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On the unitary representations of compact groups

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

Let G be a compact group and let $g \to U_g$ be a continuous (strongly or weakly, since one implies the other) unitary representation of G on a complex Hilbert space $\mathfrak{H}$. By definition, the dimensionality of the representation is the Hilbert dimension of $\mathfrak{H}$. Denote by $\mathscr{B}(\mathfrak{H})$ the von Neumann algebra of all bounded linear operators of $\mathfrak{H}$. Furthermore, let $\mathscr{U}$ be the von Neumann algebra generated by $\{U_g \colon g \in G\}$. The representation $g \to U_g$ is said to be *irreducible* if $\mathscr{U} = \mathscr{B}(\mathfrak{H})$.

One of the fundamental theorems of representation theory is that a continuous irreducible unitary representation of a compact group G is finite-dimensional. Several methods have been introduced in the literature to prove this theorem. Most of them use the analytic geometry of Hilbert spaces and their compact operators, and some of them use the elegant but highly intricate machinery of Hilbert algebras or von Neumann algebras ([1], [3], [4], [5]). All of them use, however, invariant integration on G in one way or another. In [4], invariant integration served also for the basis of a general theory to create a mapping $T \to \tilde{T}$ of $\mathscr{B}(\mathfrak{H})$ onto $\mathscr{U}'$, the algebraic commutant of $\mathscr{U}$, which yielded, in final analysis, a non-elementary, but an easy access to the fundamental facts of the representation theory of compact groups on Hilbert spaces, including the above mentioned theorem (cf. [4], 2, §4). Later, Karl H. Hofmann, apparently unaware of [4], came up with an explicit form of $\tilde{T}$ as

$$\tilde{T} = \int_G U_g^{-1} T U_g \, dg \quad (T \in \mathscr{B}(\mathfrak{H})), \tag{1}$$

where dg is the normalized Haar measure on G, and (1) is taken, for instance, in the sense of the weak operator topology of $\mathscr{B}(\mathfrak{H})$. Furthermore, using certain considerations in topological vector spaces, he observed that the mapping $T \to \tilde{T}$ carried compact operators into compact operators, a fact which rendered the proof of the theorem elegant, elementary, and easy (cf. [2]).

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We are going to give a proof here which is technically different from Hofmann's. In fact, denote by $\mathscr{B}^+(\mathfrak{H})$ the convex cone of the non-negative elements of $\mathscr{B}(\mathfrak{H})$ and consider an orthonormal basis $(e_i)_{i\in I}$ of $\mathfrak{H}$. For $T\in\mathscr{B}^+(\mathfrak{H})$, put

$$\text{(2)} \qquad \operatorname{tr}(T) = \sum_{i\in I}(Te_i|e_i),$$

where the symbol $(\cdot|\cdot)$ stands for the inner product of $\mathfrak{H}$. For further references, we precise that the summation in (2) is understood in the following manner. Let $\mathscr{F}$ be the family of all finite subsets J of I[1]. Then, by definition,

$$\text{(3)} \qquad \sum_{i\in I}(Te_i|e_i) = \sup_{J\in\mathscr{F}}\Big\{\sum_{i\in J}(Te_i|e_i)\Big\}.$$

The extended valued function $\operatorname{tr}(\cdot)$ which does not depend on the particular choice of (e_i) is linear, unitarily invariant, and is known as the canonical trace of $\mathscr{B}^+(\mathfrak{H})$. If P is an orthogonal projection of $\mathfrak{H}$, then $\operatorname{tr}(P)=\dim P\mathfrak{H}$. This implies that $\mathfrak{H}$ is finite-dimensional if and only if $\operatorname{tr}(I)<\infty$, where I is the identity operator of $\mathfrak{H}$. Now, choose, for instance, an arbitrary one-dimensional projection P of $\mathfrak{H}$, i.e. $\operatorname{tr}(P)=1$, and observe that for every $g\in G$ we have

$$\text{(4)} \qquad \sum_{i\in I}(U_g^{-1}PU_ge_i|e_i) = \operatorname{tr}(U_g^{-1}PU_g) = \operatorname{tr}(P) = 1.$$

For every $J\in\mathscr{F}$, let $f_J(g)=1-\sum_{i\in J}(U_g^{-1}PU_ge_i|e_i)$. Then, $(f_J)_{J\in\mathscr{F}}$ forms a downward directed family of continuous functions on G such that $\inf_{J\in\mathscr{F}} f_J=0$. Then, an elementary property of Radon integrals tells us that (4) can be termwise integrated (cf. [5], III, 2, §6):

$$\text{(5)} \qquad \operatorname{tr}(P) = 1 = \int_G \operatorname{tr}(P)\,dg = \sum_{i\in I}\int_G(U_g^{-1}PU_ge_i|e_i)\,dg = \sum_{i\in I}(\tilde{P}e_i|e_i) = \operatorname{tr}(\tilde{P}).$$

From this, we conclude that $\tilde{P}\neq 0$. Furthermore, the translation invariance of the Haar measure and (1) imply that $\tilde{P}\in\mathscr{U}'$ and $\tilde{P}>0$. Now, if the representation $g\to U_g$ is irreducible, then $\mathscr{U}'=(\mathscr{B}(\mathfrak{H}))'=(cI)_{c\in\mathbf{C}}$ ($\mathbf{C}$ is the complex number field), hence $\tilde{P}=c_0I$ with $c_0\neq 0$. Then this and (5) imply $\operatorname{tr}(I)=1/c_0<\infty$, i.e., $\mathfrak{H}$ is finite-dimensional. The proof is complete.

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[1]) With respect to the inclusion of the elements of $\mathscr{F}$ as a partial ordering, $\mathscr{F}$ is an upward directed set.

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