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Local upper estimates for the eigenfunctions of a linear differential operator

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Dedicated to Professor Károly Tandori on his 60th birthday

Let $G\subset\mathbf{R}$ be an arbitrary open interval, $n\in\mathbf{N}$, $q_1, \ldots, q_n\in L^1_{\mathrm{loc}}(G)$ arbitrary complex functions, and consider the differential operator

$$Lu = u^{(n)}+q_1u^{(n-1)}+\ldots+q_nu.$$

We recall the definition of the eigenfunctions of higher order:

Given a complex number λ, the function $u\colon G\to\mathbf{C}$, $u\equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. A function $u\colon G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \ldots)$ of the operator L with the eigenvalue λ if the following two conditions are satisfied:

— u, together with its first $n-1$ derivatives is absolute continuous on every compact subinterval of G;

— there exists an eigenfunction u^* of order $m-1$ of the operator L with the eigenvalue λ such that for almost all $x\in G$

$$(Lu)(x) = \lambda u(x)+u^*(x). \tag{1}$$

Let u be an eigenfunction of order m $(m=0, 1, \ldots)$ of the operator L with some eigenvalue λ. Let us index the n-th roots of λ such that

$$\operatorname{Re}\mu_1 \geqq \ldots \geqq \operatorname{Re}\mu_n. \tag{2}$$

It is known (see the references below) that to any compact subinterval K of G there exists a constant $G=G_m$ such that

$$\|u\|_{L^\infty(K)} \leqq C\big(1+\max_{1\leqq p\leqq n}|\operatorname{Re}\mu_p|\big)\|u\|_{L^1(K)} \tag{3}$$

and for $|\lambda|$ sufficiently large

$$\|u^{(i)}\|_{L^\infty(K)} \leqq C\big(1+\max_{1\leqq p\leqq n}|\operatorname{Re}\mu_p|\big)\|u^{(i)}\|_{L^1(K)} \quad (i = 1, \ldots, n-1). \tag{4}$$

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Furthermore, if $q_1; \dots, q_n \in L^p_{loc}(G)$ for some $p \in [1, \infty]$ then

(5) $$\|u^*\|_{L^{p'}(K)} \leqq C(1+|\mu_1|)^{n-1}(1+\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|)\|u\|_{L^{p'}(K)}.$$

The constant C does not depend on the choice of u.

Remarks. (i) If $n \geqq 3$ then the quantity $\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|$ may be replaced by $|\mu_1| = |\lambda|^{1/n}$: they are equivalent. One can see easily by counterexamples that the above estimates are the best possible.

(ii) The estimates (3), (4), (5) were proved in [3] for the case $n=2$ and $q_1 \equiv 0$ (see also [2]), in [5] for the case $n \geqq 3$ and $q_1 \equiv 0$; in the general case $q_1 \not\equiv 0$, using the results of the paper [5], they were proved in [6] for the case $n \geqq 3$ and in [7] for the case $n \leqq 2$.

The aim of this paper is to show that if we replace the compact interval K on the right side of the estimates (3), (4), (5) by another compact interval, strictly containing K, then the terms $(1+\max_{1\leqq p\leqq n} |\operatorname{Re}\mu_p|)$ can be omitted. This phenomenon plays an important role in the local investigation of spectral expansions.

Remarks. (i) The first results of this type were proved by V. A. Il'in [1] and were used to prove a general local basis theorem. For the proof he used the following condition: putting

(6) $$\mu = \begin{cases} [(-1)^{n/2}\lambda]^{1/2} & \text{if } n \text{ is even,} \\ [i\lambda]^{1/n} & \text{if } n \text{ is odd and } \operatorname{Im}\lambda \leqq 0, \\ [-i\lambda]^{1/n} & \text{if } n \text{ odd and } \operatorname{Im}\lambda > 0 \end{cases}$$

where

$$[re^{i\varphi}]^{1/n} = r^{1/n}e^{i\varphi/n}, \quad -\frac{\pi}{2} < \varphi \leqq \frac{3\pi}{2},$$

the existence of a constant C was proved for any fixed band

(7) $$|\operatorname{Im}\mu| \leqq C_1 \quad (C_1 \text{ is constant}).$$

Also, the coefficients of the differential operator were assumed to be sufficiently smooth. As we shall see, the above conditions can be omitted.

(ii) In the proof of Theorem 2 of this paper we shall use a formula obtained in [5] for the coefficients of which very simple explicit formulas were found by by Joó [4]. This will play an important role in the proof.

In the sequel we shall use the following notations:

(8) $$n' = \left[\frac{n+1}{2}\right], \; N' = n'(m+1), \; N = n(m+1), \; \mu = \mu_n, \; \varrho = |\operatorname{Re}\mu|.$$

Obviously

(9) $$\varrho = \min \{|\mathrm{Re}\, \mu_p|\colon p = 1, \dots, n\}$$

and

(10) $$\varrho = \mathrm{Re}\, \mu \quad \text{if} \quad n \quad \text{is} \quad \text{even.}$$

Remark. Suppose n is odd and consider the operator

$$\tilde{L}u = u^{(n)} + \tilde{q}_1 u^{(n-1)} + \dots + \tilde{q}_n u$$

on the interval $\tilde{G} := -G$ where $\tilde{q}_s(y) := (-1)^s q_s(-y)$. Then, for any eigenfunction u of order m of the operator L with some eigenvalue λ, the function $\tilde{u}(y) := u(-y)$ is an eigenfunction of order m of the operator $\tilde{L}$ with the eigenvalue $-\lambda$. This correspondence makes us possible to consider always the case $\mathrm{Re}\, \mu \geqq 0$ i.e. $\varrho = \mathrm{Re}\, \mu$.

In the sequel $u = u_m$ will denote an arbitrary eigenfunction of order m of the operator L with some eigenvalue λ. Let us introduce recursively the continuous functions

$$u_j\colon G \to \mathbf{C}, \quad u_j = Lu_{j+1} - \lambda u_{j+1} \quad \text{a.e. on } G$$

for $0 \leqq j \leqq m-1$. Then u_j is an eigenfunction of order j of the operator L with the eigenvalue λ and $u_{m-1} = u^*$.

1. Local "anti a priori" estimates. In this section we shall prove the following result:

Theorem 1. *Assume $q_1 \equiv 0$ and $q_2, \dots, q_n \in L^p_{\mathrm{loc}}(G)$ for some $p \in [1, \infty]$. Then to any $m \in \{0, 1, \dots\}$ and to arbitrary compact intervals $K_1, K_2 \subset G$, $K_1 \subset \mathrm{int}\, K_2$, there exists a constant C such that for any eigenfunction u of order m of the operator L with some eigenvalue $\lambda = \mu^n$,*

(11) $$\|u^*\|_{L^p(K_1)} \leqq C(1+|\mu|)^{n-1} \|u\|_{L^p(K_2)}.$$

The proof will be based on the following assertion:

Proposition 1. *Given $0 \neq \mu \in \mathbf{C}$ and $t \in \mathbf{R}$ arbitrarily, there exist numbers $d(\mu, t)$, $d_k(\mu, t)$ and continuous functions $D_r(\mu, t, \cdot)$ such that for any eigenfunctions u of order m of the operator L with the eigenvalue $\lambda = \mu^n$,*

(12) $$t^n d(\mu, t) u_{m-1}(x) =$$

$$= \sum_{k=N'-N+1}^{N'} d_k(\mu, t) u_m(x+kt) + \sum_{r=0}^{m} \sum_{s=1}^{n} \int_{x+(N'-N+1)t}^{x+N't} D_r(\mu, t, x-\tau)\, q_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau$$

whenever $x+(N'-N+1)t \in G$ and $x+N't \in G$. Furthermore, introducing the no-

tation

(13)
$$P(\mu, t) = (\mu t)^{n(1+\dots+m)} \exp\Big(\sum_{i=n'-n+1}^{n'} \big(i(m+1)+\dots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t \Big),$$

there exist positive constants C_1, C_2 *and to any fixed positive number* A *a positive constant* C *such that*

$$|d_k(\mu, t)| \leqq C|\mu|^{n-1}|P(\mu, t)|e^{-|k|\varrho t} \quad \textit{for all } k, \tag{14}$$

$$|D_r(\mu, t, x-\tau)| \leqq C|\mu|^{r(1-n)}|P(\mu, t)|e^{-\varrho|x-\tau|} \tag{15}$$

for all $x+(N'-N+1)t \leqq \tau \leqq x+N't$,

$$\sup_{t/2 \leqq t_0 \leqq t} |d(\mu, t_0)| > C_1|P(\mu, t_0)| \tag{16}$$

whenever

$$\operatorname{Re}\mu \geqq 0, \quad 0 \leqq t \leqq A \quad \textit{and} \quad |\mu t| \geqq C_2. \tag{17}$$

First we deduce Theorem 1 from Proposition 1. For $m=0$ the theorem is obvious because $u^* \equiv 0$. Assume $m \geqq 1$ and that the theorem is true for $m-1$. Let us fix a compact interval $K \subset G$ such that

$$K_1 \subset \operatorname{int} K \quad \text{and} \quad K \subset \operatorname{int} K_2$$

and put

$$\varepsilon = (N')^{-1} \text{ dist } (K_1, \partial K).$$

It suffices to consider the case $\operatorname{Re}\mu \geqq 0$ in view of (10) and the Remark after (10). For $|\mu|$ sufficiently large we can fix a number $t \in [\varepsilon/2, \varepsilon]$ by Proposition 1 such that

$$|d_k(\mu, t)| \leqq C|\mu|^{n-1}|d(\mu, t)|, \quad |D_r(\mu, t, x-\tau)| \leqq C|\mu|^{r(1-n)}|d(\mu, t)|.$$

Fixing t by this manner, we have by (12) for any $x \in K_1$

$$|u_{m-1}(x)| \leqq C|\mu|^{n-1} \sum_{k=N'-N+1}^{N'} |u_m(x+kt)| + C \sum_{r=0}^{m} \sum_{s=2}^{n} |\mu|^{r(1-n)} \|q_s\|_{L^p(K)} \|u_{m-r}^{(n-s)}\|_{L^{p'}(K)}$$

whence

$$\|u_{m-1}\|_{L^{p'}(K_1)} \leqq C|\mu|^{n-1}\|u_m\|_{L^{p'}(K)} + C \sum_{r=0}^{m} \sum_{s=2}^{n} |\mu|^{r(1-n)} \|u_{m-r}^{(n-s)}\|_{L^{p'}(K)}. \tag{18}$$

(Here and in the sequel C denotes diverse constants which do not depend on the choice of u.) Being $|\mu|$ large, by Theorem 2 of [5] we have

$$\|u_{m-r}^{(n-s)}\|_{L^{p'}(K)} \leqq C|\mu|^{n-s}\|u_{m-r}\|_{L^{p'}(K)} \leqq C|\mu|^{n-2}\|u_{m-r}\|_{L^{p'}(K)}.$$

On the other hand, using the inductive hypothesis, for $r \geqq 2$

$$\|u_{m-r}\|_{L^{p'}(K)} \leqq C|\mu|^{(r-1)(n-1}\|u_{m-1}\|_{L^{p'}(K_2)}.$$

Finally, using again Theorem 2 of [5] we obtain

$$\|u_{m-1}\|_{L^{p'}(K_2)} \leqq C|\mu|^n \|u_m\|_{L^{p'}(K_2)}.$$

Therefore we obtain from (18) the estimate

$$\|u_{m-1}\|_{L^{p'}(K_1)} \leqq C|\mu|^{n-1}\|u_m\|_{L^{p'}(K_2)}$$

i.e. (11) is proved for $|\mu|$ sufficiently large. But for $|\mu|$ bounded (11) follows immediately from (5) and the theorem is proved.

Let us now turn to the proof of Proposition 1. Putting

$$K_0(\mu, y) = \sum_{p=1}^{n} \frac{\mu_p}{n\lambda} e^{\mu_p y}, \quad K_r(\mu, y) = \int_0^y K_0(\mu, \xi) K_{r-1}(\mu, y-\xi)\, d\xi \quad (r = 1, 2, \dots)$$

and for any fixed $x \in G$

(19)

$$v_m(y) = u_m(y) + \sum_{r=0}^{m} \int_x^y K_r(\mu, y-\tau) \sum_{s=0}^{n} q_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau, \quad v_{m-1}(y) = v_m^{(n)}(y) - \lambda v_m(y),$$

it follows from the results of the paper [5] that v_m is an eigenfunction of order m of the operator $L_0 v = v^{(n)}$ (defined on G) with the eigenvalue λ, and $v_{m-1}(x) = u_{m-1}(x)$.

Consequently the function $v_m(y)$ is a linear combination of the functions

$$y \mapsto (r!)^{-1}(\mu_p(y-x))^r e^{\mu_p(y-x)} \quad (r = 0, \dots, m,\ p = 1, \dots, n);$$

therefore the determinant

$$\begin{vmatrix} \dots & v_m(x+kt) & \dots & t^n u_{m-1}(x) \\ & \vdots & & \vdots \\ \dots & \dfrac{(k\mu_p t)^r}{r!} e^{k\mu_p t} & \dots & C_r(\mu_p t)^n \\ & \vdots & & \vdots \end{vmatrix}$$

$$\left(r = 0, \dots, m, \quad p = 1, \dots, n, \quad k = N'-N+1, \dots, N', \quad C_r = (1-\delta_{r0})\binom{n}{r}\right)$$

vanishes whenever $x+(N'-N+1)t \in G$ and $x+N't \in G$. Developing the determinant according to its first row, in view of (19) we obtain (with obvious notations) the formula (12).

Let us set for brevity

$$z_1^* = \sum_{i=1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t,$$

$$(21) \qquad z_2^* = \sum_{i=n'-n+1}^{0} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t,$$

$$z^* = z_1^*+z_2^*.$$

We shall also use the notation

$$w_1 \overset{r}{\leqq} w_2 \Leftrightarrow \operatorname{Re} w_1 \leqq \operatorname{Re} w_2.$$

First we prove (14). One can see easily that each term of the development of the minor defining $d_k(\mu, t)$ can be estimated by an expression of type

$$C|\mu|^{n-1+n(1+\ldots+m)}|e^z|.$$

In view of (13) and (21) it suffices to show that we can always choose z such that

$$(22) \qquad \operatorname{Re}(z-z^*) \leqq -|k|\varrho t.$$

Introducing the notation $k=(m+1)l_1-l_2$, $l_1\in\{n'-n+1, \ldots, n'\}$, $l_2\in\{0, \ldots, m\}$, we can choose

$$z = z_2^*+\sum_{i=l_1+1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+$$

$$+\big(l_1(m+1)+\ldots+(k+1)+(k-1)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{=1}^{l_1-1} \big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

is $k\geqq 1$, and

$$z = z_1^*+\sum_{i=n'-n+1}^{l_1-1} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\big((l_1(m+1)+1)+\ldots+(k+1)+(k-1)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=l_1+1}^{0} \big((i(m+1)+1)+\ldots+((i-1)(m+1)+2)\big)\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $k\leqq 0$. Using (2) hence we obtain

$$z-z^* = \big((l_1-1)(m+1)-k\big)\mu_{n'+1-l_1}-(m+1)\sum_{i=1}^{l_1-1}\mu_{n'-1-i}t \overset{r}{\leqq}$$

$$\overset{r}{\leqq} \big((l_1-1)(m+1)-k\big)(\mu_{n'+1-l_1}-\mu_{n'})t-k\mu_{n'}t \overset{r}{\leqq} -k\mu_{n'}t = -|k|\mu_{n'}t$$

if $k \geqq 1$, and

$$z-z^* = \big(l_1(m+1)+1-k\big)\mu_{n'+1-l_1}t+(m+1)\Big(\sum_{i=l_1-1}^{0}\mu_{n'+1-i}t\Big)-\mu_{n'+1}t \overset{r}{\leqq}$$

$$\overset{r}{\leqq} \big(l_1(m+1)+1-k\big)(\mu_{n'+1-l_1}-\mu_{n'+1})t-k\mu_{n'+1}t \overset{r}{\leqq} -k\mu_{n'+1}t = |k|\mu_{n'+1}t$$

if $k \leqq 0$. In view of (9) hence (22) follows in both cases and (14) is proved.

Now we prove (15). Let us fix $r \in \{0, \dots, m\}$ arbitrarily and let $l \in \{N'-N+2, \dots, \dots, N'\}$ be such that

(23) $$x+(l-1)t \leqq \tau \leqq x+lt.$$

Then $D_r(\mu, t, x-\tau)$ is defined by the determinant which differs from the determinant (20) in the first row:

in case $\tau \geqq x$ the element $v_m(x+kt)$ is replaced by $K_r(\mu; x-\tau+kt)$ if $l \leqq k \leqq N'$, all the other elements are replaced by 0;

in case $\tau \leqq x$ the element $v_m(x+kt)$ is replaced by, $-K_r(\mu, x-\tau+kt)$ if $N'-N+1 \leqq k \leqq l-1$, all the other elements are replaced by 0.

One can see easily by induction on r that with some constants $c_{rp\alpha}$

$$K_r(\mu, x-\tau+kt) = \sum_{p=1}^{n}\mu_p^{1-rn-n}\sum_{\alpha=0}^{r}c_{rp\alpha}\big(\mu_p(x-\tau+kt)\big)^{\alpha}e^{\mu_p(x-\tau+kt)}.$$

In view of (17) it suffices to show that for any fixed $q \in \{1, \dots, n\}$ and $\beta \in \{0, \dots, r\}$, *if we replace in the first row of the determinant* (20)

in case $\tau \geqq x$ the element $v_m(x+kt)$ by $k^{\beta}e^{\mu_q(x-\tau+kt)}$ if $l \leqq k \leqq N'$, all the other element by 0;

in case $\tau \leqq x$ the element $v_m(x+kt)$ by $-k^{\beta}e^{\mu_q(x-\tau+kt)}$ if $N'-N+1 \leqq k \leqq l-1$, all the other elements by 0,

then this new determinant can be estimated by

$$C|\mu|^{n-1}|P(\mu, t)|e^{-\varrho|x-\tau|}.$$

One can see esaily that those terms of this determinant the factors of which choosen from the first row and from the row corresponding to $p=q$ and $r=\beta$ are in case $\tau \geqq x$ in one of the l-th, ..., N'-th columns, in case $\tau \leqq x$ in one of the $(N'-N+1)$-th, ..., $(l-1)$-th columns, pairwise eliminate each other. All the other terms can be estimated by

$$C|\mu|^{n-1+n(1+\dots+m)}|e^z|;$$

it suffices to show that here one can always choose z such that

(24) $$\operatorname{Re}(z-z^*) \leqq -\varrho|x-\tau|.$$

Let us consider first the case $\tau \geqq x$. Putting

$$l = (m+1)l_1-l_2,\ l_1 \in \{1, \dots, n'\},\quad l_2 \in \{0, \dots, m\},$$

we can take

$$z = z_2^* + \sum_{i=l_1+1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+\mu_q(x-\tau)+(l-1)\mu_q t+$$

$$+\big(l_1(m+1)+\ldots+l+(l-2)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=1}^{l_1-1}\big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

if $q \leqq n'+1-l_1$, and

$$z = z_2^* + \sum_{i=l_1+1}^{n'} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{n'+1-i}t+\mu_q(x-\tau+lt)+$$

$$+\big(l_1(m+1)+\ldots+(l+1)+(l-1)+\ldots+(l_1-1)(m+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=1}^{l_1-1}\big((i(m+1)-1)+\ldots+(i-1)(m+1)\big)\mu_{n'+1-i}t$$

if $q > n'+1-l_1$. Now using (2) and (23), in both cases

$$z-z^* \overset{r}{\leqq} \mu_{n'+1-l_1}\big(x-\tau+(l_1-1)(m+1)t\big)-(m+1)\sum_{i=1}^{l_1-1}\mu_{n'+1-it} \overset{r}{\leqq}$$

$$\overset{r}{\leqq} (\mu_{n'+1-l_1}-\mu_{n'})\big(x-\tau+(l_1-1)(m+1)t\big)+\mu_{n'}(x-\tau) \overset{r}{\leqq} \mu_{n'}(x-\tau) = -\mu_{n'}|x-\tau|$$

whence (24) follows.

Let us now consider the case $\tau \leqq x$. Putting $l-1=(m+1)l_1-l_2$, $l_1 \in \{n'-n+1, \ldots, 0\}$, $l_2 \in \{0, \ldots, m\}$, we can take

$$z = z_1^* + \sum_{i=n'-n+1}^{l_1-1} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)_{n'+1-1}t+\mu_q(x-\tau)+l\mu_q t+$$

$$+\big((l_1(m+1)+1)+\ldots+(l+1)+(l-1)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=l_1+1}^{0}\big((i(m+1)+1)+\ldots+((i-1)(m+1)+2)\big)\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $q \geqq n'+1-l_1$, and

$$z = z_1^* + \sum_{i=n'-n+1}^{l_1-1} \big(i(m+1)+\ldots+((i-1)(m+1)+1)\big)\mu_{\cdot+1-i}t+\mu_q\big(x-\tau+(l-1)t\big)+$$

$$+\big((l_1(m+1)+1)+\ldots+l+(l-2)+\ldots+((l_1-1)(m+1)+1)\big)\mu_{n'+1-l_1}t+$$

$$+\sum_{i=l_1+1}^{0}\big((i(m+1)+1)+\ldots+((i-1)(m+1)+2)\big)\mu_{n'+1-i}t-\mu_{n'+1}t$$

if $q < n'+1-l_1$. Using again (2) and (23), in both cases

$$z-z^* \overset{r}{\leqq} \mu_{n'+1-l_1}(x-\tau+(l_1(m+1)+1)t)-(m+1)\Big(\sum_{i=l_1+1}^{0}\mu_{n'+1-i}t\Big)-\mu_{n'+1}t \overset{r}{\leqq}$$

$$\overset{r}{\leqq} (\mu_{n'+1-l_1}-\mu_{n'+1})(x-\tau+(l_1(m+1)+1)t)+\mu(x-\tau) \overset{r}{\leqq} \mu_{n'+1}(x-\tau) = \mu_{n'+1}|x-\tau|$$

whence (24) follows and (15) is proved.

Finally we prove (16). One can see by induction on m that

$$|d(\mu, t)| = |\mu t|^{m(m+1)/2}|e^{\mu t}|^{(m+1)(m+2)/2}$$

if $n=1$, and

$$|d(\mu, t)| = |\mu t|^{m(m+1)n/2} \prod_{1\leqq p<q\leqq n} |e^{\mu_p t}-e^{\mu_q t}|^{(m+1)^2}$$

if $n\geqq 2$. In case $n=1$ (16) hence follows at once because $|d(\mu, t)|=|P(\mu, t)|$. In case $n\geqq 2$, taking into account that $e^{(\mu_1+\ldots+\mu_n)t}=1$, we obtain

$$|d(\mu, t)| = |P(\mu, t)| \prod_{1\leqq p<q\leqq n} |1-e^{(\mu_q-\mu_p)t}|^{(m+1)^2}.$$

Taking into account that

$$\operatorname{Re} z \leqq -1/2 \Rightarrow |1-e^z| \geqq 1-e^{-1/2},$$

we have for any $t_0\in[t/2, t]$

$$|d(\mu, t_0)| \geqq |P(\mu, t_0)|(1-e^{-1/2})^{n(n-1)/2} \prod_{\substack{1\leqq p<q\leqq n\\ \operatorname{Re}(\mu_q-\mu_p)t>-1}} |1-e^{(\mu_q-\mu_p)t_0}|^{(m+1)^2}.$$

If we choose C_2 sufficiently large, the condition (17) implies for all the pairs (p, q) in this product

$$|\operatorname{Im}(\mu_q-\mu_p)t| > 2\pi$$

and then, in view of the inequality

$$\operatorname{Re} z > -1 \Rightarrow |1-e^z| \geqq e^{-1}|\sin(\operatorname{Im} z)|$$

(16) reduces to the following lemma:

Lemma. *Given* $a_1, \ldots, a_{k_0}\in\mathbf{R}$, $k_0\in\mathbf{N}$ *such that* $|a_k|>2\pi$ *for all* $k=1, \ldots, k_0$, *we have*

$$\sup_{1/2\leqq b\leqq 1} \min_{k=1}^{k_0} |\sin(ba_k)| \geqq \sin(\pi/(12k_0)).$$

Indeed, for any $k\in\{1, \ldots, k_0\}$ the measure of the set

$$\{b\in[1/2, 1]:\ |\sin(ba_k)| < \sin(\pi/(12\,k_0))\}$$

is less than or equal to $(3k_0)^{-1}$ whence the lemma follows.

The proof of Proposition 1 (and also of Theorem 1) is finished.

Remark. In case $n \leqq 2$ Theorem 1 remains valid under the weaker condition $q_1 \in L^p_{loc}(G)$, too. Indeed, we proved in [7] that in case $n \leqq 2$ there exists a positive constant R such that for all the eingenfunctions u of order m of the operator L with some eigenvalue λ,

$$\|u\|_{L^\infty(K_1)} \leqq Ce^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L^\infty(K_2)}. \tag{25}$$

Using (3), (5) and (25),

$$\begin{aligned}\|u^*\|_{L^{p'}(K_1)} &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)\|u\|_{L^{p'}(K_1)} \leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)\|u\|_{L^\infty(K_1)} \leqq \\ &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)e^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L^\infty(K_2)} \leqq \\ &\leqq C(1+|\mu|)^{n-1}(1+|\mathrm{Re}\,\mu_1|)^2e^{-R|\mathrm{Re}\,\mu_1|}\|u\|_{L_1(K_2)} \leqq \\ &\leqq C(1+|\mu|)^{n-1}\|u\|_{L^1(K_2)} \leqq C(1+|\mu|)^{n-1}\|u\|_{L^{p'}(K_2)}.\end{aligned}$$

Conjecture. *The condition $q_1 \equiv 0$ in Theorem 1 can be replaced by the weaker condition $q_1 \in L^p_{loc}(G)$ in case $n \geqq 3$, too.*

2. Local uniform estimates. We shall prove the following result:

Theorem 2. *Assume $q_1 \equiv 0$. Then to any $m \in \{0, 1, \ldots\}$ and to any compact intervals $K_1, K_2 \subset G$, $K_1 \subset \mathrm{int}\, K_2$, there exists a constant C such that for any eigenfunction u of order m of the operator L with some eigenvalue λ,*

$$\|u\|_{L^\infty(K_1)} \leqq C\|u\|_{L^1(K_2)}. \tag{26}$$

For $|\lambda|$ sufficiently large we have also

$$\|u^{(i)}\|_{L^\infty(K_1)} \leqq C\|u^{(i)}\|_{L^1(K_2)} \quad (i = 1, \ldots, n-1). \tag{27}$$

We need the following assertion:

Proposition 2. *There exist continuous functions f_k, F_r such that for any eigenfunction u_m of order m of the operator L with some eigenvalue $\lambda = \mu^n$,*

$$\begin{aligned}&\sum_{k=N'-N}^{N'} f_k(\mu, t)u_m^{(i)}(x+kt) = \\ &= \sum_{r=0}^{m} \int_{x+(N'-N)t}^{x+N't} D_3^i F_r(\mu, t, x-\tau) \sum_{s=1}^{n} q_s(\tau)u_{m-r}^{(n-s)}(\tau)\,d\tau \quad (i = 0, \ldots, n-1)\end{aligned} \tag{28}$$

whenever $x+(N'-N)t \in G$ and $x+N't \in G$. Furthermore, introducing the notation

$$Q(\mu, t) = \exp\left((m+1)(\mu_1 + \ldots + \mu_{n'})t\right), \tag{29}$$

to any fixed positive number A there exists a constant C such that

(30) $$|f_0(\mu, t)-Q(\mu, t)| \leqq C|Q(\mu, t)|e^{\mathrm{Re}(\mu_{n'+1}-\mu_{n'})t},$$

(31) $$|f_{m+1}(\mu, t)-e^{-(m+1)\mu_{n'}t}Q(\mu, t)| \leqq C|e^{-(m+1)\mu_{n'}t}Q(\mu, t)|e^{\mathrm{Re}(\mu_{n'}-\mu_{n'-1})t}$$

(32) $$|f_k(\mu, t)| \leqq C|Q(\mu, t)|e^{-|k|\varrho t},$$

(33) $$|D_3^i F_r(\mu, t, x-\tau)| \leqq C|\mu|^{i+(r+1)(1-n)}|Q(\mu, t)|e^{-\varrho|x-\tau|}$$

whenever

(34) $$\mathrm{Re}\,\mu \geqq 0, \quad 0 \leqq t \leqq A \quad \textit{and} \quad |\mu| \geqq 1.$$

First we deduce Theorem 2 from Proposition 2. As in Theorem 1, it suffices to consider the case $\mathrm{Re}\,\mu\geqq 0$. Let us fix a compact interval $K\subset G$ such that

$$K_1\subset \mathrm{int}\, K \quad \text{and} \quad K\subset \mathrm{int}\, K_2$$

and put

$$R = (m+1+N')^{-1} \text{ dist } (K_1, \partial K)$$

Let us fix $B_1>0$ such that

(35) $$\mathrm{Re}\,\mu \geqq B_1 \text{ and } t \geqq R/2 \Rightarrow |f_0(\mu, t)| \geqq 2^{-1}|Q(\mu, t)|$$

and then $B_2, B_3>0$ such that

(36) $$|\mu| \geqq B_2 \Rightarrow \|u\|_{L^\infty(K_2)} \leqq B_3|\mu|^{1-i}\|u^{(i)}\|_{L^1(K_2)}$$

and

(37) $$|\mu| \geqq B_2, \mathrm{Re}\,\mu \leqq B_1 \quad \text{and} \quad t \geqq R/2 \Rightarrow |f_{m+1}(\mu, t)| \geqq 2^{-1}|e^{-(m+1)\mu_{n'}t}Q(\mu, t)|.$$

This is possible by (30), (31) and by Theorems 3, 4 in [6] (if we are interested only in the estimate (26), it suffices to use Theorem 2 in [5] instead of the results of the paper [6]). Now we distinguish three cases.

If $|\mu|\leqq B_2$ then (26) follows from (3).

If $|\mu|>B_2$ and $\mathrm{Re}\,\mu\geqq B_1$ then we apply the formula (28) with any $x\in K_1$ and $R/2\leqq t\leqq R$; in view of (22), (33) and (35) we obtain

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N\leqq k\leqq N' \\ k\neq 0}} |u_m^{(i)}(x+kt)| + C\sum_{r=0}^{m}\sum_{s=2}^{n} |\mu|^{i+(r+1)(1-n)}\|q_s\|_{L^1(K)}\|u_{m-r}^{(n-s)}\|_{L^\infty(K)}.$$

Using Theorem 2 in [5], Theorem 1 from the preceeding section and (36),

$$|\mu|^{i+(r+1)(1-n)}\|u_{m-r}^{(n-s)}\|_{L^\infty(K)} \leqq C|\mu|^{i+1-s}\|u_m\|_{L^\infty(K_2)} \leqq C|\mu|^{2-s}\|u_m^{(i)}\|_{L^1(K_2)} \leqq C\|u_m^{(i)}\|_{L^1(K_2)};$$

therefore

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N\leqq k\leqq N' \\ k\neq 0}} |u_m^{(i)}(x+kt)| + C\|u_m^{(i)}\|_{L^1(K_2)}.$$

Applying the transformation $\int_{R/2}^{R} dt$ we obtain

$$|u_m^{(i)}(x)| \leqq C\|u_m^{(i)}\|_{L^1(K_2)}$$

whence

$$\|u_m^{(i)}\|_{L^\infty(K_1)} \leqq C\|u_m^{(i)}\|_{L^1(K_2)}$$

and (26), (27) are proved.

If $|\mu|>B_2$ and $\operatorname{Re}\mu<B_1$, then we apply the formula (28) with any $x\in K_1$ and $R/2\leqq t\leqq R$ (we put x in place of $x+(m+1)t$); using (32), (33) and (37) we obtain

$$|u_m^{(i)}(x)| \leqq C \sum_{\substack{N'-N-m-1\leqq k\leqq N'-m-1\\ k\neq 0}} |u_m^{(i)}(x+kt)|+$$

$$+C\sum_{r=0}^{m}\sum_{s=2}^{n} |\mu|^{i+(r+1)(1-n)}\|q_s\|_{L^1(K)}\|u_{m-r}^{(n-s)}\|_{L_\infty(K)};$$

hence we can conclude (26), (27) similarly as in the preceeding case. The theorem is proved.

Now we prove Proposition 2. Let us denote by $S_k(\mu, t)$ the elementary symmetric polynomial of order k of $e^{\mu_1 t}, \ldots, e^{\mu_n t}$ with the main coefficient $(-1)^{n-k}$ if $k\in\{0, \ldots, n\}$; otherwise we put $S_k(\mu, t)=0$. Define

$$f_{k+N'-N}(\mu, t) = S_{n-k}(\mu, t)$$

if $m=0$, and

$$f_{k+N'-N}(\mu, t) = \sum_{r_1\in Z}\cdots\sum_{r_m\in Z} S_{n-r_1}(\mu, t)\ldots S_{n-r_m}(\mu, t)S_{n-k+r_1+\ldots r_m}(\mu, t)$$

if $m\geqq 1$. It was shown by Joó [4] that for any eigenfunction v_m of order m of the operator $L_0 v=v^{(n)}$ with some eigenvalue $\lambda=\mu^n$,

$$\sum_{k=N'-N}^{N'} f_k(\mu, t)v_m(x+kt) \equiv 0;$$

hence for $i\in\{0, \ldots, n-1\}$

$$\sum_{k=N'-N}^{N'} f_k(\mu, t)v_m^{(i)}(x+kt) \equiv 0. \tag{38}$$

Using the notations of the preceeding section, let us define v_m by the formula (19). Then we have (see also [5])

$$v_m^{(i)}(y) = u_m^{(i)}(y)+\sum_{r=0}^{m}\int_x^y D_2^i K_r(\mu, y-\tau)\sum_{s=1}^{n} q_s(\tau)u_{m-r}^{(n-s)}(\tau)\,d\tau. \tag{39}$$

(38) and (39) imply (28) (with obvious notations).

The estimates (30), (31), (32) follow easily from the explicit expressions of the functions f_k. To prove (33) we note that the formula (38) can be obtained if we develop the determinant

(40)
$$\begin{vmatrix} \dots & v_m(x+kt) & \dots \\ & \vdots & \\ \dots & \dfrac{(k\mu_p t)^r}{r!} e^{k\mu_p t} & \dots \\ & \vdots & \end{vmatrix}$$

$$(r = 0, \dots, m, \quad p = 1, \dots, n, \quad k = N'-N, \dots, N')$$

according to the first row and then we simplify the obtained formula by a suitable expression $R(\mu, t)$. Repeating the proof of the estimate (15) in Proposition 1, we obtain (33) under the condition $R(\mu, t)\neq 0$. But this condition can be omitted because for any fixed $\mu\neq 0$, both sides of (33) are continuous in t and the set

$$\{t\in\mathbf{R}\colon\ R(\mu, t) = 0\}$$

is discrete.

The proposition (and also the theorem) is proved.

Remark. In case $n\leqq 2$ the condition $q_1\equiv 0$ in Theorem 2 can be omitted. Indeed, using (25) and (3),

$$\|u\|_{L^\infty(K_1)} \leqq Ce^{-R|\mathrm{Re}\,\mu_1|}\,\|u\|_{L^\infty(K_2)} \leqq Ce^{-R|\mathrm{Re}\,\mu_1|}(1+|\mathrm{Re}\,\mu_1|)\,\|u\|_{L^1(K_2)} \leqq C\|u\|_{L^1(K_2)}.$$

Conjecture. *The condition* $q_1\equiv 0$ *in Theorem 2 can be omitted in case* $n\geqq 3$, *too.*

Finally we note another version of Theorem 2 which is a little weaker than the above conjecture:

Theorem 3. *Assume* $q_1, \dots, q_n\in L^p_{\mathrm{loc}}(G)$ *for some* $p\in[1, \infty]$. *Then to any compact intervals* $K_1, K_2\subset G$, $K_1\subset \mathrm{int}\, K_2$, *there exists a constant C such that for any eigenfunction u of order* 0 *of the operator L with some eigenvalue* λ,

(41)
$$\|u\|_{L^\infty(K_1)} \leqq C\|u\|_{L^{p'}(K_2)}.$$

For $|\lambda|$ *sufficiently large we have also*

(42)
$$\|u^{(i)}\|_{L^\infty(K_1)} \leqq C\|u^{(i)}\|_{L^{p'}(K_2)} \quad (i = 1, \dots, n-1).$$

Proof. We repeat the proof of Theorem 2 with the following changes:

In case $|\mu|>B_2$ and $\mathrm{Re}\,\mu\geqq B_1$ we have now

$$|u_0^{(i)}(x)| \leqq C \sum_{\substack{n'-n\leqq k\leqq n' \\ k\neq 0}} |u_0^{(i)}(x+kt)| + C\sum_{s=1}^{n} |\mu|^{i+1-n}\|q_s\|_{L^p(K)}\|u_0^{(n-s)}\|_{L^{p'}(K)}.$$

Using Theorem 2 in [5] and (36),

$$|\mu|^{i+1-n}\|u_0^{(n-s)}\|_{L^{p'}(K)} \leqq C|\mu|^{i+1-s}\|u_0\|_{L^{p'}(K)} \leqq C|\mu|^i\|u_0\|_{L^{p'}(K)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K)};$$

therefore

$$|u_0^{(i)}(x)| \leqq C \sum_{\substack{n'-n\leqq k\leqq n' \\ k\neq 0}} |u_0^{(i)}(x+kt)| + C\|u_0^{(i)}\|_{L^{p'}(K)},$$

$$|u_0^{(i)}(x)| \leqq C\|u_0^{(i)}\|_{L^1(K)} + C\|u_0^{(i)}\|_{L^{p'}(K)},$$

and

$$\|u_0^{(i)}\|_{L^\infty(K_1)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K)} \leqq C\|u_0^{(i)}\|_{L^{p'}(K_2)}.$$

The case $|\mu|>B_2$ and $\operatorname{Re}\mu<B_1$ is similar.

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