)$ with finitely many factors.

$\mathbf{P}_{\alpha_0}(K)$: α_0-products of algebras from K.

$\mathbf{P}_{\alpha_0 f}(K)$: α_0-products from $\mathbf{P}_{\alpha_0}(K)$ with finitely many factors.

Next we define the metric equivalence of the general product and the α_0-product. We say that the α_0-product is *metrically equivalent* to the general product if for arbitrary class K of finite algebras with rank type R, integer $m\geqq 0$ and DFT-transducer $\mathfrak{A}=(\Sigma, X_u, A, \Omega, Y_v, P, A')\in\mathrm{tr}\,(\mathbf{A})$ with $\mathbf{A}=(\mathscr{A}, \mathbf{a}, X_u, A')$ and $\mathscr{A}=(A, \Sigma)\in P_{gf}(K)$ there are a $\mathscr{B}=(B, \Sigma)\in\mathbf{P}_{\alpha_0 f}(K)$, $\mathbf{B}=(\mathscr{B}, \mathbf{b}, X_u, B')$ $(\mathbf{b}\in B^u, B'\subseteqq B)$ and $\mathfrak{B}= =(\Sigma, X_u, B, \Omega, Y_v, P', B')\in\mathrm{tr}\,(\mathbf{B})$ such that $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$.

Before showing that the α_0-product is metrically equivalent to the general product we recall the following result from [1].

Theorem 1. *For arbitrary class K of algebras with rank type R the equality*

$$\mathbf{HSP}_g(K) = \mathbf{HSP}_{\alpha_0}(K) = \mathbf{HSPP}_{\alpha_0}(K)$$

holds.

Using Theorem 1 we prove

Theorem 2. *The α_0-product is metrically equivalent to the general product.*

Proof. It is enough to show that for arbitrary ranked alphabet Σ of rank type R, integers $m, n\geqq 0$, Σ-algebra $\mathscr{A}=(A, \Sigma)$ in $\mathbf{P}_{gf}(K)\cap K_\Sigma$ and vector $\mathbf{a}=(a_1, \ldots, a_n)\in A^n$ there are a $\mathscr{B}=(B, \Sigma)$ in $\mathbf{P}_{\alpha_0 f}(K)\cap K_\Sigma$ and a vector $\mathbf{b}=(b_1, \ldots, b_n)\in B^n$ such that $(\mathscr{B}, \mathbf{b})$ can be mapped m-homomorphically onto $(\mathscr{A}, \mathbf{a})$. If $\mathscr{A}\in\mathbf{P}_{gf}(K)\cap K_\Sigma$ then, by Theorem 1, $\mathscr{A}$ is in $\mathbf{HSPP}_{\alpha_0 f}(K)\cap K_\Sigma$. Therefore, there is a $\mathscr{C}=(C, \Sigma)\in\mathbf{PP}_{\alpha_0 f}(K)\cap K_\Sigma$ such that a subalgebra $\mathscr{C}'=(C', \Sigma)$ of $\mathscr{C}$ can be mapped homomorphically onto $\mathscr{A}$ under a homomorphism ψ. Let us write $\mathscr{C}$ in the form $\mathscr{C}=\prod(\mathscr{A}_i|i\in I)$ $(\mathscr{A}_i=(A_i, \Sigma)\in\mathbf{P}_{\alpha_0 f}(K), i\in I)$, and for every j $(=1, \ldots, n)$ take a $c_j\in\psi^{-1}(a_j)$. Set $\mathbf{c}=(c_1, \ldots, c_n)$. Denote by J a minimal subset of I such that for arbitrary $c, c' \in C_{\mathbf{c}}^{(m)}$ there is a $j \in J$ with $\mathrm{pr}_j(c) \neq \mathrm{pr}_j(c')$. Let $\mathcal{B} = (B, \Sigma) = \prod(\mathcal{A}_j | j \in J)$, and define $b_i \in B$ by $\mathrm{pr}_j(b_i) = \mathrm{pr}_j(c_i)$ $(j \in J, i = 1, \ldots, n)$. Moreover, set $\mathbf{b} = (b_1, \ldots, b_n)$. Then $\mathcal{B} \in \mathbf{P}_{\alpha_0 f}(K) \cap K_\Sigma$ and $(\mathcal{B}, \mathbf{b})$ is m-isomorphic to $(\mathcal{C}, \mathbf{c})$. Therefore $(\mathcal{B}, \mathbf{b})$ can be mapped m-homomorphically onto $(\mathcal{A}, \mathbf{a})$.

2. Root-to-frontier tree automata

First of all we fix a finite rank type R such that $0 \notin R$. Moreover, $F_\Sigma(X_n \cup \xi)$ will denote the set of all trees from $F_\Sigma(X_n \cup \xi)$ whose frontiers contain the auxiliary variable ξ exactly once.

Let us define the path, path (p), leading from the root of a tree $p \in F_\Sigma(X_n \cup \xi)$ to the leaf ξ in the following way.

(i) If $p = \xi$ then path $(p) = \xi$.

(ii) If $p = \sigma(p_1, \ldots, p_m)$ $(\sigma \in \Sigma_m, m \in R)$ and $p_j \in F_\Sigma(X_n \cup \xi)$ then path $(p) = (\sigma, j)(\text{path}(p_j))$.

Next we recall some concepts concerning ascending algebras which are not so well known (cf. [3]).

Let Σ be an operator domain with $\Sigma_0 = \emptyset$. A (deterministic) *ascending Σ-algebra* $\mathcal{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an m-ary ascending operation $\sigma^{\mathcal{A}}: A \to A^m$, where m is the arity of σ. The mapping $\sigma \to \sigma^{\mathcal{A}}$ will not be mentioned explicitly, but we write $\mathcal{A} = (A, \Sigma)$. If Σ is not specified then we speak about an ascending algebra. $\mathcal{A}$ is *finite* if A is finite and Σ is a ranked alphabet. Moreover, $\mathcal{A}$ has rank type R if Σ is of rank type R.

Take two ascending Σ-algebras $\mathcal{A} = (A, \Sigma)$ and $\mathcal{B} = (B, \Sigma)$. $\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$ if

(i) $B \subseteq A$, and

(ii) $\sigma^{\mathcal{B}}(b) = \sigma^{\mathcal{A}}(b)$ for arbitrary $\sigma \in \Sigma$ and $b \in B$.

Again consider the ascending algebras $\mathcal{A}$ and $\mathcal{B}$ above. Moreover, let $\psi: A \to B$ be a mapping. ψ is a *homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ if the equality

$$\sigma^{\mathcal{B}}(\psi(a)) = (\psi(a_1), \ldots, \psi(a_m))$$

holds for arbitrary $\sigma \in \Sigma$ and $a \in A$, where $(a_1, \ldots, a_m) = \sigma^{\mathcal{A}}(a)$. If there is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$ then $\mathcal{B}$ is a *homomorphic image* of $\mathcal{A}$.

Next we define the concept of the product of ascending algebras.

Let Σ and Σ^i $(i \in I)$ be ranked alphabets of rank type R, and consider the ascending Σ^i-algebras $\mathcal{A}_i = (A_i, \Sigma^i)$ $(i \in I)$. Furthermore, let

$$\varphi = \{\varphi^m: \prod(A_i | i \in I) \times \Sigma_m \to \prod(\Sigma_m^i | i \in I) | m \in R\}$$

be a family of mappings. Then by the *general product* of $\mathcal{A}_i$ $(i\in I)$ with respect to Σ and φ we mean the ascending Σ-algebra

$$\mathcal{A}=(A,\Sigma)=\prod(\mathcal{A}_i|i\in I)[\Sigma,\varphi]$$

with $A=\prod(A_i|i\in I)$ and for arbitrary $m\in R$, $\sigma\in\Sigma_m$ and $a\in A$

$$\big(\mathrm{pr}_i(\sigma^{\mathcal{A}}(a)_1),\ldots,\mathrm{pr}_i(\sigma^{\mathcal{A}}(a)_m\big)=\sigma_i^{\mathcal{A}_i}(\mathrm{pr}_i(a))\quad(i\in I)$$

where $\sigma^{\mathcal{A}}(a)_j$ is the j^{th} component of $\sigma^{\mathcal{A}}(a)$ and $\sigma_i=\mathrm{pr}_i(\varphi^m(a,\sigma))$ $(i\in I)$. In the sequel we shall assume that I is given together with a linear ordering $\leqq$. (If we have more than one index set then the same notations $\leqq$ will be used for the linear ordering of each of them. This will not cause any confusion.)

To define the concept of the α_0-product of ascending algebras let us introduce the notation $\varphi_i(a,\sigma)=\mathrm{pr}_i(\varphi^m(a,\sigma))$ for arbitrary $i\in I$, $a\in A$ and $\sigma\in\Sigma_m$. We say that the product $\mathcal{A}$ above is an *α_0-product* if for arbitrary $i\in I$, φ_i is independent of its j^{th} component $(j\in I)$ whenever $i\leqq j$.

In this section the symbols $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_{gf}$, $\mathbf{P}_{\alpha_0}$ and $\mathbf{P}_{\alpha_0 f}$ introduced in Section 1 will be used in their original sense and they also denote the corresponding operators for ascending algebras. This double use will not cause any difficulties since their concrete meaning will be clear from the context.

We say that (regarding ascending algebras) the α_0-product is *metrically equivalent* to the product if for arbitrary class K of finite ascending algebras with rank type R, integer $m\geqq 0$, uniform deterministic root-to-frontier transducer $\mathfrak{A}=(\Sigma,X_u,A,\Omega,Y_v,a_0,P)\in\mathrm{tr}\,(\mathbf{A})$ with $\mathbf{A}=(\mathcal{A},a_0,X_u,\mathbf{a})$ and $\mathcal{A}\in\mathbf{P}_{gf}(K)$ there are a $\mathcal{B}=(B,\Sigma)\in\mathbf{P}_{\alpha_0 f}(K)$, $\mathbf{B}=(\mathcal{B},b_0,X_u,\mathbf{b})$ $(b_0\in B,\ \mathbf{b}\in P(B)^u)$ and $\mathfrak{B}=(\Sigma,X_u,B,\Omega,Y_v,b_0,P')\in\mathrm{tr}\,(\mathbf{B})$ such that $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$.

We introduce some more terminology.

For every operator domain Σ (of rank type R), $\bar\Sigma$ will denote the operator domain $\{(\sigma,k)|\sigma\in\Sigma_m,\ 1\leqq k\leqq m,\ m\in R\}$ of unary operators.

Take a Σ-algebra $\mathcal{A}=(A,\Sigma)$ of rank type R. Correspond to $\mathcal{A}$ the $\bar\Sigma$ algebra $s(\mathcal{A})=(A,\bar\Sigma)$ given by $(\sigma,k)^{s(\mathcal{A})}(a)=\mathrm{pr}_k(\sigma^{\mathcal{A}}(a))$ $(\sigma\in\Sigma_m,\ 1\leqq k\leqq m,\ a\in A)$.

Obviously, s is a one-to-one mapping of K_Σ onto $K_{\bar\Sigma}$, where K_Σ is the class of all ascending Σ-algebras. Moreover, we have

Statement 1. *For arbitrary operator domain Σ of rank type R and Σ-algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{B}_i$ $(i\in I)$ we have*

(i) *$\mathcal{A}=\prod(\mathcal{A}_i|i\in I)$ if and only if $s(\mathcal{A})=\prod(s(\mathcal{A}_i)|i\in I)$,*

(ii) *$\mathcal{B}$ is a subalgebra of $\mathcal{A}$ if and only if $s(\mathcal{B})$ is a subalgebra of $s(\mathcal{A})$,*

(iii) *$\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ if and only if $s(\mathcal{B})$ is a homomorphic image of $s(\mathcal{A})$.*

Next we define a restricted form of products for the above $\bar{\Sigma}$-algebras. Take a family $\mathscr{A}_i=(\mathscr{A}_i, \bar{\Sigma}^i)$ $(i\in I)$ of $\bar{\Sigma}^i$-algebras, where every Σ^i $(i\in I)$ is an operator domain of rank type R. Moreover, let Σ be an operator domain with rank type R. Then a general product (α_0-product)

$$\mathscr{A}=(A, \bar{\Sigma})=\prod(\mathscr{A}_i|i\in I)[\bar{\Sigma}, \varphi]$$

is a *restricted product (restricted α_0-product)* if for arbitrary $i\in I$, $a\in A$ and (σ, k), $(\sigma, l)\in\bar{\Sigma}$, $\varphi_i(a, (\sigma, k))=(\sigma_i, k)$ and $\varphi_i(a, (\sigma, l))=(\sigma_i, l)$ $(\sigma_i\in\Sigma^i)$.

The following result is also obvious.

Statement 2. *The formations of the restricted product and the restricted α_0-product are transitive. Moreover, the direct product preserves both the restricted product and the restricted α_0-product.*

For arbitrary $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$ denote $\bar{\mathbf{Q}}$ the restricted version of $\mathbf{Q}$. Moreover, $\mathbf{Q}_f$ will stand for $\mathbf{P}_{gf}$ if $\mathbf{Q}=\mathbf{P}_g$, and $\mathbf{Q}_f=\mathbf{P}_{\alpha_0 f}$ if $\mathbf{Q}=\mathbf{P}_{\alpha_0}$. We use the notation $\bar{\mathbf{Q}}_f$ for the restricted form of $\mathbf{Q}_f$. Take a set K of ascending algebras with rank type R. Then $\bar{K}$ is defined by $\bar{K}=\{s(\mathscr{A})|\mathscr{A}\in K\}$.

Statement 3. *For arbitrary class K of ascending algebras with rank type R, algebra $\mathscr{A}$ of rank type R and $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$ the following conditions hold.*

(i) $\mathscr{A}\in\mathbf{Q}(K)$ *if and only if* $s(\mathscr{A})\in\bar{\mathbf{Q}}(\bar{K})$.

(ii) $\mathscr{A}\in\mathbf{Q}_f(K)$ *if and only if* $s(\mathscr{A})\in\bar{\mathbf{Q}}_f(\bar{K})$.

Next we prove

Lemma 1. *Let K be a class of ascending algebras with rank type R, and take a $\mathbf{Q}\in\{\mathbf{P}_g, \mathbf{P}_{\alpha_0}\}$. Then* $\mathbf{HSQ}(K)=\mathbf{HSPQ}_f(K)$.

Proof. The inclusion $\mathbf{HSPQ}_f(K)\subseteq\mathbf{HSQ}(K)$ is obvious.

Let us show the converse inclusion. By Statements 1 and 3, $s(\mathbf{HSQ}(K))=$ $=\mathbf{HS}\bar{\mathbf{Q}}(\bar{K})$ and $s(\mathbf{HSPQ}_f(K))=\mathbf{HSP}\bar{\mathbf{Q}}_f(\bar{K})$, where s is extended to classes of ascending algebras in an obvious way. We show that $\mathbf{HS}\bar{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}\subseteq\mathbf{HSP}\bar{\mathbf{Q}}_f(\bar{K})\cap K_{\bar{\Sigma}}$ for every operator domain Σ of rank type R. This will imply $\mathbf{HSQ}(K)=$ $=s^{-1}(\mathbf{HS}\bar{\mathbf{Q}}(\bar{K}))\subseteq s^{-1}(\mathbf{HSP}\bar{\mathbf{Q}}_f(\bar{K}))=\mathbf{HSPQ}_f(K)$.

By Statement 2, $\mathbf{HS}\bar{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}$ is an equational class. Assume that an equation $p(x)=q(x)$ $(p, q\in F_{\bar{\Sigma}}(x))$ does not hold in $\mathbf{HS}\bar{\mathbf{Q}}(\bar{K})\cap K_{\bar{\Sigma}}$. Let us write p and q in a more detailed form $p=\sigma_k(\dots(\sigma_1(x))\dots)$, $q=\omega_l(\dots(\omega_1(x))\dots)$ $(\sigma_i, \omega_j\in\bar{\Sigma}$, $i=1, \dots, k$, $j=1, \dots, l)$, and assume that $l\leqq k$. Moreover, set $p_i=\sigma_i(\dots(\sigma_1(x))\dots)$ $(i=0, \dots, k)$ and $q_i=\omega_i(\dots(\omega_1(x))\dots)$ $(i=1, \dots, l)$, where $p_0=q_0=x$. There are an

$$\mathscr{A}=(A, \bar{\Sigma})=\prod(\mathscr{A}_i|i\in I)[\bar{\Sigma}, \varphi]\in\bar{\mathbf{Q}}(\bar{K})$$

and an $a_0 \in A$ such that $p(a_0) \neq q(a_0)$. (Here and in the sequel the above notation means that $\mathscr{A}$ is formed by the product represented by $\overline{\mathbf{Q}}$ and every $\mathscr{A}_i$ $(i \in I)$ is in $\overline{K}$.) Set $A' = \{a_0 p_i | i=1, \dots, k\} \cup \{a_0 q_i | i=1, \dots, l\}$. Denote by I_1 a minimal subset of I such that for arbitrary two distinct elements $a, b \in A'$ there is an $i \in I_1$ with $\mathrm{pr}_i(a) \neq \mathrm{pr}_i(b)$. Moreover, let I_{j+1} $(1 \leqq j \leqq k)$ be a minimal extension of I_j under which for arbitrary $i \in I_j$, $a, b \in A'$ and $\sigma \in \Sigma$ if $\varphi_i(a, \sigma) \neq \varphi_i(b, \sigma)$ then there is a $t \in I_{j+1}$ such that φ_i depends on its tth component and $\mathrm{pr}_t(a) \neq \mathrm{pr}_t(b)$. (We write φ_i for φ_i^1.) Set $J = I_{k+1}$ and restrict the ordering of I to J. Obviously, J is finite. Take the product

$$\bar{\mathscr{A}} = (\bar{A}, \bar{\Sigma}) = \prod (\mathscr{A}_j | j \in J)[\bar{\Sigma}, \varphi']$$

where φ' is given as follows. For arbitrary $\bar{a} \in \bar{A}$, $1 \leqq j \leqq k$, $i \in I_j$ and $\sigma \in \bar{\Sigma}$, $\varphi'_i(\bar{a}, \sigma) = \varphi_i(a, \sigma)$ if there is an $a \in A'$ such that $\mathrm{pr}_{I_{j+1}}(\bar{a}) = \mathrm{pr}_{I_{j+1}}(a)$. (Here and in the sequel if $I' \subseteq I$ and $a_1, a_2 \in \prod (A_i | i \in I)$ then $\mathrm{pr}_{I'}(a_1) = \mathrm{pr}_{I'}(a_2)$ means that $\mathrm{pr}_i(a_1) = \mathrm{pr}_i(a_2)$ for every $i \in I'$.) In all other cases φ' is given arbitrarily in accordance with the definition of the product represented by $\overline{\mathbf{Q}}$. φ' is obviously well defined. It is also clear that $\bar{\mathscr{A}} \in \overline{\mathbf{Q}}_f(\overline{K})$.

For every $m = 1, \dots, k+1$ introduce the relation $\bar{a} \sim_m a$ $(\bar{a} \in \bar{A}, a \in A')$ if and only if $\mathrm{pr}_{I_m}(\bar{a}) = \mathrm{pr}_{I_m}(a)$, and let $\bar{a}_0 \in \bar{A}$ satisfy $\bar{a}_0 \sim_{k+1} a_0$. Then $p_i(\bar{a}_0) \sim_{k+1-i} p_i(a_0)$ and $q_j(\bar{a}_0) \sim_{k+1-j} q_j(a_0)$ for arbitrary i $(=0, \dots, k)$ and j $(=0, \dots, l)$. In particular, $p(\bar{a}_0) \sim_1 p(a_0)$ and $q(\bar{a}_0) \sim_1 q(a_0)$. Therefore $p(\bar{a}_0) \neq q(\bar{a}_0)$, that is $p(x) = q(x)$ does not hold in $\mathbf{HSP}\overline{\mathbf{Q}}_f(\overline{K}) \cap K_{\bar{\Sigma}}$.

The case when an equation $p(x) = q(y)$ $\big(p \in F_{\bar{\Sigma}}(x),\ q \in F_{\bar{\Sigma}}(y)\big)$ is not valid in $\mathbf{HS}\overline{\mathbf{Q}}(\overline{K}) \cap K_{\bar{\Sigma}}$ can be treated similarly.

Lemma 2. *For arbitrary class K of ascending algebras with rank type R the equality* $\mathbf{HSP}_g(K) = \mathbf{HSP}_{\alpha_0}(K)$ *holds.*

Proof. The inclusion $\mathbf{HSP}_{\alpha_0}(K) \subseteq \mathbf{HSP}_g(K)$ is obviously valid.

To prove $\mathbf{HSP}_g(K) \subseteq \mathbf{HSP}_{\alpha_0}(K)$, by Statements 1 and 3 it is enough to show $\mathbf{HS}\overline{\mathbf{P}}_g(\overline{K}) \subseteq \mathbf{HS}\overline{\mathbf{P}}_{\alpha_0}(\overline{K})$. Take an operator domain Σ of rank type R, and consider an equation $p(x) = q(x)$ $(p, q \in F_{\bar{\Sigma}}(x), h(q) \leqq h(p) = k)$ which does not hold in $\mathbf{HS}\overline{\mathbf{P}}_g(\overline{K}) \cap K_{\bar{\Sigma}}$. Then there are an

$$\mathscr{A} = (A, \Sigma) = \prod (\mathscr{A}_i = (A_i, \Sigma^i) | i \in I)[\Sigma, \psi] \in \overline{\mathbf{P}}_g(\overline{K})$$

and an $a \in A$ such that $p(a) \neq q(a)$. Take $J = \{1, \dots\ k+1\}$ with the natural ordering and order $J \times I$ in the following way: for arbitrary two $(j_1, i_1), (j_2, i_2) \in J \times I$, $(j_1, i_1) \leqq (j_2, i_2)$ if and only if $j_1 < j_2$, or $j_1 = j_2$ and $i_1 \leqq i_2$. Consider the restricted α_0-product

$$\mathscr{B} = (B, \bar{\Sigma}) = \prod (\mathscr{A}_{(j,i)} | (j, i) \in J \times I)[\bar{\Sigma}, \varphi'],$$

where $\mathscr{A}_{(j,i)}=\mathscr{A}_i$ $((j,i)\in J\times I)$, and for arbitrary $b\in B$ and $\sigma\in\bar{\Sigma}$, $\varphi_{(1,i)}(b,\sigma)$ $(i\in I)$ is arbitrary, and $\varphi_{(j,i)}(b,\sigma)=\varphi_i(b_{j-1},\sigma)$ $(1<j\leqq k+1, i\in I)$, where $b_t\in\prod(A_i|i\in I)$ is given by the equality $\mathrm{pr}_{(t,i)}(b)=\mathrm{pr}_i(b_t)$ $(t=1,\dots,k+1, i\in I)$. Introduce the notation $b=(b_1,\dots,b_{k+1})$ where $b_1,\dots,b_{k+1}$ are defined by the previous equalities. Taking $b=(a,\dots,a)$ and an $r\in F_{\bar{\Sigma}}(x)$ with $h(r)\leqq k$, one can show easily by induction on $h(r)$, that the equality

$$r^{\mathscr{B}}(b)=\big(c_1,\dots,c_t,r^{\mathscr{A}}(a),\dots,r^{\mathscr{A}}(a)\big)$$

holds, where $t=h(r)$ and $c_1,\dots,c_t\in\prod(A_i|i\in I)$. Especially, $p^{\mathscr{B}}(b)=\big(c_1,\dots,\dots,c_k,p^{\mathscr{A}}(a)\big)$ and $q^{\mathscr{B}}(b)=\big(c_1',\dots,c_k',q^{\mathscr{A}}(a)\big)$ $\big(c_1,\dots,c_k,c_1',\dots,c_k'\in\prod(A_i|i\in I)\big)$. Therefore, $p^{\mathscr{B}}(b)\neq q^{\mathscr{B}}(b)$, that is $p(x)=q(x)$ does not hold in $\mathbf{HS\bar{P}}_{\alpha_0}(\bar{K})\cap K_{\bar{\Sigma}}$.

The case when an equation of form $p(x)=q(y)$ $\big(p(x)\in F_{\bar{\Sigma}}(x),\ q(y)\in F_{\bar{\Sigma}}(y)\big)$ is not valid in $\mathbf{HS\bar{P}}_g(\bar{K})\cap K_{\bar{\Sigma}}$ can be treated similarly. Thus we got that $\mathbf{HS\bar{P}}_g(\bar{K})\cap K_{\bar{\Sigma}}\subseteq\mathbf{HS\bar{P}}_{\alpha_0}(\bar{K})\cap K_{\bar{\Sigma}}$, which implies the inclusion $\mathbf{HSP}_g(K)\cap K_\Sigma\subseteq\subseteq\mathbf{HSP}_{\alpha_0}(K)\cap K_\Sigma$. This ends the proof of Lemma 2.

For arbitrary class K of ascending algebras with rank type R let $\mathbf{1}(K)$ denote the subclass consisting of all ascending algebras from K generated by single elements. The members of $\mathbf{1}(K)$ will be written as systems $(\mathscr{A},a)$ where $\mathscr{A}\in K$ and a is a generating element of $\mathscr{A}$.

Theorem 1. *The general product is metrically equivalent to the α_0-product if and only if for arbitrary class K of finite ascending algebras with rank type R the equality*

$$(*)\qquad \mathbf{1HSP}_g(K)=\mathbf{1HSP}_{\alpha_0}(K)$$

holds.

Proof. Assume that $(*)$ is valid. Take a system $(\mathscr{A},a)\in\mathbf{1SP}_{gf}(K)$ with $\mathscr{A}=(A,\Sigma)$. By $(*)$ and Lemma 1, $(\mathscr{A},a)\in\mathbf{1HSPP}_{\alpha_0 f}(K)$. Then there are a $\mathscr{B}=(B,\Sigma)\in\mathbf{PP}_{\alpha_0 f}(K)$ and a $b\in B$ such that for the subalgebra $\mathscr{B}'=(B',\Sigma)$ of $\mathscr{B}$ generated by b the system $(\mathscr{B}',b)$ can be mapped homomorphically onto $(\mathscr{A},a)$. (This terminology means that b is mapped into a under the given homomorphism of $\mathscr{B}'$ onto $\mathscr{A}$.) Let us write $\mathscr{B}$ in the form

$$\mathscr{B}=\prod(\mathscr{B}_i|\in I)\ \ (\mathscr{B}_i=(\mathscr{B}_i,\Sigma^i)\in\mathbf{P}_{\alpha_0 f}(K),\quad i\in I).$$

Take an integer $m\geqq 0$, and consider $B_b^{(m)}$. Denote by J a minimal subset of I such that for arbitrary two distinct elements $b_1,b_2\in B_b^{(m)}$ there is a $j\in J$ with $\mathrm{pr}_j(b_1)\neq\neq\mathrm{pr}_j(b_2)$. Obviously, J is finite. Define $\bar{b}\in\prod(B_j|j\in J)$ by $\mathrm{pr}_J(\bar{b})=\mathrm{pr}_J(b)$. Let $\bar{\mathscr{B}}=(\bar{B},\Sigma)$ be the ascending subalgebra of $\prod(\mathscr{B}_j|j\in J)$ generated by $\bar{b}$. Then $(\bar{\mathscr{B}},\bar{b})\in\mathbf{1SP}_{\alpha_0 f}(K)$ and it is m-isomorphic to $(\mathscr{B}',b)$. Thus $(\bar{\mathscr{B}},\bar{b})$ can be mapped m-homomorphically onto $(\mathscr{A},a)$. This ends the proof of the sufficiency.

In order to prove the necessity assume that the α_0-product is metrically equivalent to the product. Take a class K of finite ascending algebras with rank type R. Set $L=\mathbf{HSP}_g(K)$ and $\bar{L}=\mathbf{HSP}_{\alpha_0}(K)$. We show that $(*)$ holds, i.e., $\mathbf{1}L=\mathbf{1}\bar{L}$. To this, by Statements 1 and 3 it is enough to prove that for arbitrary operator domain Σ of rank type R if an equation $\bar{p}(x)=\bar{q}(x)$ $(\bar{p}, \bar{q}\in F_{\bar{\Sigma}}(x))$ does not hold in $s(L)\cap K_{\bar{\Sigma}}$ then it is not valid in $s(\bar{L})\cap K_{\bar{\Sigma}}$ since this implies that the free algebras in the equational classes $s(L)\cap K_{\bar{\Sigma}}$ and $s(\bar{L})\cap K_{\bar{\Sigma}}$ generated by single elements are isomorphic.

Thus assume that $\bar{p}(x)=\bar{q}(x)$ $(\bar{p}, \bar{q}\in F_{\bar{\Sigma}}(x))$ does not hold in $s(L)\cap K_{\bar{\Sigma}}$. Then, by Lemma 1, there is an $(\bar{\mathscr{A}}, a_0)\in \mathbf{1S\bar{P}}_{gf}(\bar{K})$ $(\bar{\mathscr{A}}=(A, \bar{\Sigma}), a_0\in A)$ such that $\bar{p}(a_0)\neq\bar{q}(a_0)$. Take $\mathscr{A}=(A, \Sigma)$ with $s(\mathscr{A})=\bar{\mathscr{A}}$. Then $(\mathscr{A}, a_0)\in\mathbf{1SP}_{gf}(K)$. Consider the transducer $\mathfrak{A}=(\Sigma, X_n, A, \Omega, A\times X_n, a_0, P)$ where $n>1$ is an arbitrary natural number, $\Omega_l=A\times\Sigma_l$ $(l\in R)$ and P consists of the following productions:

(1) $$ax_i\to(a, x_i)\quad(a\in A,\ x_i\in X_n),$$

(2) $$a\sigma\to(a, \sigma)(a_1\xi_1, \dots, a_l\xi_l)\quad\big(a\in A, \sigma\in\Sigma_l, l\in R,\quad \sigma^{\mathscr{A}}(a)=(a_1, \dots, a_l)\big).$$

Take two trees $p, q\in F_\Sigma(X_n\cup\xi)$ such that $\bar{p}=\text{path}\,(p)$ and $\bar{q}=\text{path}\,(q)$. Let $m\geq h(p), h(q)$. Then, by our assumptions, there is a $(\mathscr{B}, b_0)\in\mathbf{1SP}_{\alpha_0 f}(K)$ $(\mathscr{B}=(B, \Sigma), b_0\in B)$ such that for a $\mathfrak{B}=(\Sigma, X_n, B, \Omega, A\times X_n, b_0, P')\in\text{tr}\,(\mathbf{B})$ $\big(\mathbf{B}=(\mathscr{B}, B_0; X_n, \mathbf{b})\big)$ we have $\tau_{\mathfrak{A}}\overset{m}{=}\tau_{\mathfrak{B}}$. One can easily show by induction on the height of a tree that for every $r\in F_\Sigma(X_n\cup\xi)$ with $h(r)\leq m$ and $\text{path}\,(r)=\bar{r}$ the derivations

$$a_0r\Rightarrow^*_{\mathfrak{A}}r'(a\xi)\quad\text{and}\quad b_0r\Rightarrow^*_{\mathfrak{B}}r''(b\xi)$$

hold, where $r', r''\in F_\Omega(A\times X_n\cup\xi)$, $a=\bar{r}^{\bar{\mathscr{A}}}(a_0)$, $b=\bar{r}^{\bar{\mathscr{B}}}(b_0)$ $\big(\bar{\mathscr{B}}=s(\mathscr{B})\big)$ and $\text{path}\,(r'')$ is a subword of $\text{path}\,(r')$. In particular,

$$a_0p\Rightarrow^*_{\mathfrak{A}}p'(a_1\xi),\quad b_0p\Rightarrow^*_{\mathfrak{B}}p''(b_1\xi),$$

and

$$a_0q\Rightarrow^*_{\mathfrak{A}}q'(a_2\xi),\quad b_0q\Rightarrow^*_{\mathfrak{B}}q''(b_2\xi)$$

where $p', p'', q', q''\in F_\Omega(A\times X_n\cup\xi)$, $a_1=\bar{p}^{\bar{\mathscr{A}}}(a_0)$, $b_1=\bar{p}^{\bar{\mathscr{B}}}(b_0)$, $a_2=\bar{q}^{\bar{\mathscr{A}}}(a_0)$ and $b_2=\bar{q}^{\bar{\mathscr{B}}}(b_0)$. By our assumptions, $a_1\neq a_2$. Assume that $b_1=b_2$. Take the trees $p(x_1)$ and $q(x_1)$. Then

$$a_0p(x_1)\Rightarrow^*_{\mathfrak{A}}p'\big((a_1, x_1)\big),\quad a_0q(x_1)\Rightarrow^*_{\mathfrak{A}}q'\big((a_2, x_1)\big)$$

and

$$b_0p(x_1)\Rightarrow^*_{\mathfrak{B}}p''(s),\quad b_0q(x_1)\Rightarrow^*_{\mathfrak{B}}q''(s)$$

where s is the right side of the rule $b_1x_1\to s$ in P'. Therefore, at least one of the equalities $p'\big((a_1, x_1)\big)=p''(s)$ and $q'\big((a_2, x_1)\big)=q''(s)$ does not hold contradicting the choice of $\mathfrak{B}$. Thus we got that $\bar{p}^{\bar{\mathscr{B}}}(b_0)\neq\bar{q}^{\bar{\mathscr{B}}}(b_0)$, that is the equality $\bar{p}(x)=\bar{q}(x)$ is not valid in $s(\bar{L})\cap K_{\bar{\Sigma}}$, which ends the proof of Theorem 1.

From Theorem 1, by Lemma 2, we obtain

Theorem 2. *Regarding ascending algebras the α_0-product is metrically equivalent to the general product.*

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