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Limit cases in the strong approximation of orthogonal series

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In honour of Professor K. Tandori on his 60th birthday

Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We consider the orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty.$$

By the Riesz—Fischer theorem series (1) converges in the metric L^2 to a square-integrable function $f(x)$. Denote $s_n(x)$ the n-th partial sum of (1).

In [1] we proved that *if* $0<\gamma<1$ *and*

(2) $$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty,$$

then

$$\frac{1}{n+1} \sum_{k=0}^{n} (s_k(x)-f(x)) = o_x(n^{-\gamma})$$

almost everywhere in (a, b).

G. SUNOUCHI [6] generalized our theorem to strong approximation in the following way: *If* $0<\gamma<1$ *and* (2) *holds, then*

(3) $$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |s_\nu(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma})$$

also holds almost everywhere for any $\alpha>0$ *and* $0<p<\gamma^{-1}$, *where* $A_n^\alpha = \binom{n+\alpha}{n}$.

We generalized this result in [2] in the following ways:

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First we showed that *the assumptions of Sunouchi's theorem imply, for any increasing sequence* $\{\nu_k\}$ *of the natural numbers, that*

$$(4)\qquad C_n(f,\alpha,p,\{\nu_k\};x):=\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma})$$

also holds almost everywhere in (a, b). In the other words we proved that the conditions of Sunouchi's theorem imply the very strong approximation with the same order. Since we speak on *strong, very strong* and *extra strong* (or *mixed*) approximation according as in the investigated means the following partial sums $s_k(x)$, $s_{\nu_k}(x)$ $(\nu_k<\nu_{n+1})$ or $s_{\mu_k}(x)$ (where $\{\mu_k\}$ is a permutation of a subsequence of the natural numbers, or briefly a mixed sequence) appear, respectively.

Secondly we replaced the partial sums in (3) by (C,δ)-means, where δ would also take negative values.

Very recently in a joint paper with H. Schwinn [5] we have attained to the following four theorems:

Theorem A. *If* $0<p\gamma<\beta$ *then for any increasing* $\{\nu_k\}$ *of the natural numbers condition* (2) *implies that*

$$(5)\qquad h_n(f,\beta,p,\{\nu_k\};x):=\left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{\nu_k}(x)-f(x)|^p\right\}^{1/p}=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

Theorem B. *If* α *and* γ *are positive numbers,* $0<p\gamma<1$, *and* $\{\nu_k\}$ *is an increasing sequence, then condition* (2) *implies* (4) *almost everywhere in* (a, b).

The novelty of these theorems is that the restriction $\gamma<1$, which appeared in the previous theorems, is omitted. The following theorems, holding this advantage, extend these results to the case of extra strong approximation under a slight restriction of other type.

Theorem C. *Let* $\{\mu_k\}$ *be a fixed permutation of a subsequence of the natural numbers, moreover let* $\gamma>0$ *and* $0<p\gamma<\min(\beta, 1)$. *Then condition* (2) *implies that*

$$(6)\qquad h_n(f,\beta,p,\{\mu_k\};x)=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

Theorem D. *Let* $\{\mu_k\}$ *be a fixed permutation of some subsequence of the natural numbers, let* $\gamma>0$ *and* $0<p\gamma<\min(\alpha, 1)$. *Then* (2) *yields that*

$$(7)\qquad C_n(f,\alpha,p,\{\mu_k\};x)=o_x(n^{-\gamma})$$

holds almost everywhere in (a, b).

In a recent paper [4] we started to investigate the order of approximation of the means h_n defined in (5) under the assumption $\beta = p\gamma$; i.e. we investigated the limit case of the restrictions of the parameters. We obtained, among others, that in the special case $p=2$, condition (2) with $\gamma=\beta/2$ implies only

$$h_n(f, \beta, 2, \{\nu_k\};\ x) = O_x(n^{-\gamma});$$

and in the case $p \neq 2$, condition (2) does not ensure even this order of approximation. In order to obtain the order $O_x(n^{-\gamma})$, new conditions were required instead of (2).

More precisely, we proved (Proposition 2 of [4])

Theorem E. *Let $\{\nu_k\}$ be an arbitrary sequence. Then for any positive β the following pairs of condition*

$$(8) \qquad 0 < p \leqq 2 \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{\beta-1}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2} < \infty$$

or

$$(9) \qquad p \geqq 2 \quad \textit{and} \quad \sum_{n=1}^{\infty} n^{(2/p)(\beta-1)+1} c_n^2 < \infty$$

imply

$$(10) \qquad h_n(f, \beta, p, \{\nu_k\};\ x) = O_x(n^{-\beta/p}) \quad (\gamma = \beta/p)$$

almost everywhere in (a, b).

The aim of the present paper is to study whether Theorems B, C and D have extensions for the limit cases of the restrictions of the parameters similar to Theorem E. For Theorem E can be interpreted as an extension of Theorem A to the case $p\gamma = \beta$.

We shall also investigate what happens if we retain condition (2) but the parameter γ takes the limit value of those in the previous theorems. In these cases, as expected, the order of strong approximation will increase by a factor $(\log n)^{1/p}$.

Now we formulate our theorems:

Theorem 1. *For any positive α and for any increasing sequences $\{\nu_k\}$ of the natural numbers the following pairs of conditions*

$$(11) \qquad 0 < p \leqq 2 \quad \textit{and} \quad \sum_{n=1}^{\infty}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2} < \infty$$

or

$$(12) \qquad p \geqq 2 \quad \textit{and} \quad \sum_{n=1}^{\infty} c_n^2 n < \infty$$

imply

$$(13) \qquad C_n(f, \alpha, p;\ \{\nu_k\};\ x) = O_x(n^{-1/p})$$

almost everywhere in (a, b).

Theorem 2. *If* $\beta>0$, $\bar\beta=\min(\beta, 1)$ *and* $\{\mu_k\}$ *is an arbitrary permutation of some subsequence of the natural numbers, then each of the conditions* (8) *and* (9) *with* $\bar\beta$ *instead of* β *implies that*

$$h_n(f, \beta, p, \{\mu_k\}; x) = O_x(n^{-\bar\beta/p}) \tag{14}$$

holds almost everywhere in (a, b).

Theorem 3. *If* $\alpha>0$, $\bar\alpha=\min(\alpha, 1)$ *and* $\{\mu_k\}$ *is an arbitrary permutation of some subsequence of the natural numbers, then each of the conditions* (11) *and* (12) *implies that*

$$C_n(f, \alpha, p, \{\mu_k\}; x) = O_x(n^{-\bar\alpha/p}) \tag{15}$$

holds almost everywhere in (a, b).

In the following theorems the conditions on the coefficients will be of the same forms as condition (2). The results to be presented can be considered as extensions of Theorems A—D.

Theorem 4. *If* p *and* β *are positive numbers then for any increasing sequence* $\{\nu_k\}$ *condition*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\beta/p} < \infty \quad (\gamma = \beta/p) \tag{16}$$

implies that

$$h_n(f, \beta, p, \{\nu_k\}; x) = o_x(n^{-\beta/p}(\log n)^{1/p}) \tag{17}$$

holds almost everywhere in (a, b).

Theorem 5. *If* α *and* p *are positive numbers then for any increasing sequence* $\{\nu_k\}$ *condition*

$$\sum_{n=1}^{\infty} c_n^2 n^{2/p} < \infty \quad (\gamma = 1/p) \tag{18}$$

implies that

$$C_n(f, \alpha, p, \{\nu_k\}; x) = o_x(n^{-1/p}(\log n)^{1/p}) \tag{19}$$

holds almost everywhere in (a, b).

Theorem 6. *If* p *and* β *are positive numbers, and* $\bar\beta=\min(\beta, 1)$, *then for any permutation* $\{\mu_k\}$ *of some subsequence of the natural numbers condition*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\bar\beta/p} < \infty \quad (\gamma = \bar\beta/p) \tag{20}$$

implies that

$$h_n(f, \beta, p, \{\mu_k\}; x) = o_x(n^{-\bar\beta/p}(\log n)^{1/p}) \tag{21}$$

holds almost everywhere in (a, b).

Theorem 7. *If α and p are positive numbers, and $\bar{\alpha}=\min(\alpha, 1)$, then for any sequence $\{\mu_k\}$ given in Theorem 6 condition*

(22) $$\sum_{n=1}^{\infty} c_n^2 n^{2\bar{\alpha}/p} < \infty \quad (\gamma = \bar{\alpha}/p)$$

implies that

(23) $$C_n(f, \alpha, p, \{\mu_k\};\ x) = o_x\big(n^{-\bar{\alpha}/p}(\log n)^{1/p}\big)$$

holds almost everywhere in (a, b).

§ 1. Lemmas

To prove the theorems we require the following lemmas:

Lemma 1 ([2, Lemma 5]). *Let $\{\lambda_n\}$ be a monotone sequence of positive numbers such that*

$$\sum_{n=1}^{m} \lambda_{2^n}^2 \leqq K\lambda_{2^m}^2\ ^{*)}.$$

If

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^2 < \infty,$$

then we have

$$s_{2^n}(x) - f(x) = o_x(\lambda_{2^n}^{-1})$$

almost everywhere in (a, b)

Lemma 2 ([5, Lemma 4]). *Denote*

$$\sigma_n^*(x) = \begin{cases} c_0\varphi_0(x) & \text{if} \quad n = 0, \\ \dfrac{1}{n-2^{m-1}} \displaystyle\sum_{k=2^m}^{n} (s_k(x) - s_{2^m}(x)) & \text{if} \quad 2^m \leqq n < 2^{m+1};\ m = 0, 1, \ldots. \end{cases}$$

Then for any positive p and $m \geqq 1$

$$\int_a^b \Big\{\frac{1}{2^m} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq K(p) \sum_{n=2^m+1}^{2^{m+1}} c_n^2.$$

Lemma 3 ([5, Lemma 5]). *Let $\gamma > 0$ and $p \geqq 2$. Then condition (2) implies that*

$$\sum_{m=0}^{\infty} \sum_{k=2^m}^{2^{m+1}-1} k^{p\gamma-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p$$

is finite almost everywhere in (a, b).

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*) $K, K_1, K_2, \ldots$ will denote positive constants not necessarily the same at each occurrence. Similarly $K(\alpha), K_1(\alpha), \ldots$ denote constants depending on the parameter α.

Lemma 4 ([5, Lemma 6]). *Under the assumptions of Lemma* 3 *we have*

$$\sum_{k=1}^{\infty} k^{p\gamma-1}|\sigma_k^*(x)|^p<\infty$$

almost everywhere in (a, b).

Lemma 5 ([5, Lemma 7]). *Condition* (2) *with any positive* γ *implies*

$$\sigma_n^*(x)=o_x(n^{-\gamma})$$

almost everywhere in (a, b).

Lemma 6 ([4, Lemma 3]). *Let* $\varkappa>0$ *and* $\{\lambda_n\}$ *be an arbitrary sequence of positive numbers. Assuming that condition*

$$\sum_{n=1}^{\infty}\lambda_n\Big\{\sum_{k=n}^{\infty}c_k^2\Big\}^{\varkappa}<\infty \tag{1.1}$$

implies a "certain property $T=(\{s_n(x)\})$*" of the partial sums* $s_n(x)$ *of* (1) *for any orthonormal system, then* (1.1) *implies that the partial sums* $s_{m_k}(x)$ *of* (1) *also have the same property* T *for any increasing sequence* $\{m_k\}$, *i.e. if*

$$(1.1)\Rightarrow T(\{s_n(x)\})\quad then\quad (1.1)\Rightarrow T(\{s_{m_k}(x)\})$$

for any increasing sequence $\{m_k\}$.

Lemma 7. *Let* $\gamma>0$, $p\geqq 2$ *and* $p\gamma\leqq 1$. *For a given sequence* $\{\mu_k\}$ *of distinct positive integers we define another sequence* $\{m_k\}$ *as follows:* $m_k=2^m$ *if* $2^m\leqq\mu_k<2^{m+1}$. *Then* (2) *implies that the sum*

$$\mu_1(x):=\sum_{k=1}^{\infty}k^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p$$

is finite almost everywhere in (a, b).

Proof. The case $p\gamma<1$ has been proved in [5] (see Lemma 8). If $p\gamma=1$ then

$$\mu_1(x)=\sum_{m=0}^{\infty}\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma_i^*(x)|^p,$$

whence, by Lemma 2 and $p\geqq 2$, we obtain that

$$\int_a^b(\mu_1(x))^{2/p}dx\leqq K\sum_{m=0}^{\infty}2^{2m/p}\sum_{n=2^m+1}^{2^{m+1}}c_n^2<\infty,$$

which prove our lemma with $\gamma=1/p$.

Lemma 8. *Let $\gamma>0$, $p\geqq 2$ and $p\gamma\leqq 1$. Then for any sequence $\{\mu_k\}$ of distinct positive integers the sum*

$$\mu_2(x):=\sum_{k=1}^{\infty} k^{p\gamma-1}|\sigma_{\mu_k}^*(x)|^p$$

is finite almost everywhere in (a, b) if (2) *holds.*

Proof. If $p\gamma<1$ then our lemma is proved in [5, Lemma 9]. The case $p\gamma=1$ follows from Lemma 4 with $\gamma=1/p$, and so the proof is complete.

Lemma 9 ([3, Lemma 2]). *Suppose that γ is a real number and that* (2) *holds. Then for any sequence $\{\mu_k\}$ of distinct positive integers we have the inequality*

$$\int_a^b \{\sum_{k=0}^{\infty} \mu_k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\}\,dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma},$$

where $m_k=2^m$ if $2^m\leqq\mu_k<2^{m+1}$.

Lemma 10. *Suppose that $\gamma>0$, $0<p\leqq 2$ and $\beta=p\gamma$, and that* (2) *holds. Using the notations of Lemma* 9 *we have that*

$$\int_a^b \left\{\sup_{0\leqq n<\infty}\left(\frac{(\log n)^{p/2}}{\log(n+2)}\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p\right)^{1/p}\right\}^2 dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} \tag{1.2}$$

holds if $\beta\leqq 1$; if $\beta>1$ then we only have

$$\int_a^b \left\{\sup_{0\leqq n<\infty}\left(\frac{(\log n)^{p/2}}{\log(n+2)}\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p\right)^{1/p}\right\}^2 dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma}. \tag{1.3}$$

Proof. First we prove (1.3). If $p=2$ then a simple integration gives (1.3). If $p<2$ we use the following form of Hölder's inequality

$$\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p \leqq \{\sum_{k=1}^{n} k^{2(\beta/p)-1}|s_k(x)-s_{m_k}(x)|^2\}^{p/2}\times \tag{1.4}$$
$$\times\{\sum_{k=1}^{n} k^{(1-2(\beta/p))p/(2-p)+2(\beta-1)/(2-p)}\}^{(2-p)/2}.$$

The sum in the second factor does not exceed $K\log n$, and so by (1.4)

$$\sum_{k=1}^{n} k^{\beta-1}|s_k(x)-s_{m_k}(x)|^p \leqq K_1(\log n)^{1-p/2}\{\sum_{k=1}^{\infty} k^{2(\beta/p)-1}|s_k(x)-s_{m_k}(x)|^2\}^{p/2},$$

whence by Lemma 9 with $\mu_k=k$ and $\gamma=\beta/p$ we obtain (1.3).

The proof of (1.2) for $p=2$ is also obtained by integration, but here we require that $\beta\leqq 1$. Namely,

$$\sum_{k=1}^{\infty} k^{\beta-1}\int_a^b |s_{\mu_k}(x)-s_{m_k}(x)|^2\,dx \leqq K\sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_k<2^{m+1}} k^{\beta-1}E_{2^m}^2\leqq$$

$$\leqq K_1\sum_{m=0}^{\infty}E_{2^m}^2\cdot 2^{m\beta}=K_1\sum_{m=0}^{\infty}E_{2^m}^2 2^{m2\gamma}\leqq K_2\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}.$$

In the case $p<2$ we distinguish two cases according as $\gamma\geqq 1/2$ or $0<\gamma<1/2$. If $\gamma\geqq 1/2$ then we use the Hölder's inequality in the following form:

$$\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p=\sum_{k=1}^{n} k^{\beta-1}\mu_k^{p(1/2-\gamma)}\mu_k^{p(\gamma-1/2)}|s_{\mu_k}(x)-s_{m_k}(x)|^p\leqq \tag{1.5}$$

$$\leqq\Big\{\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)}\Big\}^{1-p/2}\Big\{\sum_{k=1}^{n}\mu_k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2}.$$

Next we estimate the sum appearing in the first factor:

$$\sum\nolimits_1=\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)}.$$

If $\gamma=1/2$ then $\beta=p/2$, and so $\sum_1\leqq K\log n$. If $\gamma>1/2$ then $2(\beta-1)/(2-p)>-1$, and therefore with $2^{l-1}<n\leqq 2^l$

$$\sum\nolimits_1\leqq\sum_{m=0}^{\infty}\sum_{\substack{2^m\leqq\mu_k<2^{m+1}\\ k\leqq n}} k^{2(\beta-1)/(2-p)}\mu_k^{p(1-2\gamma)/(2-p)}=\sum_{m=0}^{l}+\sum_{m=l+1}^{\infty}=\sum\nolimits_2+\sum\nolimits_3; \tag{1.6}$$

furthermore

$$\sum\nolimits_2\leqq\sum_{m=0}^{l}2^{mp(1-2\gamma)/(2-p)}\sum_{k=2}^{2^m}k^{2(\beta-1)/(2-p)}\leqq \tag{1.7}$$

$$\leqq K_1\sum_{m=0}^{l}2^{mp(1-2\gamma)/(2-p)}2^{m(1+2(\beta-1)/(2-p))}\leqq K_1\sum_{m=0}^{l}1\leqq K_2\log n$$

and

(1.8)

$$\sum\nolimits_3\leqq\sum_{m=l+1}^{\infty}2^{mp(1-2\gamma)/(2-p)}\sum_{k=1}^{n}k^{2(\beta-1)/(2-p)}\leqq Kn^{1+2(\beta-1)/(2-p)}\cdot 2^{lp(1-2\gamma)(2-p)}\leqq K_1.$$

The estimates (1.5)—(1.8) and Lemma 9 give (1.2) for $\gamma\geqq 1/2$.

Finally we prove (1.2) for $0<\gamma<1/2$ and $p<2$. As in (1.5) we use again Hölder's inequality with k instead of μ_k. We obtain that

$$\sum_{k=1}^{n} k^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p \leqq$$

$$\text{(1.9)} \qquad \leqq \Big\{\sum_{k=1}^{n} k^{2(\beta-1)/(2-p)}k^{p(1-2\gamma)/(2-p)}\Big\}^{1-p/2}\Big\{\sum_{k=1}^{n} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2} \leqq$$

$$\leqq K(\log n)^{1-p/2}\Big\{\sum_{k=1}^{n} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}^{p/2}.$$

If we can show that

$$\text{(1.10)} \qquad \int_a^b \Big\{\sum_{k=1}^{\infty} k^{2\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)|^2\Big\}\,dx \leqq K\sum_{n=1}^{\infty} c_n^2 n^{2\gamma};$$

then (1.9) and (1.10) will yield (1.2) with $0<\gamma<1/2$, too.

Now we verify (1.10) as follows:

$$\sum_{k=1}^{\infty} k^{2\gamma-1}\int_a^b |s_{\mu_k}(x)-s_{m_k}(x)|^2\,dx = \sum_{k=1}^{\infty} k^{2\gamma-1}\sum_{n=m_k+1}^{\mu_k} c_n^2 \leqq$$

$$\leqq \sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_k<2^{m+1}} k^{2\gamma-1}\sum_{n=2^m+1}^{2^{m+1}} c_n^2 \leqq \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} c_n^2\sum_{k=1}^{2^m} k^{2\gamma-1} \leqq K\sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}} c_n^2 n^{2\gamma}.$$

Herewith we completed the proof.

§ 2. Proof of the theorems

Proof of Theorem 1. If $\alpha=1$ then (13) follows from Theorem E with $\beta=1$, since $h_n(f, 1, p, \{v_k\}; x)=C_n(f, 1, p, \{v_k\}; x)$. On the other hand, in respect to the following elementary fact:

$$\text{(2.1)} \qquad \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha} \leqq \frac{K}{n} \quad \text{for any} \quad \alpha \geqq 1,$$

we have for $\alpha>1$ that

$$\text{(2.2)} \qquad C_n(f, \alpha, p, \{v_k\}; x) \leqq KC_n(f, 1, p, \{v_k\}; x),$$

so (13) is proved for any $\alpha\geqq 1$.

Now let $0<\alpha<1$. We put $C_n(x):=C_n(f, \alpha, p, \{k\}; x)$ and $2^m\leqq n<2^{m+1}$ $(m\geqq 2)$. Then

$$\text{(2.3)} \qquad C_n(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{2^{m-1}} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p} + \right.$$

$$\left. + \left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right) = K\big(C_n^{(1)}(x)+C_n^{(2)}(x)\big).$$

Here the first term $C_n^{(1)}(x)$ is of the order $O_x(n^{-1/p})$, for after simplification it becomes a part of the mean $C_n(f, 1, p, \{k\}; x)$.

Now we estimate $C_n^{(2)}(x)$ as follows:

$$(2.4)\qquad C_n^{(2)}(x) \leqq K\Bigg(\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_{2^{m-1}}(x)-f(x)|^p\Big\}^{1/p}+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_{2^m}(x)-f(x)|^p\Big\}^{1/p}+\Big\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|\sigma_k^*(x)|^p\Big\}^{1/p}\Bigg)=:K\sum_{i=1}^{5} D_n^{(i)}(x).$$

By Lemmas 1 and 5 we have

$$(2.5)\qquad D_n^{(2)}(x)+D_n^{(4)}(x)+D_n^{(5)}(x)=o_x(n^{-1/p}),$$

since it is almost trivial that conditions (11) and (12), separately, imply

$$(2.6)\qquad \sum_{n=1}^{\infty} c_n^2 n^{2/p}<\infty.$$

The implication (11)⇒(2.6) can be proved as follows: By $p/2\leqq 1$ we have

$$\sum_{n=1}^{\infty} c_n^2 n^{2/p} \leqq K(p)\sum_{m=1}^{\infty} m^{(2/p)-1}\sum_{n=m}^{\infty} c_n^2 \leqq K_1(p)\sum_{m=1}^{\infty} 2^{m2/p}\sum_{n=2^m+1}^{\infty} c_n^2 \leqq$$

$$\leqq K_2(p)\Big(\sum_{m=1}^{\infty} 2^m\Big\{\sum_{n=2^m+1}^{\infty} c_n^2\Big\}^{p/2}\Big)^{2/p} \leqq K_3(p)\Big(\sum_{n=1}^{\infty}\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{p/2}\Big)^{2/p}.$$

In order to estimate $D_n^{(1)}(x)$ and $D_n^{(3)}(x)$ we use the Hölder inequality with q being chosen such that $q>1$ and $(\alpha-1)q>-1$. Then

$$(2.7)\qquad \begin{aligned} D_n^{(1)}(x) &\leqq \frac{1}{(A_n^\alpha)^{1/p}}\Big\{\sum_{k=2^{m-1}+1}^{2^m-1}(A_{n-k}^{\alpha-1})^q\Big\}^{1/pq}\Big\{\sum_{k=2^{m-1}+1}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\Big\}^{1/pq'} \leqq \\ &\leqq K\Big\{\frac{1}{2^m}\sum_{k=2^{m-1}}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\Big\}^{1/pq'}=:D_m^*(x). \end{aligned}$$

Furthermore, by Lemma 2 and (2.6), we obtain that

$$\sum_{m=1}^{\infty}\int_a^b (2^{m/p}D_m^*(x))^2\,dx \leqq K_1\sum_{m=1}^{\infty} 2^{2m/p}\sum_{n=2^{m-1}+1}^{2^m} c_n^2<\infty,$$

which by (2.7) implies

$$D_n^{(1)}(x) \leqq D_m^*(x) = o_x(2^{-m/p}) = o_x(n^{-1/p}).$$

$D_n^{(3)}(x)$ can be estimated similarly by $D_{m+1}^*(x)$, and so $D_n^{(3)}(x) = o_x(n^{-1/p})$

also holds almost everywhere in (a, b).

Collecting the given estimates we obtain the following result

$$C_n(f, \alpha, p, \{k\}; x) = O_x(n^{-1/p})$$

almost everywhere in (a, b).

Hence, using Lemma 6 with $\varkappa=p/2$, $\lambda_n=1$ for $0<p\leqq 2$; and with $\varkappa=1$, $\lambda_n=1$ for $p\geqq 2$; furthermore with the property T given by

$$T(\{s_n(x)\}) := C_n(f, \alpha, p, \{k\}; x) = O_x(n^{-1/p}),$$

we obtain the statement of Theorem 1 immediately. The proof is complete.

Proof of Theorem 2. First we prove the case $\beta\leqq 1$, then $\bar{\beta}=\beta$, and so (14) means that

$$\sum\nolimits_1 := \sum_{k=1}^{\infty} k^{\beta-1} |s_{\mu_k}(x) - f(x)|^p \tag{2.8}$$

converges almost everywhere in (a, b).

To verify (2.8) we first consider the case $0<p<2$. Then, using the notation $E_n^2 = \sum_{k=n+1}^{\infty} c_k^2$, we have

$$\int_a^b \left(\sum\nolimits_1\right) dx = \sum_{m=0}^{\infty} \sum_{2^m \leqq \mu_k < 2^{m+1}} k^{\beta-1} \int_a^b |s_{\mu_k}(x) - f(x)|^p \, dx \leqq$$

$$\leqq K \sum_{m=0}^{\infty} \sum_{2^m \leqq \mu_k < 2^{m+1}} k^{\beta-1} \left(\int_a^b |s_{\mu_k}(x) - f(x)|^2 \, dx\right)^{p/2} \leqq$$

$$\leqq K_1 \sum_{m=0}^{\infty} 2^{m\beta} E_{2^m}^p \leqq K_2 \sum_{n=1}^{\infty} n^{\beta-1} E_n^p < \infty,$$

whence, by the Beppo Levi theorem, the convergence of series (2.8) follows almost everywhere in (a, b).

If $p\geqq 2$, then the following obvious estimate

$$\sum\nolimits_1 \leqq \left\{\sum_{k=1}^{\infty} k^{2(\beta-1)/p} |s_{\mu_k}(x) - f(x)|^2\right\}^{p/2} := \left(\sum\nolimits_2\right)^{p/2} \tag{2.9}$$

shows that it is enough to prove that condition (9) with $\bar\beta$ implies the finiteness of $\sum_2$ almost everywhere. But, by $-1<2/p(\beta-1)<0$,

$$\int (\textstyle\sum_2)\, dx = \sum_{m=0}^{\infty} \sum_{2^m \le k < 2^{m+1}} k^{2(\beta-1)/p} \int_a^b |s_{\mu_k}(x)-f(x)|^2\, dx \le$$

$$\le K \sum_{m=0}^{\infty} E^{m(1+2(\beta-1)/p)} E_{2^m}^2 \le K_1 \sum_{n=1}^{\infty} c_n^2 n^{1+2(\beta-1)/p} < \infty,$$

so the series (2.9) converges, which completes the proof for $\beta \le 1$.

Since

$$h_n(f, \beta, p, \{\mu_k\}; x) \le h_n(f, \bar\beta, p, \{\mu_k\}; x) \tag{2.10}$$

always holds, thus the proof of Theorem 2 is complete.

Proof fo Theorem 3. On account of the following inequality

$$C_n(f, \alpha, p, \{\mu_k\}; x) \le C(f, \bar\alpha, p, \{\mu_k\}; x)$$

we may assume that $\alpha \le 1$, then $\bar\alpha = \alpha$. Furthermore the case $\alpha=1$ is the same as the case $\beta=1$ of Theorem 2, so we may assume that $0<\alpha<1$. In this case we can choose a number $q>1$ such that $(\alpha-1)q<-1$; and if we now use the Hölder's inequality with this q and $q'=q/(q-1)$ then

$$C_n(f, \alpha, p, \{\mu_k\}; x) \le \left\{\frac{1}{(A_n^\alpha)^q} \sum_{k=0}^{n} (A_{n-k}^{\alpha-1})^q\right\}^{1/pq} \left\{\sum_{k=0}^{n} |s_{\mu_k}(x)-f(x)|^{pq'}\right\}^{1/pq'} \le$$

$$\le K n^{-\alpha/p} \left\{\sum_{k=0}^{n} |s_{\mu_k}(x)-f(x)|^{pq'}\right\}^{1/pq'}$$

holds. This will prove our theorem if we can show that

$$\sum_{k=0}^{\infty} |s_{\mu_k}(x)-f(x)|^{pq'} \tag{2.11}$$

converges almost everywhere in (a, b). But, by the special case $\beta=1$ of Theorem 2, our assumptions (11) and (12) imply the convergence of the series

$$\sum_{k=0}^{\infty} |s_{\mu_k}(x)-f(x)|^{p}$$

almost everywhere, and so on account of $q'>1$ the series (2.11) converges almost everywhere, too. This has completed the proof.

Proof of Theorem 4. If $p=2$ then Theorem E yields a sharper estimate than (17). Thus we have to prove our theorem only for $p\ne 2$. First we prove (17) for $0<p<2$. By Lemma 6 it is also clear that it will be enough to prove (17) in the speical case $\nu_k=k$. Then

$$h_n^p(f, \beta, p, \{k\};\ x) \leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-s_{m_k}(x)|^p+\right. \tag{2.12}$$

$$\left.+\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\right\}.$$

Here the second sum, by Lemma 1, does not exceed $o_x(n^{-\beta}\log n)$.

In the estimation of the first sum we can use the statement (1.3) of Lemma 10. So we obtain that this sum has the order $O_x(n^{-\beta}(\log n)^{1-p/2})$ which is even better than the required $o_x(n^{-\beta}\log n)$.

These estimations and (2.12) obviously imply (17) for $0<p<2$ and $\nu_k=k$. If $p>2$ then we use the following estimation with the assumption $2^m\leqq n<2^{m+1}$:

$$h_n(f, \beta, p, \{k\};\ x) \leqq K\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{\nu=2^\nu}^{2^{\nu+1}-1} k^{\beta-1}|s_k(x)-f(x)|^p\Big\}^{1/p} \leqq$$

$$\leqq K_1\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1}|s_k(x)-s_{2^\nu}(x)-\sigma_k^*(x)|^p\Big\}^{1/p}+$$

$$+\Big\{n^{-\beta}\sum_{\nu=0}^{m}\sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1}|s_{2^\nu}(x)-f(x)|^p\Big\}^{1/p}+\Big\{n^{-\beta}\sum_{k=1}^{2^{m+1}} k^{\beta-1}|\sigma_k^*(x)|^p\Big\}^{1/p} = K_1\sum_{i=1}^{3}D_n^{(i)}(x).$$

Using Lemma 3 and 4 with $\gamma=\beta/p$ we obtain that

$$D_n^{(1)}(x) = O_x(n^{-\beta/p}) \quad\text{and}\quad D_n^{(3)}(x) = O_x(n^{-\beta/p}),$$

furthermore by Lemma 1

$$D_n^{(2)}(x) = o_x(n^{-\beta/p}(\log n)^{1/p}).$$

Summing up our partial estimations, we get that

$$h_n(f, \beta, p, \{k\};\ x) = o_x(n^{-\beta/p}(\log n)^{1/p}),$$

and this, by Lemma 6, conveys the assertion of Theorem 4.

Proof of Theorem 5. On account of Lemma 6 we have to prove (19) only for the special case $\nu_k=k$. In the special case $p=2$ Theorem 1 gives a better estimate than (19) does. Hence it is sufficient to consider the cases $p\neq 2$. We can follow the line of the proof of Theorem 1. Using the notations introduced there, we have

$$C_n(f, \alpha, p, \{k\};\ x) \leqq K\big(C_n^{(1)}(x)+C_n^{(2)}(x)\big), \tag{2.13}$$

where $C_n^{(1)}(x)$ has the order $o_x(n^{-1/p}(\log n)^{1/p})$ since

$$C_n^{(1)}(x) \leqq Kh_n(f, 1, p, \{k\};\ x)$$

and so Theorem 4 conveys the order of approximation given above.

The sum $C_n^{(2)}(x)$ can be estimated exactly the same way as in the proof of Theorem 1, namely the condition (2.6) which was used during the estimation of $C_n^{(2)}(x)$ is the same as (18). So we have $C_n^{(2)}(x)=o_x(n^{-1/p})$. Collecting these estimations, by (2.13), we obtain (19) for $v_k=k$; and this was to be proved.

Proof of Theorem 6. On account of the obvious inequality (2.10) we may assume that $\beta\leqq 1$ and so $\bar{\beta}=\beta$. We may also omit the proof of the case $p=2$, for then Theorem 2 gives a sharper estimation than (21) claims. In the subsequent steps of the proof we distinguish two cases according to $0<p<2$ or $p>2$.

In the case $0<p<2$ we start with the following estimation

$$h_n^p(f,\beta,p,\{\mu_k\};x)\leqq K\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^n (k+1)^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)|^p+\right.$$

(2.14)

$$\left.+\frac{1}{(n+1)^\beta}\sum_{k=0}^n (k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\right\}=\sum\nolimits_1+\sum\nolimits_2.$$

Here the first sum, by the statement (1.2) of Lemma 10, has the following order

$$\sum\nolimits_1 = O_x\big(n^{-\beta}(\log n)^{1-p/2}\big). \tag{2.15}$$

To estimate $\sum_2$ we assume $2^{l-1}<n<2^l$. Then, by Lemma 1, we have

$$\sum\nolimits_2=\sum_{m=0}^{\infty}\Bigg(\sum_{\substack{2^m\leqq\mu_k<2^{m+1}\\ k\leqq n}}\frac{(k+1)^{\beta-1}}{(n+1)^\beta}\Bigg)o_x(2^{-m\beta})=\sum_{m=0}^{l}+\sum_{m=l+1}^{\infty}=\sum\nolimits_3+\sum\nolimits_4. \tag{2.16}$$

A simple consideration gives that

$$\sum\nolimits_3=n^{-\beta}\sum_{m=0}^{l}\Big(\sum_{k=1}^{2^m}k^{\beta-1}\Big)o_x(2^{-m\beta})=o_x(n^{-\beta}\log n) \tag{2.17}$$

and

$$\sum\nolimits_4=n^{-\beta}\sum_{m=l+1}^{\infty}\Big(\sum_{k=1}^{n}k^{\beta-1}\Big)o_x(2^{-m\beta})=o_x(2^{-l\beta})=o_x(n^{-\beta}). \tag{2.18}$$

Collecting the estimations (2.14)—(2.18) we obtain that

$$h_n^p(f,\beta,p,\{\mu_k\};x)=o_x(n^{-\beta}\log n)$$

holds almost everywhere in (a,b), and this proves (21) for $p<2$.

If $p>2$ then we use the following estimation:

$$h_n^p(f,\beta,p,\{\mu_k\};x)\leqq K\Bigg(\frac{1}{(n+1)^\beta}\sum_{k=0}^n (k+1)^{\beta-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p+$$

$$+\frac{1}{(n+1)^\beta}\sum_{k=0}^n (k+1)^{\beta-1}|\sigma_{\mu_k}^*(x)|^p+\frac{1}{(n+1)^\beta}\sum_{k=0}^n (k+1)^{\beta}|s_{m_k}(x)-f(x)|^p\Bigg)=$$

$$=K\big(\sum\nolimits_5+\sum\nolimits_6+\sum\nolimits_2\big).$$

Above we have verified that $\sum_2 = o_x(n^{-\beta}\log n)$.
To estimate $\sum_5$ we apply Lemma 7, whence

$$\sum\nolimits_5 = O_x(n^{-\beta})$$

follows. Similarly Lemma 8 gives that

$$\sum\nolimits_6 = O_x(n^{-\beta}).$$

Summing up these partial results, we again arrive at (21), and this completes the proof.

Proof of Theorem 7. Using the same arguments as we did at the beginning of the proof of Theorem 3 we may assume $0<\alpha<1$. Then we can use the Hölder inequality with $1/\alpha$ and $1/(1-\alpha)$ and obtain that

$$\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p \leqq \Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha}\Big\{\sum_{k=0}^{n}(A_{n-k}^{\alpha-1})^{1/(1-\alpha)}\Big\}^{1-\alpha}.$$

Hence we obtain that

$$\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} \leqq Kn^{-\alpha/p}(\log n)^{1/p-\alpha/p}\left\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\right\}^{\alpha/p}.$$

To prove (23) it suffices to verify that

$$(\log n)^{-\alpha/p}\Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha/p} = o_x(1) \tag{2.19}$$

holds almost everywhere in (a, b). If we apply Theorem 6 with $\beta=1$ and p/α (instead of p), then (21) gives that

$$\Big\{\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^{p/\alpha}\Big\}^{\alpha/p} = o_x\big((\log n)^{\alpha/p}\big),$$

and so (2.19) is fulfilled, indeed. Theorem 7 is hereby proved.

References

[1] L. Leindler, Über die Rieszschen Mittel allgemeiner Orthogonalreihen, *Acta Sci. Math.*, **24** (1963), 129—138.

[2] L. Leindler, On the strong approximation of orthogonal series, *Acta Sci. Math.*, **32** (1971), 41—50.

[3] L. Leindler, On the extra strong approximation of orthogonal series, *Analysis Math.*, **8** (1982), 125—133.
[4] L. Leindler, On the strong approximation of orthogonal series with large exponent, *Analysis Math.*, **8** (1982), 173—179.
[5] L. Leindler and H. Schwinn, On the strong and extra strong approximation of orthogonal series, *Acta Sci. Math.*, **45** (1983), 293—304.
[6] G. Sunouchi, Strong approximation by Fourier series and orthogonal series, *Indian J. Math.*, **9** (1967), 237—246.

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