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Stability properties of the equilibrium under the influence of unbounded damping

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

It is a well-known phenomenon that damping can make mechanical equilibria asymptotically stable. However, as is to be expected, in the presence of too large damping the system can remain far from the equilibrium position. For example, the equation $\ddot{x}+(2+e^t)\dot{x}+x=0$ due to LASALLE [7] admits nonvanishing solutions $x=a(1+e^{-t})$ (a=const.). Considering the second order nonlinear differential equation

$$\ddot{x}+f(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x}+g(x) = 0,$$

BALLIEU and PEIFFER [1] investigated which conditions on f assure attractivity or nonattractivity of the origin. They have proved that if $f(\cdot, x, \dot{x})$ is "not too large" then the equilibrium is attractive, and if it is "large enough" then the equilibrium is not attaractive. Now the following question arises: what happens in the second case? Experience suggests (see also LaSalle's example) that the deviation x tends to a finite limit (possibly different from zero) and the velocity $\dot{x}$ tends to zero as $t\to\infty$. In other words, the point asymptotically stops (possibly far from the original equilibrium position).

In this paper we study the conditions of the asymptotic stop by Lyapunov's direct method and differential inequalities. In [12] the second author gave conditions assuring x-stability of the equilibrium state and the convergence of the deviation $x(t)$ as $t\to\infty$. Recently [4] the first author got conditions for the convergence to zero of the velocities in a mechanical system. Here it will be pointed out that the two methods can be combined to get conditions for the asymptotic stop.

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After some preliminaries (Section 2) we present a theorem for general differential systems which guarantees the stability of the zero solution with respect to a part of the variables, the convergence of this part to a finite limit and the convergence to zero of the further variables along the solutions as $t\to\infty$ (Section 3). In the final two sections we apply this result to establish stability properties of equilibria of dissipative mechanical systems and of the zero solution of nonlinear second order differential equations. The paper is concluded by the example of the mathematical plain pendulum with changing length.

2. Preliminaries

Consider the system of differential equations

$$\dot{x} = X(t, x), \tag{2.1}$$

where $t\in R_+:=[0, \infty)$ and $x\in R^k$ with a norm $|x|$. Let a partition $x=(y, z)$ ($y\in R^m$, $z\in R^n$; $1\leqq m\leqq k$, $n=k-m$) be given. Assume that the function X is defined and continuous on the set $\Gamma:=R_+\times R^m\times D$, where $D\subset R^n$ is open and contains the origin, and $X(t, 0)\equiv 0$, i.e. $x=0$ is a solution of (2.1). We denote by $x(t)=x(t;\, t_0, x_0)$ any solution with $x(t_0)=x_0$. We always assume the solutions to be *y-continuable* which means that if $x(t)=\big(y(t), z(t)\big)$ is a solution of (2.1) and $|z(t)|$ is bounded in $[t_0, T)$, then $x(t)$ can be continued to the closed interval $[t_0, T]$.

The zero solution of (2.1) is said to be:

z-stable if for every $\varepsilon>0$, $t_0\in R_+$ there exists a $\delta(\varepsilon, t_0)>0$ such that $|x_0|<\delta(\varepsilon, t_0)$ implies $|z(t;\, t_0, x_0)|<\varepsilon$ for $t\geqq t_0$;

asymptotically z-stable if it is z-stable and, in addition, for every $t_0\in R_+$ there exists a $\sigma(t_0)>0$ such that $|x_0|<\sigma(t_0)$ implies $|z(t;\, t_0, x_0)|\to 0$ as $t\to\infty$.

Instead of x-stability we will say simply stability.

With a continuously differentiable function $V\colon \Gamma\to R$ we associate a function $\dot{V}\colon \Gamma\to R$ by the definition

$$\dot{V}(t, x) := \frac{\partial V(t, x)}{\partial x} X(t, x) + \frac{\partial V(t, x)}{\partial t}$$

which is called the *derivative of V with respect to* (2.1). Here as well as in the sequel, for two vectors $a, b\in R^k$ by ab we denote the scalar product of a and b.

Denote by L^γ the class of the Lebesgue measurable functions $f\colon R_+\to R$ with

$$\|f\|_\gamma := \begin{cases} \Big[\int_0^\infty |f|^\gamma\Big]^{1/\gamma} < \infty & (0<\gamma<\infty) \\ \operatorname*{sup\,ess}_{s\in R_+} |f(s)| & (\gamma=\infty). \end{cases}$$

For a continuously differentiable function $f\colon R_+\to R$, by the *function of the positive* and *negative variation* of the function f we mean

$$\int_0^t [\dot f(s)]_+ ds, \quad \int_0^t [\dot f(s)]_- ds,$$

respectively, where

$$[a]_+ := \max\{a, 0\} \quad [a]_- := \max\{-a, 0\} \quad (a\in R).$$

One of the basic notion of the main theorem will be the integral positivity. A continuous function $f\colon R_+\to R_+$ is called *integrally positive* if $\int_I f=\infty$ whenever $I=\bigcup_{i=1}^{\infty}[a_i, b_i]$, and $a_i<b_i<a_{i+1}$, $b_i-a_i\geqq\delta>0$ hold for all $i=1, 2, \ldots$ with some positive constant δ.

In the proofs of the theorem we will need the following

Lemma 2.1. *If the functions $f\colon R_+\to R_+$, $g\colon R_+\to(0, \infty)$ are continuous, f is integrally positive, and there exists an α $(0<\alpha\leqq\infty)$ such that*

$$\frac{f^{1+1/\alpha}}{g}\in L^\alpha,$$

then g is integrally positive.

Proof. Suppose that the statement is not true, i.e. there exists a sequence of intervals $[a_i, b_i]$ possessing the properties in the definition of the integral positivity, and such that

$$\sum_{i=1}^{\infty}\int_{a_i}^{b_i} g<\infty. \tag{2.2}$$

Suppose that $\alpha<\infty$ and introduce the notations $p:=1+1/\alpha$, $q:=\alpha+1$. By Hölder's inequality we get the estimate

$$\int_{a_i}^{b_i} f=\int_{a_i}^{b_i} g^{1/p}\frac{f}{g^{1/p}}\leqq\Big[\int_{a_i}^{b_i} g\Big]^{1/p}\Big[\int_{a_i}^{b_i}\frac{f^q}{g^{q/p}}\Big]^{1/q} \tag{2.3}$$

for all $i=1, 2, \ldots$. For every fixed natural number N the application of the Cauchy inequality yields

$$\sum_{i=1}^{N}\int_{a_i}^{b_i} f\leqq\Big[\sum_{i=1}^{N}\int_{a_i}^{b_i} g\Big]^{1/p}\Big[\sum_{i=1}^{N}\int_{a_i}^{b_i}\frac{f^{\alpha+1}}{g^\alpha}\Big]^{1/q}.$$

In consequence of (2.2) we have

$$\sum_{i=1}^{\infty}\int_{a_i}^{b_i} f<\infty,$$

in contradiction to the fact that f is integrally positive.

In the case of $\alpha=\infty$, instead of (2.3) we start from the estimate

$$\int_{a_i}^{b_i} f = \int_{a_i}^{b_i} g\frac{f}{g} \leqq \left\|\frac{f}{g}\right\|_\infty \int_{a_i}^{b_i} g,$$

which leads to a contradiction because of (2.2). The lemma is proved.

3. The main theorem

Consider the system of the differential equations

$$\dot{y} = Y(t, y, z), \quad \dot{z} = Z(t, y, z), \tag{3.1}$$

where $y\in R^m$, $z\in R^n$, $t\in R_+$, and right-hand sides Y, Z satisfy the assumptions in Section 2.

Theorem 3.1. *Suppose that there are continuous functions $V_1, V_2\colon \Gamma\to R$; $c, r\colon R_+^2\to R_+$; $b\colon R_+^3\to R_+$; $a, \varphi, \psi, \chi\colon R_+\to R_+$ and real numbers α, β $(0\leqq\beta\leqq\alpha)$ satisfying the following conditions on the set Γ:*

(i) *functions V_1, V_2 are continuously differentiable and $V_1(t, x)\geqq 0$, $V(t, x):=$ $:=V_1(t, x)+V_2(t, x)\geqq 0$, $V(t, 0)\equiv 0$;*

(ii) *$\dot{V}(t, x)\leqq -\varphi(t)V_1^\alpha(t, x)+r(t, V(t, x))$, where φ is integrally positive; the function $r(t, \cdot)$ is nondecreasing for every $t\in R_+$ and the zero solution of the equation $\dot{u}=r(t, u)$ is stable;*

(iii)
$$|Z(t, x)| \leqq \psi(t)V_1^\beta(t, x)a(V(t, x))$$

and the function a is nondecreasing;

(iv)
$$\left[\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial y}Y\right]_+(t, y, z) \leqq b(t, |z|, V(t, y, z)),$$

so that for every $t\in R_+$ the function $b(t, \cdot, \cdot)$ is nondecreasing in its both variables, and for every $r_1, r_2>0$ the primitive $\int_0^t b(s, r_1, r_2)\,ds$ is uniformly continuous on R_+;

(v)
$$\left|\frac{\partial V_2(t, y, z)}{\partial z}\right| \leqq \chi(t)c(|z|, V(t, y, z)),$$

where c is nondecreasing in its both variables;

(vi) $\psi/\varphi^{\beta/\alpha}\in L^{\alpha/(\alpha-\beta)}$, *and the function*

$$\int_t^{t+1}\left(\frac{\chi^\alpha\psi^\alpha}{\varphi^\beta}\right)^{1/(\alpha-\beta)} \quad \textit{if} \quad \alpha>\beta$$

$$\frac{\chi\psi}{\varphi} \quad \textit{if} \quad \alpha=\beta$$

is bounded on R_+.

Then every solution $(y(t), z(t))$ *of* (3.1) *with sufficiently small initial values exists for large* t, $z(t)\to\text{const.}$, $V_1(t, y(t), z(t))\to 0$ *as* $t\to\infty$, *and the zero solution is z-stable.*

Proof. If the initial value u_0 is sufficiently small then the maximal solution of the initial value problem

$$\dot{u}=r(t,u) \quad u(t_0)=u_0\geqq 0$$

exists for all $t\geqq t_0$, and being nondecreasing it has a finite limit $u_\infty(t_0, u_0)$ as $t\to\infty$. Since the zero solution of the equation $\dot{u}=r(t,u)$ is stable, we have

$$\lim_{u_0\to 0} u_\infty(t_0, u_0)=0.$$

For any solution $x\colon [t_0, T)\to R^k$ $(t_0<T\leqq\infty)$ of (3.1) introduce the notations

$$v_1(t):=V_1(t, x(t)), \quad v_2(t):=V_2(t, x(t))$$

$$v(t)=v_1(t)+v_2(t), \quad w(t):=v(t)+u_\infty(t_0, u_0)-u(t; t_0, u_0),$$

where $u_0:=v(t_0)$.

In view of condition (ii) function v satisfies the estimate $\dot{v}(t)\leqq r(t, v(t))$, hence by the theory of differential inequalities ([6], Theorem 1.4.1) we have $v(t)\leqq$ $\leqq u(t; t_0, u_0)$ for all $t\in[t_0, T)$. Therefore,

$$\dot{w}(t)\leqq -\varphi(t)v_1^\alpha(t)\leqq 0 \quad (t_0<t<T), \tag{3.3}$$

which together with $w(t)\geqq 0$ implies

$$\int_{t_0}^{T}\varphi(t)v_1^\alpha(t)\,dt<\infty.$$

Besides, the function w and, consequently, v as well, is bounded also above.

In order to establish z-stability and the existence of the limit of $z(t)$ we estimate the variation of $z(t)$ over an interval $[A, B]$ making use of the Hölder inequality:

$$(3.4)\qquad |z(A)-z(B)| \leqq \int_A^B |Z(t, x(t))|\,dt \leqq \int_A^B \psi(t)v_1^\beta(t)a(v(t))\,dt \leqq$$

$$\leqq a(u_\infty(t_0, u_0)) \int_A^B \frac{\psi}{\varphi^{\beta/\alpha}} \varphi^{\beta/\alpha} v_1^\beta \leqq a(u_\infty(t_0, u_0)) \left\| \frac{\psi}{\varphi^{\beta/\alpha}} \right\|_{\alpha/(\alpha-\beta)} \left[\int_A^B \varphi v_1^\alpha\right]^{\beta/\alpha} \leqq$$

$$\leqq a(u_\infty(t_0, u_0)) \left\| \frac{\psi}{\varphi^{\beta/\alpha}} \right\|_{\alpha/(\alpha-\beta)} (w(A)-w(B)).$$

First we apply (3.4) to prove the continuability of the solutions and the z-stability of the zero solution. Let $\varepsilon>0$, $t_0\in R_+$ be given such that $|z|<\varepsilon$ implies $z\in D$. Fix an $x_0=(y_0, z_0)$ $(|z_0|<\varepsilon)$ and denote

$$T^* := \sup\{T\colon\ t_0 < T,\ |z(t;\ t_0, x_0)| < \varepsilon \quad \text{for} \quad t\in[t_0, T]\}.$$

We prove $T^*=\infty$ provided $|x_0|$ is sufficiently small. By (3.4)

$$|z(t_0)-z(t)| \leqq c_1 a(u_\infty(t_0, V(t_0, x_0)))u_\infty(t_0, V(t_0, x_0))$$

for every $t\in(t_0, T)$ with an appropriate constant $c_1>0$ independent of the solution. Because of

$$\lim_{x_0\to 0} V(t_0, x_0) = 0, \quad \lim_{u_0\to 0} u_\infty(t_0, u_0) = 0$$

we can choose a $0<\delta(\varepsilon, t_0)<\varepsilon/3$ such that $|x_0|<\delta$ implies $|z_0-z(t;\ t_0, x_0)|<\varepsilon/3$. Consequently, $|z(t;\ t_0, x_0)|<2\varepsilon/3$ for all $t\in[t_0, T)$. By y-continuability of the solutions and the definition of T^* this means that $T^*=\infty$, i.e. the solutions $x(t;\ t_0, x_0)$ with $|x_0|<\delta$ can be continued to all $t\geqq t_0$, and the zero solution of (3.1) is z-stable.

On the other hand, function $w(t)$ has a finite limit as $t\to\infty$, thus $w(A)-w(B)\to 0$ as $A, B\to\infty$. Hence, by (3.4), $|z(A)-z(B)|\to 0$ as $A, B\to\infty$, i.e. $z(t)$ has also a finite limit as $t\to\infty$.

It remains to prove that $V_1(t, x(t))\to 0$ as $t\to\infty$. To this end, take an $\varepsilon_0>0$, $t_0\in R_+$ and consider a solution $\xi(t)=(\eta(t), \zeta(t))$ of (3.1) with $|\xi(t_0)|<\delta(\varepsilon, t_0)$. We know that $|\zeta(t)|\leqq c_2$, $v(t)\leqq c_3$ with appropriate constants c_2, c_3 and

$$(3.5)\qquad \int_{t_0}^\infty \varphi(t)v_1^\alpha(t)\,dt<\infty.$$

Suppose now that $v_1(t)\not\to 0$ as $t\to\infty$. This assumption together with inequality (3.5) and the fact that φ is integrally positive imply the existence of a $\gamma>0$ such that for every $T \geqq 0$ there are $A=A(T)$, $B=B(T)$ $(T<A<B)$ with the properties

$$v_1(A) = 2\gamma/3, \quad v_1(B) = \gamma/3$$
$$\gamma/3 \leqq v_1(t) \leqq 2\gamma/3 \ \left(A(T) \leqq t \leqq B(T)\right)$$
$$B(T)-A(T) \to 0 \quad (T \to \infty). \tag{3.6}$$

On the other hand, the sum $v_1(t)+v_2(t)$ has a finite limit; consequently, there exists a $\tau>0$ such that the positive variation of $v_2(t)$ over $[A(T), B(T)]$ is greater than τ for every T. But, by conditions (iii)—(v) of the theorem we have

$$\tau \leqq \int_A^B [\dot v_2(t)]_+ dt = \int_A^B [\dot V_2(t, \xi(t))]_+ dt \leqq \int_A^B b(t, |\xi(t)|, v(t))\, dt + \tag{3.7}$$
$$+ \int_A^B \chi(t)\psi(t) v_1^\beta(t) c(|\xi(t)|, v(t)) a(v(t))\, dt \leqq$$
$$\leqq \int_A^B b(t, c_2, c_3)\, dt + c(c_2, c_3) a(c_3) \int_A^B \chi\psi v_1^\beta.$$

Using the Hölder inequality, for the last integral we get the estimate

$$\int_A^B \frac{\chi\psi}{\varphi^{\beta/\alpha}} \varphi^{\beta/\alpha} v_1^\beta \leqq \left[\int_A^B \frac{(\chi\psi)^{\alpha/(\alpha-\beta)}}{\varphi^{\beta/(\alpha-\beta)}}\right]^{(\alpha-\beta)/\alpha} \left[\int_A^B \varphi v_1^\alpha\right]^{\beta/\alpha}. \tag{3.8}$$

(3.5)—(3.8) and condition (iv) imply

$$0 < \tau \leqq \int_{A(T)}^{B(T)} [\dot v_2(t)]_+ \, dt \to 0 \quad (T \to \infty),$$

which is a contradiction proving that $v_1(t) \to 0$ as $t \to \infty$. The theorem is proved.

Remark 3.1. It can be proved that the zero solution of the equation $\dot u = r(t, u)$ is stable (see condition (ii) in Theorem 3.1) if and only if

$$\int_0^\infty r(t, u)\, dt < \infty$$

for every sufficiently small $u \geqq 0$, and the function $u(t) \equiv 0$ is the unique solution of the initial value problems

$$\dot u = r(t, u), \quad u(t_0) = 0 \quad (t_0 \geqq 0).$$

Remark 3.2. It is clear from the proof that in condition (iv) the "positive part" $[\cdot]_+$ on the left-hand side can be replaced by the "negative part" $[\cdot]_-$, the theorem remains true.

Remark 3.3. By Lemma 2.1, in consequence of condition (vi) in the theorem we can require of function ψ to be integrally positive instead of φ.

Remark 3.4. Analysing the proof of the theorem one can easily see that estimates (3.7)—(3.8) are not needed if we know the function $v_2(t)$ to be uniformly continuous on $[t_0, \infty)$. Consequently, if it is a priori known that the function $V_2(t, y(t), z(t))$ is uniformly continuous on $[t_0, \infty)$ for every solution $(y(t), z(t))$ of (3.1), such that $z(t) \to \text{const.}$, $V(t, y(t), z(t))$ is bounded as $t \to \infty$ and

$$\int_0^\infty \varphi(t) V_1^\alpha(t, y(t), z(t))\, dt < \infty,$$

then after dropping conditions (iv)—(v) and (3.2) the theorem remains true. In the next section we show how this condition can be checked directly in a special case.

4. Applications to damped mechanical systems

Consider a holonomic scleronomic mechanical system of r degrees of freedom. Assume that there act upon the system potential and dissipative forces depending also on the time. Let the motions be described by the Lagrangian equation ([10], Appendix II)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q, \tag{4.1}$$

where $q \in D \subset R^r$ is the vector of the generalized coordinates (D is open and contains the origin of R^r), $\dot q \in R^r$ is the vector of the generalized velocities; $T = T(q, \dot q)$ denotes the kinetic energy which is a quadratic form of the velocities; $\Pi = \Pi(t, q)$ is the potential energy, the vector $Q = Q(t, q, \dot q)$ is the resultant of frictional and gyroscopic forces, i.e. $Q(t, q, \dot q)\dot q \leqq 0$ for all $t \geqq 0$, $q \in D$, $\dot q \in R^r$. Suppose that

$$\Pi(t, 0) \equiv 0, \quad \frac{\partial \Pi}{\partial q}(t, 0) \equiv 0,$$

which means that $q = \dot q = 0$ is an equilibrium state of the system.

Many authors have investigated the conditions of the asymptotic stability of the equilibrium state (see [10, 11, 9, 2]). As the simple example in the Introduction shows, for this property it is necessary to bound above the damping in some way.

In this section we examine what happens under the action of damping not restricted above at all. It will turn out that if the damping is sufficiently large then the system asymptotically stops, i.e. for every motion of (4.1) $q(t) \to \text{const.}$ (maybe different from the origin), $\dot q(t) \to 0$ as $t \to \infty$.

Theorem 4.1. *Suppose that there exist constants* $h, \gamma > 0$ *and continuous functions* $\varphi\colon R_+ \to R_+$, $r\colon R_+^2 \to R_+$ *such that the following conditions are satisfied on the set* $\Gamma := \{(t, q, \dot q)\colon t \in R_+,\ |q| \leqq h,\ |\dot q| \leqq h\}$:

(i) $$\Pi(t, q) \geqq 0;$$

(ii) $$\left[\frac{\partial \Pi}{\partial t}(t, q)\right]_+ \leqq r\big(t, \Pi(t, q)\big),$$

where the function $r(t, \cdot)$ *is nondecreasing for every* $t \in R_+$, *and the zero solution of the equation* $\dot u = r(t, u)$ *is stable;*

(iii) *the dissipation is complete and large enough in the sense that the inequality*

$$Q(t, q, \dot q)\dot q \leqq -\varphi(t)|\dot q|^{1+\gamma}$$

holds so that

(4.2) $$\int_0^\infty \varphi^{-1/\gamma} < \infty;$$

(iv) *the function*

$$\int_t^{t+1} \max\left\{\left|\frac{\partial \Pi}{\partial q}(s, q)\right| : |q| \leqq h\right\}^{(1+\gamma)/\gamma} \varphi^{-1/\gamma}(s)\, ds$$

is bounded on R_+.

Then the equilibrium state $q = \dot q = 0$ *of* (4.1) *is stable, asymptotically* $\dot q$*-stable, and along every motion with sufficiently small initial values* $|q(t_0)|$, $|\dot q(t_0)|$ *the vector of the generalized coordinates has a finite limit as* $t \to \infty$ *(i.e. the system asymptotically stops).*

Proof. It is known ([10], Appendix II) that the matrix in the quadratic form of the kinetic energy is positive definite, so by introducing the new variable $y := \dot q$ system (4.1) can be rewritten into the explicit form

(4.3) $$\dot y = Y(t, y, q), \quad \dot q = y.$$

Take the auxiliary functions

$$V_1(y, q) := T(q, y) \quad V_2(t, y) := \Pi(t, q).$$

An easy computation shows (see [10]) that the derivative of $V(t, y, q) := V_1(y, q) + V_2(t, y)$ with respect to (4.3) reads

(4.4) $$\dot V(t, y, q) = Q(t, q, y)y + \frac{\partial \Pi}{\partial t}(t, q) \leqq -\varphi(t)|y|^{1+\gamma} + r\big(t, \Pi(t, q)\big) \quad \big((t, y, q) \in \Gamma\big).$$

Because the kinetic energy T is a positive definite quadratic form, there are $0 < \lambda < \Lambda$ such that

$$\lambda^2|y|^2 = V_1(y, q) \leqq \Lambda^2|y|^2 \quad (|q| \leqq h, y \in R^r).$$

Therefore, from (4.4) we got the estimate

$$\dot{V}(t, y, q) \leqq -\frac{\varphi(t)}{\lambda} V_1^{(1+\gamma)/2}(y, q) + r(t, V(t, y, q)) \tag{4.5}$$

on the set Γ. By setting $\alpha := (1+\gamma)/2$, $\beta := 1/2$,

$$\psi(t) := 1/\lambda, \quad b(t, r_1, r_2) := r(t, r_2), \quad c(r_1, r_2) := 1$$

$$\chi(t) := \max\left\{\left|\frac{\partial \Pi}{\partial q}(t, q)\right| : |q| \leqq h\right\}$$

all the conditions of Theorem 3.1 are met on the set Γ (for the integral positivity of φ see Remark 3.3). Thus if we have proved y-stability of the zero solution of (4.3), the further statements of the theorem follows from Theorem 3.1.

Function V is positive definite with respect to y and in view of (4.5) it satisfies the differential inequality $\dot{u} \leqq r(t, u)$. The zero solution of the associated differential equation $\dot{u} = r(t, u)$ is stable; therefore, by C. Corduneanu's theorem [5] (see also [6]) the equilibrium state $q = \dot{q} = 0$ is q-stable. The theorem is proved.

Remark. It is worth noticing that either the stronger version

$$\left|\frac{\partial \Pi}{\partial t}(t, q)\right| \leqq r(t, \Pi(t, q)) \tag{ii'}$$

of condition (ii) or the condition that $\Pi(t, q)$ is uniformly continuous for $t \in R_+$, $|q| \leqq h$ can replace condition (iv) in the theorem.

Indeed, according to Remark 3.4 it is enough to prove that for every motion $q(t)$ with sufficiently small initial values $|q(t_0)|$, $|\dot{q}(t_0)|$ the function $\Pi(t, q(t))$ is uniformly continuous on R_+, provided that $q(t) \to$ const. as $t \to \infty$. This is obviously satisfied if $\Pi(t, q)$ is uniformly continuous. On the other hand, for any q fixed sufficiently small, from condition (ii') we get the estimate

$$|\Pi(A, q) - \Pi(B, q)| \leqq \left|\int_A^B r(t, u(t;\ t_0, \Pi(t_0, q_0))\, dt\right| =$$

$$= |u(A;\ t_0, \Pi(t_0, q_0)) - u(B;\ t_0, \Pi(t_0, q_0))| \to 0 \quad (A, B \to \infty).$$

Therefore, $\Pi(t, q) \to \Pi^*(q)$ uniformly in a small ball around the origin as $t \to \infty$, hence $\Pi(t, q(t))$ also has a finite limit, which is sufficient for the uniform continuity of $\Pi(t, q(t))$.

5. Application to second order equations

In this section we apply our main theorem to study of asymptotic behaviour of the motions of a rheonomic mechanical system of one degree of freedom. In differential equation language, consider the equation

$$(p(t)\dot{x})^{\cdot}+g(t,x,\dot{x})\dot{x}+q(t)f(x)=0 \quad (x\in R), \tag{5.1}$$

where $p, q\colon R_+\to(0,\infty)$ are continuously differentiable, and $g\colon R_+\times R^2\to R$, $f\colon R\to R$ are continuous functions, and

$$xf(x)\geqq 0 \quad (t\in R_+,\, x\in R).$$

The following two theorems illuminate how to get different conditions for the same asymptotic property of the solutions of the same equation by different choices of the auxiliary functions. The first theorem concerns the case of bounded q, the second one can be applied also to unbounded q.

Theorem 5.1. *Suppose that*

(i) *there exists a function* $\gamma\colon R_+\to R_+$ *such that*

$$\gamma(t)\leqq g(t,u,v) \quad (t\in R_+;\, u,v\in R)$$

$$2\gamma(t)+\dot{p}(t)>0, \quad \int_0^\infty \frac{1}{2\gamma+p^{\cdot}}<\infty;$$

(ii)
$$\int_0^\infty \left[\frac{\dot{q}}{q}\right]_+<\infty;$$

(iii) *either* $(2\gamma+\dot{p})/p$ *or* $1/\sqrt{p}$ *is integrally positive.*

Then the zero solution of (5.1) *is x-stable and for every solution* $x(t)$ *with sufficiently small initial values* $x(t)\to\text{const.}$ $p(t)\dot{x}^2(t)\to 0$ *as* $t\to\infty$.

Proof. Introducing the notations $y=p(t)\dot{x}$, $z=x$, we can write equation (5.1) in the form

$$\begin{cases}\dot{y}=-q(t)f(z)-g\big(t,z,y/p(t)\big)y/p(t)\\ \dot{z}=y/p(t).\end{cases} \tag{5.2}$$

Define

$$F(z):=\int_0^z f(r)\,dr\geqq 0, \quad V_1(t,y):=y^2/2p(t), \quad V_2(t,z):=q(t)F(z).$$

The derivative of the total mechanical energy $V:=V_1+V_2$ can be estimated as follows:

$$\dot{V}=-\left(\frac{2g}{p}+\frac{\dot{p}}{p}\right)V_1+\dot{q}F\leqq-\frac{2\gamma+\dot{p}}{p}V_1+\left[\frac{\dot{q}}{q}\right]_+V.$$

By Lemma 2.1, condition (iii) implies the function $(2\gamma+\dot{p})/p$ to be integrally positive. Because of (ii) the function q is bounded, so the conditions of Theorem 3.1 are met by the choices $\beta:=1/2$, $\psi:=\sqrt{2/p}$, $\chi:=q$ and

$$c(r, s) := \max\{|f(z)|:\ |z| \leqq r\}.$$

The theorem is proved.

Theorem 5.2. *Suppose that*

(i) *there exist a constant* $h>0$ *and a function* $\gamma\colon R_+\to R_+$ *such that*

$$\gamma(t) \leqq g(t, u, v) \quad (t\in R_+,\ |u| \leqq h,\ v\in R),$$

$$(\ln(pq))^{\cdot}(t)+2\frac{\gamma(t)}{p(t)} > 0;$$

(ii)
$$\int_0^\infty \frac{q}{p(\ln(pq))^{\cdot}+2\gamma} < \infty;$$

(iii) *either* $\sqrt{q/p}$ *or* $(\ln(pq))^{\cdot}+2\gamma/p$ *is integrally positive.*

Then the zero solution of (5.1) *is x-stable, and for every solution* $x(t)$ *with sufficiently small initial values* $x(t)\to\text{const.}$, $p(t)\dot{x}^2(t)/q(t)\to 0$ *as* $t\to\infty$.

Proof. After setting

$$V_1(t, y):=\frac{1}{p(t)q(t)}y^2, \quad V_2(z):=2F(z)$$

and computing

$$(V_1+V_2)^{\cdot} = -\left[(\ln(pq))^{\cdot}+\frac{2g}{p}\right]V_1$$

the proof of the theorem can be concluded in the same way as in Theorem 5.1.

Finally, in order to illuminate these results we give sufficient conditions for the asymptotic stop of a mathematical plain pendulum whose length changes by the law $l=l(t)$ (see [3]). Assume that there acts viscous friction on the material point such that the damping force is proportionate to the velocity. Let the position of the material point in the plain be described by the length $l(t)$ of the thread and the angle φ between the axis directed vertically downwards and the thread. Then the kinetic energy T, the potential energy Π and the dissipative force Q are

$$T = \frac{1}{2}m[l^2(t)\dot{\varphi}^2 \dot{l}^2(t)], \quad \Pi = mgl(t)(1-\cos\varphi)+\dot{l}, \quad Q = -h(t)l^2(t)\dot{\varphi},$$

where m is the mass of the material point, g denotes the constant of gravity and $h(t)$ is the frictional coefficient at the moment t. The motions are described by the Lagrange's equation

$$(ml^2(t)\dot{\varphi})^{\cdot}+hl^2(t)\dot{\varphi}+mgl(t)\sin\varphi = 0.$$

Corollary 5.1. *If*

(i) $$h(t)+m(\ln l(t))^{\cdot}>0 \quad (t\in R_+),$$

$$\int_0^\infty \frac{1/l^2}{h+m(\ln l)^{\cdot}}<\infty;$$

(ii) $$\int_0^\infty [(\ln l)^{\cdot}]_+<\infty,$$

then the equilibrium state $\varphi=\dot\varphi=0$ *is* φ*-stable; every motion* $\varphi(t)$ *has a finite limit, and* $l(t)\dot\varphi(t)\to 0$ *as* $t\to\infty$.

Corollary 5.2. *If*

(i) $$3m(\ln l(t))^{\cdot}+2h(t)>0 \quad (t\in R_+);$$

(ii) $$\int_0^\infty \frac{1/l}{3m(\ln l)^{\cdot}+2h}<\infty;$$

(iii) *either* $1/\sqrt{l(t)}$ *or* $3m(\ln l)^{\cdot}+2h$ *is integrally positive,*
then the equilibrium state $\varphi=\dot\varphi=0$ *is* φ*-stable, every motion* $\varphi(t)$ *has a finite limit and* $\sqrt{l(t)}\,\dot\varphi(t)\to 0$ *as* $t\to\infty$.

Proof. They immediately follow from Theorems 5.1 and 5.2, respectively.

One can observe that Corollary 5.2 concerns the case of nonincreasing length (perhaps $l(t)\to 0$ as $t\to\infty$), and Corollary 5.2 works mainly if $l(t)$ is nondecreasing (possibly, $l(t)\to\infty$ as $t\to\infty$).

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