Acta Sci. Math., 48 (1985), 227—242

Approximation and quasisimilarity

L. KÉRCHY

Dedicated to Professor Károly Tandori on the occasion of his 60th birthday

1. Introduction

For an arbitrary complex Hilbert space $\mathfrak{H}$ let $\mathscr{L}(\mathfrak{H})$ denote the Banach algebra of all bounded linear operators acting on $\mathfrak{H}$. For any $T\in\mathscr{L}(\mathfrak{H})$ let Alg T denote the weakly closed subalgebra of $\mathscr{L}(\mathfrak{H})$ generated by T and the identity I, while $\{T\}'$ stands for the commutant of T. We call a subspace $\mathfrak{M}$ of $\mathfrak{H}$ to be cyclic for T if $\bigvee_{n\geqq 0} T^n\mathfrak{M}=\mathfrak{H}$; $\mathfrak{M}$ is a minimal cyclic subspace if it contains no proper subspace which is also cyclic for T. The number disc T is defined as the supremum of the dimensions of all finite dimensional minimal cyclic subspaces for T.

In this paper we are going to investigate the problems of quasisimilarity invariances of the approximating property "Alg $T=\{T\}''$" and the number "disc T" in the class of cyclic C_{11}-contractions. We remark that the commutant of a C_{11}-contraction T is commutative if and only if T is cyclic. So to consider only cyclic contractions does not mean the restriction of the generality in connection with the first problem.

Our paper is organized as follows. In section 2 we discuss the approximating property "Alg T$=\{T\}''$" and describe its connection with the reflexivity problem of C_{11}-contractions. In section 3 the question of quasisimilarity invariance of "disc T" is performed. Our main goal is formulated in section 4: to construct contractions with special properties, whose study may give hope to solve the previous problems. Our construction is given in section 7 and is based on the results of sections 5 and 6. Section 5 deals with injective contractions, while in section 6 it is proved that a large amount of cyclic C_{11}-contractions possesses 0 as an "approximate reducing eigenvalue".

Received April 6, 1984.

2. The approximating property "Alg $T=\{T\}'$"

Let us consider the normalized Lebesgue measure m on the unit circle $\mathbf{T}$ of the complex plane $\mathbf{C}$ and let α be a Borel subset of $\mathbf{T}$ with positive measure: $m(\alpha)>0$. For any function φ from $L^\infty(\alpha)$ let $M_{\alpha,\varphi}$ denote the operator of multiplication by φ in $L^2(\alpha)$. (The spaces $L^p(\alpha)$, $p=2, \infty$, are defined with respect to the measure m on α.) In the case of the identical function $\varphi(\zeta)\equiv\zeta$ we will use the notation M_α for $M_{\alpha,\varphi}$, moreover $M_{\mathbf{T}}$ will simply be denoted by M. We remark that for two Borel subsets α, β of $\mathbf{T}$ the operators M_α, M_β are unitarily equivalent, $M_\alpha\cong M_\beta$, if and only if the symmetric difference $\alpha\triangle\beta$ of these sets is of measure 0, in notation $\alpha=\beta$ $[m]$ (cf. [6]).

Let us denote by $\mathcal{L}^\infty(\alpha)$ the set of multiplication operators on $L^2(\alpha)$, i.e.

$$\mathcal{L}^\infty(\alpha) = \{M_{\alpha,\varphi}\colon \varphi\in L^\infty(\alpha)\}.$$

It is known (cf. e.g. [16, Theorem 1]) that the commutant of M_α coincides with $\mathcal{L}^\infty(\alpha)$.

Let $P^\infty(\alpha)$ be the w^*-closure of the set of polynomials in $L^\infty(\alpha)$. It can be easily seen that the corresponding set of multiplication operators is exactly the closure of the set of polynomials of M_α in the weak operator topology, i.e.

$$\{M_{\alpha,\varphi}\colon \varphi\in P^\infty(\alpha)\} = \operatorname{Alg} M_\alpha.$$

Furthermore, it is a remarkable fact that $P^\infty(\alpha)=L^\infty(\alpha)$ if and only if m is not absolutely continuous with respect to the measure $\chi_\alpha dm$, where χ_α stands for the characteristic function of α (cf. [16, p. 17]). Hence, it follows that

$$\operatorname{Alg} M_\alpha = \{M_\alpha\}',$$

i.e. every operator in the commutant of M_α can be approximated by polynomials of M_α in the weak operator topology, exactly when $\alpha\neq\mathbf{T}[m]$.

Let T be a contraction acting on a complex separable Hilbert space $\mathfrak{H}$, i.e. $T\in\mathcal{L}(\mathfrak{H})$ and $\|T\|\leqq 1$. Let us assume that T is of class C_{11}, that is $\lim\limits_{n\to\infty}\|T^n h\|\neq 0\neq \lim\limits_{n\to\infty}\|T^{*n}h\|$ for every $0\neq h\in\mathfrak{H}$, and that T has a cyclic vector f, which means that every vector $h\in\mathfrak{H}$ can be approximated in the norm of $\mathfrak{H}$ by vectors of the form $p(T)f$, where $p(\lambda)$ is a complex polynomial. Moreover, for the sake of simplicity, we assume that the unitary part of T (cf. [17, Theorem I.3.2]) is absolutely continuous with respect to the Lebesgue measure. Let C_1 denote the class of such contractions.

It is known (cf. [11, p. 15]) that for any operator $T\in C_1$ there exists a unique Borel subset α of $\mathbf{T}$ such that $m(\alpha)>0$ and T is quasisimilar to M_α: $T\sim M_\alpha$, which means that appropriate quasiaffinities X, Y (i.e. operators with zero kernels and dense ranges) interwine T and M_α: $XT=M_\alpha X$, $YM_\alpha=TY$. (In connection with the theory of contractions we refer to our main reference [17].)

Now the question naturally arises:

Problem A. Is the approximating property $\operatorname{Alg} T=\{T\}'$ a quasisimilarity invariant in the class C_1?

The answer for this question is *negative,* as it was pointed out for me by Hari Bercovici. Indeed, let T be a completely non-unitary (c.n.u.) contraction from C_1 which is quasisimilar to M_α, where $\alpha\neq\mathbf{T}[m]$. We can choose T such that its spectrum covers the whole unit disc: $\sigma(T)=D^-$ (cf. [4, Proposition 3.1]). Since T is a C_{11}-contraction, its essential spectrum coincides with $\sigma(T)$. So we can infer that the Sz.-Nagy, Foiaş functional calculus for T is an isometry and $\operatorname{Alg} T$ coincides with the w^*-closure of the set of polynomials of T (cf. [1, Corollary 1]), consequently $\operatorname{Alg} T=H^\infty(T):=\{u(T): u\in H^\infty$ (the Hardy space)$\}$.

On the other hand, the commutant $\{T\}'$ of T never coincides with $H^\infty(T)$, and so $\operatorname{Alg} T\neq\{T\}'$ contrary to the fact that $\operatorname{Alg} M_\alpha=\{M_\alpha\}'$.

The following argumentation proving the inequality $\{T\}'\neq H^\infty(T)$ is slightly different from the one given by Bercovici.

Let us say that the subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathfrak{H})$ has the property (P^*) if every non-zero operator in $\mathscr{A}$ is a quasiaffinity. Since every non-zero function $u\in H^\infty$ differs from 0 a.e., it is immediate that $H^\infty(M_\alpha)=\{u(M_\alpha)\colon u\in H^\infty\}$ has the property (P^*). Let us assume that $T\in C_1$ is quasisimilar to M_α, i.e. $XT=M_\alpha X$ and $TY=YM_\alpha$ with some quasiaffinities X and Y. Then for any non-zero function $u\in H^\infty$ the relations $Xu(T)=u(M_\alpha)X$ and $u(T)Y=Yu(M_\alpha)$ imply that $u(T)$ is a quasiaffinity. (The first one implies that $u(T)$ is injective, while the second one implies that its range is dense.) So we infer that $H^\infty(T)$ possesses the property (P^*).

Let us consider now an arbitrary operator S from $\{M_\alpha\}'$. Then $YSX\in\{T\}'$, XY, $XYSXY\in\{M_\alpha\}'$ and since $\{M_\alpha\}'=\mathscr{L}^\infty(\alpha)$ is commutative, it follows that $X(YSX)Y=(XY)S(XY)=S(XY)^2=(XY)^2S$. These relations show that if $S\neq 0$, then $YSX\neq 0$, and if YSX is a quasiaffinity, then so is S also. Since $\{M_\alpha\}'=\mathscr{L}^\infty(\alpha)$ does not have obviously property (P^*), the statements above result that $\{T\}'$ does not possess (P^*) too.

Consequently, for every contraction $T\in C_1$ we have $H^\infty(T)\subsetneqq\{T\}'$.

Owing to the negative answer for Problem A we introduce the operator class C_1' consisting of those elements of C_1 for which $\operatorname{Alg} T\neq H^\infty(T)$. For example every contraction $T\in C_1$ whose spectrum does not include $\mathbf{T}$ belongs to C_1' (cf. [2, p. 337]). Now we formulate our question in the following form:

Problem A′. Is the property $\operatorname{Alg} T=\{T\}'$ a quasisimilarity invariant in C_1'?

This problem seems to be relevant in connection with the *reflexivity* problem of C_{11}-contractions. (As for the notion of reflexivity see for instance [8, chapter 9].) Indeed, the operators in some subclasses of $C_1\setminus C_1'$, e.g. operators with dominating essential spectrum (so-called (BCP)-operators) belonging to C_1 are reflexive by a recent result of Bercovici, Foiaş, Langsam and Pearcy [1]. On the other hand, it is well-known that quasisimilarity preserves the reflexivity of the commutant (cf. [3, Proposition 4.1]). So an affirmative answer for Problem A′ would reduce the reflexivity problem to certain subclasses of $C_1 \setminus C_1'$ at least in the case of contractions from C_1 which are quasisimilar to some M_α with $\alpha \neq \mathbf{T}[m]$. Conversely, a counterexample for Problem A' might be a candidate for a non-reflexive C_{11}-contraction.

3. The number "disc T"

The second problem we are interested in was posed by Nikolskii and Vasjunin. In connection with questions concerning controllable systems they have introduced the number "disc" of an arbitrary Hilbert space operator (cf. [13]). Namely, for an operator $T \in \mathscr{L}(\mathfrak{H})$ let Cyc T be the set of finite dimensional cyclic subspaces:

$$\operatorname{Cyc} T := \Big\{\mathfrak{M} \text{ subspace of } \mathfrak{H}\colon \dim \mathfrak{M} < \infty,\ \bigvee_{n \geqq 0} T^n \mathfrak{M} = \mathfrak{H}\Big\}.$$

Then $\operatorname{disc} T$ denotes the number

$$\operatorname{disc} T := \sup \big\{\min \{\dim \mathfrak{N}\colon \mathfrak{N} \in \operatorname{Cyc} T,\ \mathfrak{N} \subset \mathfrak{M}\}\colon \mathfrak{M} \in \operatorname{Cyc} T\big\}.$$

Nikolskii and Vasjunin posed the question of quasisimilarity invariance of disc T in general (cf. [13, p. 330]). In particular, it would be interesting to know the answer for the following problem:

Problem B. Is the number disc T a quasisimilarity invariant in the class C_1?

This question seems to be of considerable interest, because $\operatorname{disc} M_\alpha$ takes on different values according to the case that $\alpha \neq \mathbf{T}[m]$ or not. Namely, the following is true:

Proposition 1. $\operatorname{disc} M_\alpha = 1$ *if* $\alpha \neq \mathbf{T}[m]$, *while* $\operatorname{disc} M = 2$.

Proof. If $\alpha \neq \mathbf{T}[m]$ then $\operatorname{Alg} M_\alpha = \{M_\alpha\}' = \mathscr{L}^\infty(\alpha)$, and so $\operatorname{Lat} M_\alpha = \{\chi_\beta L^2(\alpha)\colon \beta \subset \alpha\}$, where χ_β is the characteristic function of β. This implies that $f \in L^2(\alpha)$ is cyclic for M_α if and only if $f(x) \neq 0$ a.e. On the other hand by Szegő's theorem (cf. [10]) we know that the cyclic vectors of M are the functions f such that $f(x) \neq 0$ a.e. and $\int_{\mathbf{T}} \log |f|\, dm = -\infty$.

Now, if $\alpha \neq \mathbf{T}[m]$, $\mathfrak{M} \in \operatorname{Cyc} M_\alpha$ and $\{f_i\}_{i=1}^n$ is a basis in $\mathfrak{M}$, then $\sum_{i=1}^n |f_i(x)| \neq 0$ a.e.. An elementary argumentation shows that $f(x) \neq 0$ a.e. on α for a suitable linear combination $f = \sum_{i=1}^n c_i f_i$.

Indeed, proceeding by induction on n, we can reduce the proof to the case $n=2$. So let us assume that $|f_1(x)|+|f_2(x)|\neq 0$ a.e., and let $\alpha_j=\{x\in\alpha: f_j(x)=0\}$ $(j=1, 2)$ and $\alpha'=\alpha\setminus(\alpha_1\cup\alpha_2)$. Then $m(\alpha_1\cap\alpha_2)=0$ and so it is enough to show that $f_1(x)+ +cf_2(x)\neq 0$ a.e. on α' with some non-zero complex number c. Taking different numbers c and d the sets $E_c=\{x\in\alpha': f_1(x)+cf_2(x)=0\}$ and $E_d=\{x\in\alpha': f_1(x)+df_2(x)=0\}$ will be disjoint. Therefore, for all but countably many points c of $\mathbf{C}\setminus\{0\}$ the set E_c will be of measure 0, but for such a number c we have $f_1(x)+cf_2(x)\neq 0$ a.e. on α.

Therefore, we conclude that $\operatorname{disc} M_\alpha=1$.

Let us determine now disc M! Let $\mathbf{T}^+$ and $\mathbf{T}^-$ denote the upper and lower semicircle, respectively, and let $\mathfrak{M}$ be the 2-dimensional subspace spanned by $\chi_{\mathbf{T}^+}$ and $\chi_{\mathbf{T}^-}$. Then $\mathfrak{M}\in\operatorname{Cyc} M$, but $\mathfrak{M}$ does not contain any cyclic vector of M. Therefore $\operatorname{disc} M\geqq 2$.

Now let $\mathfrak{M}\in\operatorname{Cyc} M$ be a subspace with $\dim\mathfrak{M}=n\geqq 2$. We want to show that $\min\{\dim\mathfrak{N}: \mathfrak{N}\in\operatorname{Cyc} T, \mathfrak{N}\subset\mathfrak{M}\}\leqq 2$. As before, we can infer that $f(x)\neq 0$ a.e. for some $f\in\mathfrak{M}$. It can be assumed that $\int_{\mathbf{T}}\log|f|\,dm>-\infty$ for these functions. It is well-known (cf. [9, Theorems II. 2 and 3]) that the invariant subspace lattice of M has the form:

$$\operatorname{Lat} M=\{L^2(\alpha): \alpha\subset\mathbf{T}\}\cup\{qH^2: q \text{ is a unimodular function on } \mathbf{T}\}.$$

Let us assume that there exists a function $g\in\mathfrak{M}$ such that $0<m(\{x\in\mathbf{T}: g(x)\neq 0\})<1$. Then $\mathfrak{M}_g:=\bigvee_{k\geqq 0} M^k g=L^2(\alpha)$ with $\alpha\neq\mathbf{T}[m]$, while $\mathfrak{M}_f:=\bigvee_{k\geqq 0} M^k f=qH^2$ with a unimodular q. Since, for every $h\in H^2$, $\int_{\mathbf{T}}\log|h|\,dm>-\infty$ (cf. [17, Sec. III. 1]) we infer that $\mathfrak{M}_g\vee\mathfrak{M}_f=L^2(\alpha)\vee qH^2=L^2(\mathbf{T})$.

Hence we have only to deal with the case when for every nonzero $f\in\mathfrak{M}$ we have $\int_{\mathbf{T}}\log|f|\,dm>-\infty$. We prove by induction on n that $\mathfrak{M}_{i_1}\vee\mathfrak{M}_{i_2}=L^2(\mathbf{T})$ for some $1\leqq i_1, i_2\leqq n$, $i_1\neq i_2$, where $\mathfrak{M}_i=\mathfrak{M}_{f_i}$ for $i=1, \ldots, n$. For $n=2$ this is obvious. Let us assume that it is true for $n-1$ $(n\geqq 3)$. If $\bigvee_{i=1}^{n-1}\mathfrak{M}_i=L^2(\mathbf{T})$, then we can apply our assumption. If $\bigvee_{i=1}^{n-1}\mathfrak{M}_i\neq L^2(\mathbf{T})$, then $\bigvee_{i=1}^{n-1}\mathfrak{M}_i=qH^2$ for a unimodular function q. On account of Beurling's theorem there are inner functions $u_i\in H^\infty$ such that $\mathfrak{M}_i=qu_iH^2$ for $i=1, \ldots, n-1$ (cf. [9, Sec. II. 4]). Since $(\bigvee_{i=1}^{n-1}\mathfrak{M}_i)\vee\mathfrak{M}_n= =(qH^2)\vee(q_nH^2)=L^2(\mathbf{T})$ (q_n unimodular), and the multiplication by an inner function is a unitary operator on $L^2(\mathbf{T})$, we conclude that $L^2(\mathbf{T})=u_1L^2(\mathbf{T})= =(u_1qH^2)\vee(u_1q_nH^2)=\mathfrak{M}_1\vee(q_n(u_1H^2))\subset\mathfrak{M}_1\vee\mathfrak{M}_n\subset L^2(\mathbf{T})$, and so $\mathfrak{M}_1\vee\mathfrak{M}_n=L^2(\mathbf{T})$.

Consequently we obtain that $\operatorname{disc} M=2$.

Of course the values of disc M_α ($\alpha\neq\mathbf{T}[m]$) and disc M can be computed from the general formula of the Theorem of [13]. We have given the above simple proof for the sake of the reader's convenience.*

4. Our programme

First of all we remark that in virtue of Wu's results the answers for Problems A and B are affirmative, if we assume that the defect indeces of the contraction $T\in C_1$ are finite. (Cf. [18, Corollary 4.6], [11, Corollary 1] and [13, p. 330].)

We have seen in section 2 that the answer for Problem A is negative in general. Another fact which points out that the general case is more complicated is the following.

For an arbitrary unitary operator U let us denote by $\mathrm{Lat}_1\, U$ the lattice of the reductive subspaces of U. It follows by section 2 that if $\alpha\neq\mathbf{T}[m]$, then $\mathrm{Lat}\, M_\alpha = \mathrm{Lat}_1\, M_\alpha$, while $\mathrm{Lat}\, M\neq\mathrm{Lat}_1\, M$. A natural generalization of a reductive subspace for a C_{11}-contraction T is an invariant subspace $\mathfrak{L}\in\mathrm{Lat}\, T$ such that $T|\mathfrak{L}\in C_{11}$. $\mathrm{Lat}_1\, T$ stands for the set of "C_{11}-invariant subspaces": $\mathrm{Lat}_1\, T=\{\mathfrak{M}\in\mathrm{Lat}\, T: T|\mathfrak{M}\in C_{11}\}$. It was shown in [4] (cf. Remark 3.4) that the property of reductivity "$\mathrm{Lat}\, T=\mathrm{Lat}_1\, T$" is not a quasisimilarity invariant in C_1.

In order to study Problems A′ and B in the general setting it would be very useful to have contractions which are close in a certain sense to some M_α with $\alpha\neq\mathbf{T}[m]$ and to M at the same time. The aim of the present paper is to provide such operators, which may clarify the real situation, perhaps they can be candidates to be counterexamples.

Our construction is based on theorems concerning injective contractions and the approximate reducing point spectrum, which will be proved in the next two sections.

5. Injective contractions

We begin by proving two lemmas which are refinements of [4, Lemma 3.2].

Lemma 2. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a contraction. Then for every $g, u\in\mathfrak{H}$ we have*

$$\|Tg+D_{T^*}^2u\|^2\leqq\|g\|^2+\|D_{T^*}u\|^2.$$

($D_T=(I-T^*T)^{1/2}$ and $D_{T^*}=(I-TT^*)^{1/2}$ are the defect operators of T.)

* After this paper had been submitted, prof. Vasjunin informed me that a direct proof of this proposition can be found in their paper "Control subspaces of minimal dimension, and spectral multiplicities" published in the Proceedings of the 6th Operator Theory Conference, held in Romania.

Proof. Using the identity $D_{T^*}T=TD_T$, we obtain $\langle Tg, D_{T^*}^2u\rangle=\langle D_{T^*}Tg, D_{T^*}u\rangle=\langle TD_Tg, D_{T^*}u\rangle=\langle D_Tg, T^*D_{T^*}u\rangle$. Applying this and the Schwartz inequality, it follows that

$$\|Tg+D_{T^*}^2u\|^2=\|Tg\|^2+2\operatorname{Re}\langle Tg, D_{T^*}^2u\rangle+\|D_{T^*}^2u\|^2=\|Tg\|^2+$$
$$+2\operatorname{Re}\langle D_Tg, T^*D_{T^*}u\rangle+\|D_{T^*}^2u\|^2\leqq\|Tg\|^2+2\|D_Tg\|\,\|T^*D_{T^*}u\|+$$
$$+\|D_{T^*}^2u\|^2\leqq\|Tg\|^2+\|D_Tg\|^2+\|T^*D_{T^*}u\|^2+\|D_{T^*}D_{T^*}u\|^2=\|g\|^2+\|D_{T^*}u\|^2.$$

We recall that, for an operator $T\in\mathscr{L}(\mathfrak{H})$ and a vector $f\in\mathfrak{H}$, $(f, T)\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H})$ denotes the operator defined by

$$(f, T)(\lambda\oplus g):=0\oplus(\lambda f+Tg)\qquad(\lambda\in\mathbf{C},\ g\in\mathfrak{H})\quad\text{(cf. [4])}.$$

Lemma 3. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a non-ivertible injective contraction. Then for every $\varepsilon>0$ there exists a vector $f\in\mathfrak{H}$ such that $\|f\|>1-\varepsilon$ and (f, T) is an injective contraction.*

Proof. If $u\in\mathfrak{H}\setminus\operatorname{ran}T$, then $f=D_{T^*}^2u=(I-TT^*)u\in\mathfrak{H}\setminus\operatorname{ran}T$, and so (f, T) is an injective operator. On account of Lemma 2 (f, T) is a contraction if $\|D_{T^*}u\|\leqq1$, so if $\|u\|\leqq1$. Since $\|u\|^2=\|T^*u\|^2+\|D_{T^*}u\|^2=\|T^*u\|^2+\|T^*D_{T^*}u\|^2+\|D_{T^*}^2u\|^2$, it follows that $\|f\|^2=\|u\|^2-\|T^*u\|^2-\|D_TT^*u\|^2\geqq\|u\|^2-2\|T^*u\|^2$.

The injectivity of T implies that $(\operatorname{ran}T^*)^-=\mathfrak{H}$. Taking into account that T^* is not invertible we infer that T^* is not bounded from below. Therefore for an arbitrary $0<\eta<1$ there exists a unit vector $u_0\in\mathfrak{H}$ such that $\|T^*u_0\|<\eta$.

Since T is injective, non-invertible, the closed graph theorem implies that $\operatorname{ran}T\neq\mathfrak{H}$. It follows that $\mathfrak{H}\setminus\operatorname{ran}T$ is dense in $\mathfrak{H}$. Hence for any $1>\delta>0$ there exists a vector $u\in\mathfrak{H}\setminus\operatorname{ran}T$ such that $\|u-(1-\delta)u_0\|<\delta$. Then $\|u\|\leqq1$, so with $f=D_{T^*}^2u$ the operator (f, T) will be an injective contraction. On the other hand $\|f\|^2\geqq\|u\|^2-2\|T^*u\|^2\geqq\|u\|^2-2(\|T^*u_0\|+\|u-u_0\|)^2$, and since $\|u\|\geqq1-2\delta$, $\|T^*u_0\|<\eta$ and $\|u-u_0\|<2\delta$, it follows that

$$\|f\|^2>(1-2\delta)^2-2(\eta+2\delta)^2\geqq1-16\delta-2\eta^2.$$

We infer that $\|f\|>1-\varepsilon$ if $\eta>0$ and $\delta>0$ are chosen to be small enough, and the proof is complete.

To any operator $T\in\mathscr{L}(\mathfrak{H})$ let us correspond the number

$$\nu(T):=\inf\{\max\{\|Tx\|, \|T^*x\|\}:\ x\in\mathfrak{H}, \|x\|=1\}.$$

Proposition 4. *Let $T\in\mathscr{L}(\mathfrak{H})$ be a non-invertible quasiaffine contraction with $\nu(T)<1/2$. Then for any $\varepsilon>0$ there exist vectors $f, g\in\mathfrak{H}$ such that $\|f\|, \|g\|>1-2\nu(T)-\varepsilon$, $\|f-g\|<2\nu(T)+\varepsilon$ and (f, T), $(g, T^*)\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H})$ are injective contractions.*

Proof. Let $\nu(T)<\eta<1$ be arbitrary. By the assumption there exists a unit vector $u_0\in\mathfrak{H}$ such that $\|Tu_0\|$ and $\|T^*u_0\|<\eta$. The proof of Lemma 3 shows that for any $1>\delta>0$ we can find vectors $u\in\mathfrak{H}\setminus\operatorname{ran} T$ and $v\in\mathfrak{H}\setminus\operatorname{ran} T^*$ such that $\|u-(1-\delta)u_0\|<\delta$ and $\|v-(1-\delta)u_0\|<\delta$. Then, considering the vectors $f=D_{T^*}^2u$ and $g=D_T^2v$, the operators (f, T) and (g, T^*) will be injective contractions while $\|f\|^2, \|g\|^2\geqq 1-16\delta-2\eta$. Furthermore, we have

$$\|f-g\| = \|(I-TT^*)u-(I-T^*T)v\| \leqq \|u-v\|+\|T^*u\|+$$
$$+\|Tv\| \leqq \|u-v\|+\|T^*u_0\|+\|u-u_0\|+\|Tu_0\|+\|v-u_0\| \leqq 6\delta+2\eta.$$

Consequently, we conclude that $\|f\|, \|g\|>1-2\nu(T)-\varepsilon$ and $\|f-g\|<2\nu(T)+\varepsilon$ if η is close enough to $\nu(T)$ and δ is small enough.

Remark 5. Note that if $\nu(T)=0$, that is if 0 is an "approximate reducing eigenvalue" of T, then f and g can be chosen to be arbitrarily close to each other, with norms arbitrarily close to 1.

6. C_1-contractions with approximate reducing eigenvalue 0

In this section we shall show that there is an abundance of C_1-contractions T with $\nu(T)=0$. In proving our results we need the following lemma which will be used in the next section too.

Lemma 6. *Let α be a Borel set on $\mathbf{T}$ such that $m(\alpha)>0$. Then there exists a sequence $\{\beta_n\}_{n=1}^{\infty}$ of closed arcs of $\mathbf{T}$ such that $\beta_n\cap\beta_{n+1}$ consists of exactly one point and $m(\alpha_n)>0$, where $\alpha_n=\beta_n\cap\alpha$, for all n, and $\bigcup_{n=1}^{\infty}\beta_n=\mathbf{T}$.*

Proof. Since $m(\alpha)>0$, it follows that α contains a point C of density 1. Moreover, there exists a sequence $\{C_n\}_{n=1}^{\infty}$ of different density points of α converging to C. For every n, let γ_n be one of the two closed arcs of $\mathbf{T}$ determined by C_n and C. With an appropriate choice of these arcs and passing on to a subsequence, if it is necessary, we can achieve that $\{\gamma_n\}_{n=1}^{\infty}$ be a decreasing sequence of sets. For every n, let B_n be an arbitrary point of the set $\gamma_n\setminus(\gamma_{n+1}\cup\{C_n\})$. Now we define β_n to be the closed arc with endpoints B_{n-1}, B_n and containing C_n, if $n\geqq 2$; while β_1 is the arc with endpoints B_1, C and containing C_1.

Since each β_n contains in its interior a density point of α, it follows that $m(\alpha_n)>0$ for every n. It is easy to see that the sequence $\{\beta_n\}_{n=1}^{\infty}$ possesses the other properties of the statement also.

Theorem 7. *For every Borel set $\alpha\subset\mathbf{T}$ with positive Lebesgue measure there exists a C_1-contraction T such that T is quasisimilar to M_α and $\nu(T)=0$.*

Proof. We first show that for every $\varepsilon>0$ there is a C_1-contraction T such that $T\sim M_\alpha$ and $\nu(T)<\varepsilon$.

By Lemma 6 there exist closed arcs β', β'' of $\mathbf{T}$ such that $\beta'\cup\beta''=\mathbf{T}$, $\beta'\cap\beta''$ has two points and $m(\alpha')>0$, $m(\alpha'')>0$, where $\alpha'=\beta'\cap\alpha$ and $\alpha''=\beta''\cap\alpha$. On account of [4, Proposition 3.1] we can find non-invertible C_1-contractions $T'\in\mathscr{L}(\mathfrak{H}')$ and $T''\in\mathscr{L}(\mathfrak{H}'')$, T' being quasisimilar to $M_{\alpha'}$ and T'' being quasisimilar to $M_{\alpha''}$. Then Lemma 3 ensures us vectors $f\in\mathfrak{H}'$ and $g\in\mathfrak{H}''$ such that $(f, T'^*)\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H}')$ and $(g, T'')\in\mathscr{L}(\mathbf{C}\oplus\mathfrak{H}'')$ are injective contractions. The matrices of these operators are

$$(f, T'^*)=\begin{bmatrix}0 & 0\\ f & T'^*\end{bmatrix} \quad\text{and}\quad (g, T'')=\begin{bmatrix}0 & 0\\ g & T''\end{bmatrix},$$

where f denotes also the operator of rank 1: $f\colon \mathbf{C}\to\mathfrak{H}'$, $f\colon \lambda\mapsto\lambda f$; its adjoint is $f^*\colon \mathfrak{H}'\to\mathbf{C}$, $f^*\colon h\mapsto\langle h, f\rangle$.

Let us consider the Hilbert space $\mathfrak{H}=\mathfrak{H}'\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathfrak{H}''$ and define the operator $T\in\mathscr{L}(\mathfrak{H})$ by the matrix

$$T=\begin{bmatrix} T' & 0 & 0 & 0 & 0\\ f^* & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\varepsilon & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2}\varepsilon & 0 & 0\\ 0 & 0 & 0 & g & T''\end{bmatrix}.$$

It is easy to see that T is a C_{11}-contraction. Since $\|T(0\oplus0\oplus1\oplus0\oplus0)\|=$ $=\|T^*(0\oplus0\oplus1\oplus0\oplus0)\|=(1/2)\varepsilon$, it follows that $\nu(T)<\varepsilon$. Moreover, by [4, Theorem 1.7] the residual part R_T of T (that is the residual part of its unitary dilation) is unitarily equivalent to $R_{T'}\oplus R_{T''}$: $R_T\cong R_{T'}\oplus R_{T''}$. This implies that $R_T\cong M_{\alpha'}\oplus M_{\alpha''}\cong$ $\cong M_\alpha$, and so T is quasisimilar to M_α (cf. [4, Proposition 1.3]).

Let us now prove the existence of a C_1-contraction T with $T\sim M_\alpha$ and $\nu(T)=0$. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of Borel sets corresponding to α by Lemma 6. Then, on account of the first part of this proof, for every n there exists an operator $T_n\in\mathscr{L}(\mathfrak{H}_n)$ such that $T_n\sim M_{\alpha_n}$ and $\nu(T_n)<1/n$. It follows that the direct sum $T=\bigoplus_{n=1}^{\infty} T_n$ of these operators is quasisimilar to M_α and $\nu(T)=0$. The proof is finished.

Now we prove that it can be achieved that the spectrum of the contraction T in the previous theorem be rather thin. We shall need the following:

Lemma 8. *Let $A\in\mathscr{L}(\mathfrak{H}')$, $B\in\mathscr{L}(\mathfrak{H}'')$ be invertible operators. Then the operator T acting on $\mathfrak{H}=\mathfrak{H}'\oplus\mathfrak{H}''$ and defined by the matrix* $\mathrm{T}=\begin{bmatrix}A & 0\\ C & B\end{bmatrix}$ *is also invertible, and* $\|T^{-1}\|\leqq\max\{\|A^{-1}\|, \|B^{-1}\|\}+\|A^{-1}\|\|B^{-1}\|$ *if* $\|C\|\leqq1$.

Proof. It is easy to verify that the inverse of T is $T^{-1}=\begin{bmatrix} A^{-1} & 0 \\ -B^{-1}CA^{-1} & B^{-1}\end{bmatrix}$. Moreover, for the norm of T^{-1} we have

$$\|T^{-1}\| \leqq \left\|\begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1}\end{bmatrix}\right\| + \left\|\begin{bmatrix} 0 & 0 \\ -B^{-1}CA^{-1} & 0\end{bmatrix}\right\| = \max\{\|A^{-1}\|, \|B^{-1}\|\}+\|B^{-1}CA^{-1}\| \leqq$$

$$\leqq \max\{\|A^{-1}\|, \|B^{-1}\|\}+\|B^{-1}\|\|A^{-1}\|.$$

Furthermore we shall use the following notation. If $\alpha\subset\mathbf{T}$ is a Borel set, $m(\alpha)>0$, then $\alpha^=$ stands for the support of the measure $\chi_\alpha dm$ and, for any $\zeta\in\alpha^=$, $D(\alpha, \zeta):=$ $:=\alpha^=\cup\{r\zeta\colon 0\leqq r\leqq 1\}$.

Theorem 9. *Let α be a Borel subset of* $\mathbf{T}$ *such that* $m(\alpha)>0$ *and let* ζ *be any point of* $\alpha^=$. *Then there exists a* C_1*-contraction* T *such that* $T\sim M_\alpha$, $v(T)=0$ *and* $\sigma(T)=D(\alpha, \zeta)$.

We remark that if $v(T)=0$, then $0\in\sigma(T)$, and that for every C_1-contraction T each closed and open part of $\sigma(T)$ intersects $\mathbf{T}$, and that $T\sim M_\alpha$ implies $\sigma(T)\supset\alpha^=$ (cf. [5]). In the light of these facts the spectrum of T in the previous theorem can not be thinner.

Proof. Let $\{\beta_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ be sequences corresponding to α by Lemma 6 such that the cluster point of the endpoints of the arcs β_n is the given ζ.

For every $n\in\mathbf{N}$, the set of natural numbers, let Ω_n denote the domain $\Omega_n=\{r\lambda\colon \lambda$ belongs to the interior of β_n and $1/(n+1)<r<1\}$, and let $\mu_n\in\Omega_n$ be a point such that $|\mu_n|<1/n$. By [5] we can find for every n a C_{11}-contraction $S_n\in\mathscr{L}(\mathfrak{K}_n)$ such that S_n is quasisimilar to M_{α_n}, $\sigma(S_n)=\alpha_n^=$, $\|(S_n-\lambda I)^{-1}\|\leqq\operatorname{dist}(\lambda, \Omega_n^-)^{-1}$ for all $\lambda\in\mathbf{C}\setminus\Omega_n^-$, and $\|(S_n-\mu_n I)x_n\|<1/n$ for a suitable unit vector $x_n\in\mathfrak{K}_n$.

Let us decompose $\mathbf{N}$ into the union of pairwise disjoint sets

$$\mathbf{N} = (\bigcup_{i\in\mathbf{N}} N_i')\cup(\bigcup_{i\in\mathbf{N}} N_i''),$$

each of which contains infinitely many points. For every $i\in\mathbf{N}$, let us define $T_i'\in\mathscr{L}(\mathfrak{H}_i')$ and $T_i''\in\mathscr{L}(\mathfrak{H}_i'')$ by $T_i'=\bigoplus_{n\in N_i'} S_n$ and $T_i''=\bigoplus_{n\in N_i''} S_n$, respectively.

Since $\inf_{n\in N_i'}\|(S_n-\mu_n I)x_n\|=\inf_{n\in N_i''}\|(S_n-\mu_n I)x_n\|=0$, it follows that T_i' and T_i'' are not invertible. So we infer by Lemma 3 that there exist vectors $f_i\in\mathfrak{H}_i'$ and $g_i\in\mathfrak{H}_i''$ such that $(f_i, T_i'^*)$ and (g_i, T_i'') are injective contractions. Now we define the operator T_i acting on the Hilbert space $\mathfrak{H}_i=\mathfrak{H}_i'\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathbf{C}\oplus\mathfrak{H}_i''$ by the matrix

$$T_i=\begin{bmatrix} T_i' & 0 & 0 & 0 & 0 \\ f_i^* & 0 & 0 & 0 & 0 \\ 0 & (i+1)^{-1} & 0 & 0 & 0 \\ 0 & 0 & (i+1)^{-1} & 0 & 0 \\ 0 & 0 & 0 & g_i & T_i'' \end{bmatrix}.$$

Finally, our operator $T\in\mathscr{L}(\mathfrak{H})$ is defined as the orthogonal sum of these operators: $T=\bigoplus_{i\in\mathbf{N}} T_i$.

It is immediate that T is a contraction. We can show as in the proof of Theorem 7 that for every i

$$T_i\sim T_i'\oplus T_i''\cong\bigoplus_{n\in N_i'\cup N_i''} S_n\sim\bigoplus_{n\in N_i'\cup N_i''} M_{\alpha_n},$$

and so

$$T\sim\bigoplus_{i\in\mathbf{N}}\Big(\bigoplus_{n\in N_i'\cup N_i''} M_{\alpha_n}\Big)\cong\bigoplus_{n\in\mathbf{N}} M_{\alpha_n}\cong M_\alpha,$$

i.e. T is quasisimilar to M_α. It follows that T belongs to the class C_1.

Since $v(T_i)<1/i$ for every i, we infer that $v(T)=0$.

Let us assume now that $\lambda\in\mathbf{C}\diagdown D(\alpha,\zeta)$. Then, for every $n\in\mathbf{N}$, the operator $S_n-\lambda I$ is invertible. Moreover, for all but at most two n, λ belongs to $\mathbf{C}\diagdown\Omega_n^-$. For these indeces $\|(S_n-\lambda I)^{-1}\|\leqq\operatorname{dist}(\lambda,\Omega_n^-)^{-1}$, and the sequence on the right side is bounded. Hence we conclude that $\{\|(S_i-\lambda I)^{-1}\|\}_{i\in\mathbf{N}}$ is bounded, and so applying Lemma 8 we obtain that $\{\|(T_i-\lambda I)^{-1}\|\}_{i\in\mathbf{N}}$ is bounded too. Therefore, we infer that $T-\lambda I$ is invertible, i.e. $\lambda\notin\sigma(T)$.

On the other hand $v(T)=0$ implies that $0\in\sigma(T)$, moreover on account of [5] we know that $\alpha^-\subset\sigma(T)$ and that every closed and open subset of $\sigma(T)$ intersects the unit circle. Consequently, we obtain that $\sigma(T)=D(\alpha,\zeta)$.

We remark that with the additional assumption that the defect number of T is 1 the statement of Theorem 7 becomes false. Namely, the following holds:

Proposition 10. *The number* $v_1=\inf\{v(T)\colon T\in C_1$ *with defect index* $1\}$ *is strictly positive.*

Proof. It is evident that we can restrict our attention to c.n.u. C_1-contractions with defect index 1. We shall consider the functional models of these contractions (cf. [17, chapter VI]).

So let H_e^∞ denote the set of outer functions ϑ in the (scalar) Hardy space H^∞ such that $|\vartheta(e^{it})|\leqq 1$ a.e. and ϑ is not a constant of absolute value 1. To any $\vartheta\in H_e^\infty$ there corresponds a Hilbert space

$$\mathfrak{H}(\vartheta)=[H^2\oplus(\Delta L^2)^-]\ominus\{\vartheta w\oplus\Delta w\colon w\in H^2\},$$

where $\Delta(e^{it})=(1-|\vartheta(e^{it})|^2)^{1/2}$, and an operator $S(\vartheta)\in\mathcal{L}(\mathfrak{H}(\vartheta))$ defined by $S(\vartheta)=$ $=P_{\mathfrak{H}(\vartheta)}U|\mathfrak{H}(\vartheta)$, where U denotes the operator of multiplication by e^{it} on $H^2\oplus L^2$ and $P_{\mathfrak{H}(\vartheta)}$ is the orthogonal projection in $H^2\oplus L^2$ onto the subspace $\mathfrak{H}(\vartheta)$. The operator $S(\vartheta)$ is a C_1-contraction of defect index 1 and being quasisimilar to M_α for $\alpha=\{e^{it}\in\mathbf{T}:\ |\vartheta(e^{it})|<1\}$ (cf. [17, Proposition VI. 3.5] and [11, Corollary 1]). Moreover, in this way we obtain all c.n.u. C_1-contractions of defect index 1 up to unitary equivalence.

So we have to prove that the infimum

$$\nu_1=\inf\{\nu(S(\vartheta)):\ \vartheta\in H_e^\infty\}$$

is not equal to zero.

Let $\vartheta\in H_e^\infty$ be an arbitrary function. The Hilbert space $\mathfrak{H}(\vartheta)$ can be decomposed into the orthogonal sums

$$\mathfrak{H}(\vartheta)=\mathfrak{D}_{S(\vartheta)}\oplus\mathfrak{D}_{S(\vartheta)}^{\perp}=\mathfrak{D}_{S(\vartheta)^*}\oplus\mathfrak{D}_{S(\vartheta)^*}^{\perp},$$

where $\mathfrak{D}_{S(\vartheta)}=(\ker D_{S(\vartheta)})^{\perp}$ and $\mathfrak{D}_{S(\vartheta)^*}=(\ker D_{S(\vartheta)^*})^{\perp}$ are 1-dimensional subspaces, the so-called defect subspaces of $S(\vartheta)$. Since $S(\vartheta)|\mathfrak{D}_{S(\vartheta)}^{\perp}$: $\mathfrak{D}_{S(\vartheta)}^{\perp}\to\mathfrak{D}_{S(\vartheta)^*}^{\perp}$ is an isometric surjection with inverse $S(\vartheta)^*|\mathfrak{D}_{S(\vartheta)^*}^{\perp}$: $\mathfrak{D}_{S(\vartheta)^*}^{\perp}\to\mathfrak{D}_{S(\vartheta)}^{\perp}$, and for appropriate unit vectors $g_0\in\mathfrak{D}_{S(\vartheta)}$ and $h_0\in\mathfrak{D}_{S(\vartheta)^*}$ we have $S(\vartheta)g_0=\vartheta(0)h_0$ and $S(\vartheta)^*h_0=$ $=\overline{\vartheta(0)}g_0$ (cf. [17, Sec. VI. 4]), it follows that $\nu_1=0$ if and only if for a sequence $\{\vartheta_n\}_{n=1}^\infty$ in H_e^∞ $\vartheta_n(0)$ and the distance of the subspaces $\mathfrak{D}_{S(\vartheta_n)}$ and $\mathfrak{D}_{S(\vartheta_n)^*}$ tend simultaneously to zero. Under the latter distance we mean the distance of the unit spheres of these subspaces, i.e.

$$\operatorname{dist}(\mathfrak{D}_{S(\vartheta_n)},\mathfrak{D}_{S(\vartheta_n)^*})=\inf\{\|x-y\|:\ x\in\mathfrak{D}_{S(\vartheta_n)},\ \|x\|=1,\ y\in\mathfrak{D}_{S(\vartheta_n)^*},\ \|y\|=1\}.$$

An easy computation shows that for any $u\oplus v\in\mathfrak{H}(\vartheta)$ $(\vartheta\in H_e^\infty)$

$$(I-S(\vartheta)S(\vartheta)^*)(u\oplus v)=u(0)P_{\mathfrak{H}(\vartheta)}(1\oplus 0),$$

and again a usual computation yields that

$$P_{\mathfrak{H}(\vartheta)}(1\oplus 0)=(1-\overline{\vartheta(0)}\vartheta)\oplus-\overline{\vartheta(0)}\Delta=:h.$$

The norm of h is $\|h\|=(1-|\vartheta(0)|^2)^{1/2}\neq 0$ and $\{h_0=h/\|h\|\}$ forms an orthonormal basis in $\mathfrak{D}_{S(\vartheta)^*}$. Then

$$g=S(\vartheta)^*h=e^{-it}\overline{\vartheta(0)}(\vartheta(0)-\vartheta)\oplus-e^{-it}\overline{\vartheta(0)}\Delta$$

belongs to $\mathfrak{D}_{S(\vartheta)}$, $\|g\|=|\vartheta(0)|\ \|h\|$ and $\{g_0=g/\|g\|\}$ is a basis in $\mathfrak{D}_{S(\vartheta)}$.

The distance of the subspaces $\mathfrak{D}_{S(\vartheta)}$ and $\mathfrak{D}_{S(\vartheta)^*}$ is by our definition

$$d(\vartheta):=\operatorname{dist}(\mathfrak{D}_{S(\vartheta)},\mathfrak{D}_{S(\vartheta)^*})=\inf\{\|\alpha h_0-g_0\|:\ \alpha\in\mathbf{C},\ |\alpha|=1\}.$$

Since $\|\alpha h_0-g_0\|^2=2\left(1-\dfrac{\operatorname{Re}(\alpha\langle h,g\rangle)}{\|h\|\,\|g\|}\right)$, it follows that $d(\vartheta)=\sqrt{2}\left(1-\dfrac{|\langle h,g\rangle|}{\|h\|\,\|g\|}\right)^{1/2}$.

A direct computation shows that $\langle h, g\rangle = -\vartheta(0)\vartheta'(0)$, and so we obtain that

$$d(\vartheta) = \sqrt{2}\left(1 - \frac{|\vartheta'(0)|}{1-|\vartheta(0)|^2}\right)^{1/2}.$$

Being an outer function ϑ has the form

$$\vartheta(\lambda) = \varkappa \exp\left[\int_{\mathbf{T}} \frac{e^{it}+\lambda}{e^{it}-\lambda} \log|\vartheta(e^{it})|\, \mathrm{dm}(t)\right] \quad (\lambda \in D),$$

where $\varkappa \in \mathbf{T}$ is a constant of absolute value 1, and D denotes the open unit disc in $\mathbf{C}$. We infer that

$$|\vartheta(0)| = \exp\left[\int_{\mathbf{T}} \log|\vartheta(e^{it})|\, dm(t)\right] \text{ and } |\vartheta'(0)| = 2|\vartheta(0)| \left|\int_{\mathbf{T}} e^{-it} \log|\vartheta(e^{it})|\, dm(t)\right|.$$

Let us assume now that $\vartheta_n(0)$ tends to 0 for a sequence $\{\vartheta_n\}_{n=1}^{\infty}$ in H_e^{∞}. Then $y_n = -\int_{\mathbf{T}} \log|\vartheta_n(e^{it})|\, dm(t)$ tends to infinity and so in virtue of $|\vartheta_n'(0)| \leqq 2\exp(-y_n) y_n$ it follows that $\vartheta_n'(0)$ converges to 0. We conclude that $\lim_{n\to\infty} \vartheta_n(0) = 0$ implies $\lim_{n\to\infty} d(\vartheta_n) = \sqrt{2}$. Therefore, the number ν_1 is not 0.

7. The construction

Let α be an arbitrary subset of $\mathbf{T}$ such that $m(\alpha) > 0$. Applying Lemma 6 we can find a sequence $\{\beta_n\}_{n=-\infty}^{\infty}$ of closed arcs of $\mathbf{T}$ such that $\beta_n \cap \beta_{n+1}$ consists of exactly one point, $m(\alpha_n) > 0$, where $\alpha_n = \beta_n \cap \alpha$, for every $n \in \mathbf{Z}$ (the set of integers), and $\bigcup_{n=-\infty}^{\infty} \beta_n$ covers the whole $\mathbf{T}$ except one point.

For every $n \in \mathbf{Z}$, Theorem 7 ensures us a C_1-contraction $T_n \in \mathscr{L}(\mathfrak{H}_n)$ with $\nu(T_n) = 0$ and being quasisimilar to M_{α_n}. Then the orthogonal sum $T' = \bigoplus_{n=-\infty}^{\infty} T_n \in \mathscr{L}(\mathfrak{H} = \bigoplus_{n=-\infty}^{\infty} \mathfrak{H}_n)$ is quasisimilar to $\bigoplus_{n=-\infty}^{\infty} M_{\alpha_n} \cong M_\alpha$.

Let us given a sequence $\{\varepsilon_n\}_{n=-\infty}^{\infty}$ of positive numbers such that $0 < \varepsilon_n < 1$ for every $n \in \mathbf{Z}$ and $\lim_{|n|\to\infty} \varepsilon_n = 0$. Such sequences will be called *admissible*. On account of Proposition 4 we can find, for every $n \in \mathbf{Z}$, vectors $f_n, g_n \in \mathfrak{H}_n$ such that $\|f_n\|$, $\|g_n\| > 1 - \varepsilon_n$, $\|f_n - g_n\| < \varepsilon_n$ and $(f_n, T_n), (g_n, T_n^*) \in \mathscr{L}(\mathbf{C} \oplus \mathfrak{H}_n)$ are injective contractions. Now we define the operator $T'' \in \mathscr{L}(\mathfrak{H})$ to be the sum $T'' = \sum_{n=-\infty}^{\infty} f_{n+1} \otimes g_n$, where for any n $f_{n+1} \otimes g_n \in \mathscr{L}(\mathfrak{H}_n, \mathfrak{H}_{n+1})$ denotes the operator of rank 1: $(f_{n+1} \otimes g_n) h = \langle h, g_n\rangle f_{n+1}$ $(h \in \mathfrak{H}_n)$, and the partial sums of the series converge in the strong operator topology.

Definition. We call the operator $S\in\mathcal{L}(\mathfrak{H})$ to be a *quasibilateral shift*, if there exists a sequence $\{\mathfrak{L}_n\}_{n=-\infty}^{\infty}$ of pairwise orthogonal subspaces and for every $n\in\mathbf{Z}$ there exist vectors $f_n, g_n\in\mathfrak{L}_n$ such that $\{\|f_n\|\}_{n=-\infty}^{\infty}$, $\{\|g_n\|\}_{n=-\infty}^{\infty}$ are bounded sequences and $S=\sum_{n=-\infty}^{\infty} f_{n+1}\otimes g_n$.

We say that the quasibilateral shift S *assymptotically approximates the bilateral shift in order* ε, where $\varepsilon=\{\varepsilon_n\}_{n=-\infty}^{\infty}$ is an admissible sequence, if the sequences $\{f_n\}_{n=-\infty}^{\infty}$, $\{g_n\}_{n=-\infty}^{\infty}$ fulfill the following relation:

$$\max\{|\|f_n\|-1|,\ |\|g_n\|-1|,\ \|f_n-g_n\|\}<\varepsilon_n$$

for every $n\in\mathbf{Z}$.

We note that every (simple) bilateral shift is unitarily equivalent to M.

It is evident that the sum $T=T'+T''$ of the contractions T', T'' obtained before is an injective contraction with dense range. With the notions introduced above our result can be formulated as follows:

Theorem 11. *Let α be a subset of $\mathbf{T}$ such that $m(\alpha)>0$ and let $\varepsilon\in\{\varepsilon_n\}_{n=-\infty}^{\infty}$ be an admissible sequence. Then there exist a C_1-contraction $T'\in\mathcal{L}(\mathfrak{H})$ which is quasisimilar to M_α and a contractive quasibilateral shift $T''\in\mathcal{L}(\mathfrak{H})$ which assymptotically approximates the bilateral shift in order ε such that their sum $T=T'+T''$ is a quasiaffine contraction on $\mathfrak{H}$.*

The contraction T is close, in different senses, both to M_α and to M. Unfortunately, we are not able to prove yet that T can be a C_{11}-contraction. However, by modifying our construction and assuming that some subspaces of $\mathfrak{H}$ are C_{11}-semiinvariant for T, we can show that T is quasisimilar to M_α.

Namely, taking into account Theorem 9 we can achieve that the spectrum of every contraction $T_n\in\mathcal{L}(\mathfrak{H}_n)$ considered be $D(\alpha_n, \zeta_n)$, where ζ_n is an arbitrary fixed point of α_n^-. Let $f_n, g_n\in\mathfrak{H}_n$ be as before, and let us introduce the operators $T_1'\in\mathcal{L}(\mathfrak{K}_1)$, $T_2'\in\mathcal{L}(\mathfrak{K}_2)$, $T''\in\mathcal{L}(\mathfrak{K}_1)$ as follows:

$$T_1'=\bigoplus_{k=-\infty}^{\infty} T_{2k},\ T_2'=\bigoplus_{k=-\infty}^{\infty} T_{2k+1},\ T''=\sum_{k=-\infty}^{\infty} f_{2(k+1)}\otimes g_{2k},$$

where $\mathfrak{K}_1=\bigoplus_{k=-\infty}^{\infty}\mathfrak{H}_{2k}$ and $\mathfrak{K}_2=\bigoplus_{k=-\infty}^{\infty}\mathfrak{H}_{2k+1}$. Now we define the quasiaffine contraction $T\in\mathcal{L}(\mathfrak{H})$ by

$$T=(T_1'+T'')\oplus T_2'.$$

Let $\mathfrak{M}$ and $\mathfrak{N}$ denote the subspaces $\mathfrak{M}=\bigoplus_{k=0}^{\infty}\mathfrak{H}_{2k}$, $\mathfrak{N}=\bigoplus_{k=-\infty}^{-1}\mathfrak{H}_{2k}$. It is evident that $T|\mathfrak{M}\in C_{1\cdot}$ and $P_{\mathfrak{N}}T|\mathfrak{N}\in C_{\cdot 1}$. Moreover, we shall prove:

Theorem 12. *If $T|\mathfrak{M}$ and $P_{\mathfrak{N}}T|\mathfrak{N}$ are C_{11}-contractions, then T is quasisimilar to M_α.*

Proof. We have to show that $T|\mathfrak{K}_1$ is quasisimilar to $\bigoplus_{k=-\infty}^{\infty} M_{\alpha_{2k}}$.

Cosidering the matrix $T|\mathfrak{K}_1=\begin{bmatrix} T|\mathfrak{M} & * \\ 0 & P_{\mathfrak{N}}T|\mathfrak{N}\end{bmatrix}$ of $T|\mathfrak{K}_1$ in the decomposition $\mathfrak{K}_1=\mathfrak{M}\oplus\mathfrak{N}$ the assumption $T|\mathfrak{M}$, $P_{\mathfrak{N}}T|\mathfrak{N}\in C_{11}$ immediately implies that $T\in C_{11}$. Therefore, T is quasisimilar to its residual part R_T (cf. [4, Proposition 1.3]). On account of [4, Theorem 1.7] R_T is unitarely equivalent to $R_{T|\mathfrak{M}}\oplus R_{P_{\mathfrak{N}}T|\mathfrak{N}}\oplus R_{T|\mathfrak{K}_2}$.

We shall prove that $R_{T|\mathfrak{M}}\cong\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}$. Applying [4, Theorem 1.7] several times we obtain that $R_{T|\mathfrak{M}}\cong\left(\bigoplus_{k=0}^{n} R_{T_{2k}}\right)\oplus R_{T|\bigoplus_{k=n+1}^{\infty}\mathfrak{H}_{2k}}\cong\left(\bigoplus_{k=0}^{n} M_{\alpha_{2k}}\right)\oplus R_{T|\bigoplus_{k=n+1}^{\infty}\mathfrak{H}_{2k}}$ for every $n\in\mathbf{N}$. Taking into account the functional model of $R_{T|\mathfrak{M}}$ (cf. [6, Theorem X.10]), we infer that $M'=\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}$ can be injected into $R_{T|\mathfrak{M}}$: $M'\overset{i}{\prec}R_{T|\mathfrak{M}}$, that is some injective operator X intertwines these operators: $XM'=R_{T|\mathfrak{M}}X$.

Next we want to show that $T|\mathfrak{M}\overset{i}{\prec}M'$. We are looking for an injection X such that $X(T|\mathfrak{M})=M'X$. Let us consider the matrix $[X_{ij}]_{i,j=0}^{\infty}$ of X with respect to the decompositions $\mathfrak{M}=\bigoplus_{k=0}^{\infty}\mathfrak{H}_{2k}$ and $\mathfrak{E}'=\bigoplus_{k=0}^{\infty}\mathfrak{E}_k$, where $\mathfrak{E}'$, $\mathfrak{E}_k$ are the domains of M' and $M_k:=M_{\alpha_{2k}}$, respectively. The commuting relation above can be expressed by the equations

$$(*)\qquad M_iX_{ij}-X_{ij}T_{2j}=(X_{i,j+1}f_{2(j+1)})\otimes g_{2j}\quad (i,j\in\mathbf{N}).$$

Since T_{2i} is quasisimilar to M_i ($i\in\mathbf{N}$ is arbitrary), we can find an intertwining quasiaffinity $X_i\in\mathscr{L}(\mathfrak{H}_{2i},\mathfrak{E}_i)$ such that $M_iX_i=X_iT_{2i}$. Let us define X_{ij} to be zero if $j>i$, and $X_{ii}=X_i$ for every $i\in\mathbf{N}$. Then equality $(*)$ holds, whenever $i\leqq j$.

Let us now assume that $0\leqq j<i$. Since $\sigma(M_i)=\alpha_{2i}^{=}$, $\sigma(T_{2j})=D(\alpha_{2j},\zeta_{2j})$ and $\alpha_{2i}^{=}\cap D(\alpha_{2j},\zeta_{2j})=\emptyset$, it follows by Rosenblum's theorem (cf. [14, Theorem 3.1]) that X_{ij} can be expressed from $(*)$ by the integral formula:

$$X_{ij}=\frac{1}{2\pi i}\int_{\Gamma_{ij}}(M_i-\lambda)^{-1}\big((X_{i,j+1}f_{2(j+1)})\otimes g_{2j}\big)(\lambda-T_{2j})^{-1}\,d\lambda,$$

where Γ_{ij} is a rectifiable Jordan curve surrounding $\alpha_{2i}^{=}$ and containing $D(\alpha_{2j},\zeta_{2j})$ in its exterior. We obtain X_{ij} for $j<i$ successively from X_i by this formula.

It is easy to see that for every $i\in\mathbf{N}$ there exists a constant K_i such that $\sum_{j=0}^{\infty}\|X_{ij}\|\leqq K_i\|X_i\|$. Since X_i's can be chosen with arbitrary small norms, we can achieve that $\sum_{i,j=0}^{\infty}\|X_{ij}\|<\infty$ hold for the operators defined above. Then the matrix $[X_{ij}]_{i,j=0}^{\infty}$ actually defines an operator $X\in\mathscr{L}(\mathfrak{M},\mathfrak{E}')$ (cf. [7, Sec. 36]), which will intertwine $T|\mathfrak{M}$ and M', and is evidently injective. Therefore, $T|\mathfrak{M}$ can be injected into M'.

Being a C_{11}-contraction, $T|\mathfrak{M}$ is quasisimilar to its residual part $R_{T|\mathfrak{M}}$, and we conclude by the chain of relations

$$R_{T|\mathfrak{M}} \sim T|\mathfrak{M} \overset{i}{\prec} M' \overset{i}{\prec} R_{T|\mathfrak{M}}$$

that $R_{T|\mathfrak{M}}$ is unitarily equivalent to M'. (Cf. [12, Lemma 6] and [17, Proposition II.3.4].)

An analogous argumentation yields that $R_{P_{\mathfrak{N}}T|\mathfrak{N}} \cong \bigoplus_{k=-\infty}^{-1} M_{\alpha_{2k}}$. Consequently $R_T \cong R_{T|\mathfrak{M}} \oplus R_{P_{\mathfrak{N}}T|\mathfrak{N}} \oplus R_{T|\mathfrak{K}_2} \cong (\bigoplus_{k=0}^{\infty} M_{\alpha_{2k}}) \oplus (\bigoplus_{k=-\infty}^{-1} M_{\alpha_{2k}}) \oplus (\bigoplus_{k=-\infty}^{\infty} M_{\alpha_{2k+1}}) \cong \bigoplus_{n=-\infty}^{\infty} M_{\alpha_n} \cong$ $\cong M_\alpha$, and so T is quasisimilar to M_α. The proof is finished.

References

[1] H. Bercovici, C. Foiaş, J. Langsam and C. Pearcy, (BCP)-operators are reflexive, *Michigan Math. J.*, **29** (1982), 371—379.

[2] H. Bercovici, C. Foiaş and C. Pearcy, Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I, *Michigan Math. J.*, **30** (1983), 335—354.

[3] H. Bercovici, C. Foiaş and B. Sz.-Nagy, Reflexive and hyper-reflexive operators of class C_0, *Acta Sci. Math.*, **43** (1981), 5—13.

[4] H. Bercovici and L. Kérchy, Quasisimilarity and properties of the commutant of C_{11} contractions, *Acta Sci. Math.*, **45** (1983), 67—74.

[5] H. Bercovici and L. Kérchy, On the spectra of C_{11} contractions, *Proc. Amer. Math. Soc.*, to appear.

[6] N. Dunford and J. Schwartz, *Linear operators. II*, Wiley-Interscience (New York—London, 1963).

[7] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand (Princeton—Toronto—London, 1967).

[8] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76** (1970), 887—933.

[9] H. Helson, *Lectures on invariant subspaces*, (New York—London, 1964).

[10] K. Hoffman, *Banach spaces of analytic functions*, (Englewood Cliffs, N. J., 1962).

[11] L. Kérchy, On the commutant of C_{11}-contractions, *Acta Sci. Math.*, **43** (1981), 15—26.

[12] L. Kérchy, On invariant subspace lattices of C_{11}-contractions, *Acta Sci. Math.*, **43** (1981), 281—293.

[13] N. K. Nikolskii and V. I. Vasjunin, Control subspaces of minimal dimension, unitary and model operators, *J. Operator Theory*, **10** (1983), 307—330.

[14] M. Rosenblum, On the operator equation $BX-XA=Q$, *Duke Math. J.*, **23** (1956), 263—269.

[15] W. Rudin, *Real and Complex Analysis*, McGraw-Hill (New York, 1966).

[16] D. Sarason, Invariant subspaces, *Topics in Operator Theory, Math. Surveys, 13*, Amer. Math. Soc. (Providence, R. I., 1974), 1—47.

[17] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland —Akadémiai Kiadó (Amsterdam—Budapest, 1970).

[18] P. Y. Wu, Bi-invariant subspaces of weak contractions, *J. Operator Theory*, **1** (1979), 261—272.

BOLYAI INSTITUTE
UNIVERSITY SZEGED
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY