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Multiplicative functions with nearly integer values

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Dedicated to Professor Károly Tandori on the occasion of his 60th birthday

We shall say that a realvalued arithmetical function $f(n)$ is completely multiplicative if $f(mn)=f(m)\cdot f(n)$ holds for each pairs of integers. Let $\|z\|$ denote the distance of z to the nearest integer, and $[z]$ denote the integer part of z.

We are interested in to determine the class of those completely multiplicative functions for which

$$\|f(n)\| \to 0 \quad (n \to \infty). \tag{1}$$

It is obvious that the validity of (1) does not depend on the sign of $f(n)$, since $\|z\|=\|-z\|$, so we may assume that $f(n)\geqq 0$.

We shall say that a real number θ is a Pisot-number, if it is an algebraic integer, $\theta>1$, and if all conjugates $\theta_2, \ldots, \theta_r$, are in the domain $|z|<1$. It is well known for a Pisot-number the relation

$$\|\theta^n\| \to 0 \tag{2}$$

holds. (See [1].)

Let now the whole set of primes $\mathcal{P}$ be divided into two disjoint subsets $\mathcal{P}_1, \mathcal{P}_2$ and $f(n)$ be defined for $p\in\mathcal{P}$ as follows:

$$f(p) = \begin{cases} 0 & \text{if} \quad p\in\mathcal{P}_1 \\ \theta^{x(p)} & \text{if} \quad p\in\mathcal{P}_2, \end{cases}$$

where $x(p)$ is a positive integer for each $p\in\mathcal{P}_2$ and $x(p)\to\infty$ if $p\to\infty$, furthermore θ is a Pisot-number. Then the completely multiplicative $f(n)$ determined by these values satisfies the relation (1).

For an algebraic α let $Q(\alpha)$ denote the simple extension of the rational number field generated by α.

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Lemma 1. *Let β be an algebraic number, $f(n)$ be completely multiplicative with values in $Q(\beta)$. Let $\beta_2, \ldots, \beta_r$ be the conjugates of β over Ω. Let $\varphi_j(n)$ denote the conjugate of $f(n)$ defined by the substitution $\beta \to \beta_j$. Then $\varphi_j(n)$ are completely multiplicative functions as well.*

Proof. Let $f(n) = r_n(\beta)$. Then $\varphi_j(n) = r_n(\beta_j)$. Since $r_{mn}(\beta) = f(mn) = f(m) \cdot f(n) = r_m(\beta) \cdot r_n(\beta)$, therefore $\varphi_j(mn) = r_{mn}(\beta_j) = r_m(\beta_j) r_n(\beta_j) = \varphi_j(m)\varphi_j(n)$.

Lemma 2. *Let β be an algebraic number and $f(n)$ a completely multiplicative function the values $f(n)$ of which are integers in $Q(\beta)$. Assume that*

$$\varphi_j(p) \to 0 \quad \text{as} \quad p \to \infty, \quad (j = 2, \ldots r), \tag{3}$$

where p runs over the set of primes. Then (1) *holds.*

Proof. It is obvious that (3) involves that $\varphi_j(n) \to 0$ $(n \to \infty)$. Furthermore $\varphi_j(n)$ are algebraic integers, and so

$$f(n) + \varphi_2(n) + \ldots + \varphi_r(n) = E_n = \text{rational integer},$$

whence (1) follows immediately.

To give a partial answer for our problem we shall use the following known theorems [1] as Lemma 3 and 4.

Lemma 3. *Let $\alpha > 1$ be an algebraic number, $\lambda \neq 0$ be a real number and*

$$\|\lambda \alpha^n\| \to 0 \quad (n \to \infty). \tag{4}$$

Then α is a Pisot-number, $\lambda = \alpha^{-N}\mu$, where $N \geqq 0$ is a suitable integer, $\mu \in Q(\alpha)$.

Lemma 4. *Let $\alpha > 1$, $\lambda \neq 0$ be a real number and*

$$\sum_{0 \leqq n < \infty} \|\lambda \alpha^n\|^2 < \infty. \tag{5}$$

Then α is an algebraic number, consequently the assertion stated in Lemma 3 holds.

Lemma 5. *Let $f(n) \geqq 0$ be a completely multiplicative function for which* (1) *holds. If $f(n_0) > 1$ for at least one n_0, then $f(n) \geqq 1$ or $f(n) = 0$ for each values of n.*

Proof. Assume in contrary that $0 < f(m_0) < 1$. Let $b = f(n_0)$, $a = f(m_0)$, $a = f(m_0)$, $x_0 = [-3 \log a] + 1$. For infinitely many k, l pairs of positive integers we have

$$\frac{-2x_0}{\log a} > k + l\frac{\log b}{\log a} > \frac{-x_0}{\log a},$$

since the length of the interval $\left(\frac{-x_0}{\log a}, \frac{-2x_0}{\log a}\right)$ is at least three. For such pairs

k, l we have $2^{-2x_0} < a^k b^l < 2^{-x_0}$, consequently

$$2^{-2x_0} < \|f(m_0^k n_0^l)\| = \|a^k b^l\| < 2^{-x_0}.$$

But this contradicts to (1).

Lemma 6. *Let $f \geqq 0$ be a completely multiplicative function satisfying* (1). *Assume that there exists an m for which $f(m) > 1$ and $f(m)$ is algebraic over Q. Let $\mathcal{P}_1$ be the set of those primes p for which $f(p) \neq 0$. Then the values $f(p)$ are Pisot-numbers for each $p \in \mathcal{P}_1$, and for every $p_1, p_2 \in \mathcal{P}_1$ we have $Q(\alpha_{p_1}) = Q(\alpha_{p_2})$, $\alpha_{p_1} = f(p_1)$, $\alpha_{p_2} = f(p_2)$.*

Proof. Let $f(m) = \alpha$. Since $\alpha > 1$, α algebraic; and $\|f(m^k)\| = \|\alpha^k\| \to 0$ $(k \to \infty)$, by Lemma 3 we get that $f(m)$ is a Pisot-number.

Let now n be an arbitrary natural number for which $f(n) \neq 0$. Since $\|f(nm^k)\| = \|f(n)\alpha^k\| \to 0$ $(k \to \infty)$, from Lemma 3 we deduce that $f(n) = \alpha^{-N}\gamma$, $N \geqq 0$, integer, $\gamma \in Q(\alpha)$. Hence $\beta = f(n) = \alpha^{-N}\gamma \in Q(\alpha)$. Since $\beta \neq 0$, from Lemma 3 we get that $\beta > 1$, and so by repeating the above argument with β instead of α, we deduce that β is a Pisot-number and $\alpha \in Q(\beta)$. The assertion is proved.

Corollary. Let $f(n) \geqq 0$ be a completely multiplicative function satisfying (1). If $1 < f(n) \in Q$ holds for at least one n, then $f(n)$ takes on integer values for every n.

Lemma 7. *Let $f(n) \geqq 0$ be a completely multiplicative function satisfying the relation*

$$\|f(n)\| \leqq \varepsilon(n), \tag{6}$$

where $\varepsilon(n)$ is a monotonically decreasing function, with

$$\sum_{k=1}^{\infty} \varepsilon^2(2^k) < \infty. \tag{7}$$

Then the following possibilities are:

a) *f takes on integer values for every n.*

b) *For a suitable n $0 < f(n) < 1$. Then $f(n) \to 0$ as $n \to \infty$.*

c) *For a suitable m $f(m) > 1$. Let $\mathcal{P}_1$ denote the whole set of those primes p for* which $f(p) \neq 0$. *Then there exists a Pisot-number Θ such that $Q(f(p)) = Q(\Theta)$ for each $p \in \mathcal{P}_1$.*

Proof. The relation (6) involves (4). If $0 < f(n) < 1$ then from Lemma 5 $f(m) \leqq 1$ for every m. If $f(m) = 1$, then $\|f(nm^k)\| = \|f(n)\|$ as $k \to \infty$; that contradicts to (1). Consequently $f(m) < 1$ for each $m > 1$. Assume that there exists a subsequence $n_1 < n_2 < \ldots$ such that $f(n_j) \to 1$. Then $f(nn_j) \to f(n)$ $(j \to \infty)$ that contradicts to (1). Consequently $f(m) \to 0$ as $m \to \infty$.

Let us considei the case c). Taking into account (6) and (7), the conditions of Lemma 4 are satisfied with $\lambda=1, \alpha=f(m)>1$. Consequently α is an algebraic number, and the assertion is an immediate consequence of Lemma 6.

Theorem 1. *Let $f(n)\geqq 0$ be a completely multiplicative function that takes on at least one algebraic value $f(n_0)=\alpha>1$. Let $\mathcal{P}_1$ denote those set of primes p for which $f(p)\neq 0$.*

If (1) holds, then the values $f(p)=\alpha_p$ are Pisot-numbers, for each $p_1, p_2\in\mathcal{P}_1$ we have $Q(\alpha_{p_1})=Q(\alpha_{p_2})$. Let Θ denote one of the values α_p $(p\in\mathcal{P}_1)$, $\Theta_2, \ldots, \Theta_r$ its conjugates, $\varphi_2(n), \ldots, \varphi_r(n)$ be defined as in Lemma 1. Then

$$\varphi_j(n)\to 0 \quad as \quad n\to\infty, \quad j=2, \ldots r. \tag{8}$$

In contrary, let us assume that the values $f(p)$ are zeros or Pisot-numbers from a given algebraic number field $\Omega(\Theta)$. If

$$\varphi_j(p)\to 0 \quad as \quad p\to\infty \quad (j=2, \ldots, r) \tag{9}$$

then (1) holds.

Proof. Let us assume that (1) holds. From Lemma 6 we get that the values $f(n)$ are zeros or Pisot-numbers taken from a given number field $\Omega(\Theta)$. Let us consider the vector

$$\Psi(n)=(\varphi_2(n), \ldots, \varphi_r(n)),$$

and denote by X the set of the limit points of $\psi(n)$ $(n\to\infty)$. Let $(x_2, \ldots, x_r)\in X$. Since

$$f(n)+\varphi_2(n)+\ldots+\varphi_r(n) = \text{rational integer},$$

$\|f(n)\|\to 0$, we get that $x_2+\ldots+x_r=$rational integer. Let m_j be such a sequence for which

$$\Psi(m_j)\to(x_2, \ldots, x_r).$$

Then $\Psi(m_j^h)\to(x_2^h, \ldots, x_r^h)$, $x_2^h+\ldots+x_r^h=$rational integer, consequently $0<|x_j|<1$ is impossible, that is $x_j=0$ or $|x_j|=1$. Let now n be fixed such that $f(n)\neq 0$. Then $\varphi_j(n)\neq 0$, $0<|\varphi_j(n)|<1$,

$$\Psi(nm_j)\to(\varphi_2(n)\, x_2, \ldots, \varphi_r(n)x_r)\in X.$$

If $x_l\neq 0$ for a suitable l, then $0\neq|\varphi_l(n)x_l|<1$, which is impossible. Consequently we have (8).

The converse assertion is an immediate consequence of Lemma 2.

Theorem 2. *Let* $f(n) \geqq 0$ *be a completely multiplicative function satisfying the conditions* (6), (7). *Let us assume that* $f(n) \nrightarrow 0$, *and that* $f(n)$ *takes on at least one nonintegral value. Then* $f(n)$ *takes on algebraic values, and the first assertion, stated in Theorem* 1, *holds.*

Proof. This is an immediate consequence of Lemma 7 and Theorem 1.

Reference

[1] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Univ. Press (1957), Ch. VIII.

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