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The asymptotic distribution of generalized Rényi statistics

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In honor of Professor Károly Tandori on his sixtieth birthday

1. Introduction and preliminaries

For each integer $n \geqq 1$ let $U_1, \ldots, U_n$ be independent Uniform (0, 1) random variables, $U_{1,n} \leqq \ldots \leqq U_{n,n}$ be their corresponding order statistics and G_n the right continuous empirical distribution function based on these n independent uniform (0, 1) random variables. We shall begin by stating some results in the literature that motivated our present investigation.

DANIELS [5] showed that for any $-\infty < x < \infty$

$$P\left\{\sup_{0 \leqq s \leqq 1} \frac{G_n(s) - s}{s} \leqq x\right\} = F(x), \tag{1}$$

where

$$F(x) = \begin{cases} \dfrac{x}{x+1}, & \text{for } 0 \leqq x < \infty, \\ 0, & \text{for } x < 0. \end{cases}$$

Let $N(t)$, $0 \leqq t < \infty$, denote a right continuous Poisson process with parameter one. PYKE [10] proved that for any $-\infty < x < \infty$

$$P\left\{\sup_{0 \leqq t < \infty} \frac{N(t) - t}{t} \leqq x\right\} = F(x). \tag{2}$$

Combining statements (1) and (2), we have for each $n \geqq 1$

$$\sup_{0 \leqq s \leqq 1} \frac{G_n(s) - s}{s} \stackrel{\mathscr{D}}{=} \sup_{0 \leqq t < \infty} \frac{N(t) - t}{t},$$

where $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution. More generally, a slight modification of

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the techniques of Mason [8] establishes that for each $0 \leqq \nu < 1/2$, as $n \to \infty$,

$$n^{\nu} \sup_{0 \leqq s \leqq 1} \frac{G_n(s) - s}{s^{1-\nu}} \xrightarrow{\mathcal{D}} \sup_{0 \leqq t < \infty} \frac{N(t) - t}{t^{1-\nu}}. \tag{3}$$

(The symbol $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.) A result closely related to (3) proven in MASON [8] is that for any $0 \leqq \nu < 1/2$, as $n \to \infty$,

$$n^{\nu} \sup_{0 \leqq s \leqq 1} \frac{|G_n(s) - s|}{(s(1-s))^{1-\nu}} \xrightarrow{\mathcal{D}} N_\nu \vee N'_\nu, \tag{4}$$

where

$$N_\nu := \sup_{0 \leqq t < \infty} \frac{|N(t) - t|}{t^{1-\nu}},$$

$N'_\nu \stackrel{\mathcal{D}}{=} N_\nu$, N_ν and N'_ν are independent random variables, and the symbol $\vee$ denotes maximum.

If a $1/2 \leqq \nu \leqq 1$ is chosen, the Poisson limit behavior in (3) and (4) breaks down. In particular, when $1/2 < \nu \leqq 1$ and n^ν is replaced by $n^{1/2}$, the limiting distribution of the left side of (4) is the same as that of

$$\sup_{0 \leqq s \leqq 1} \frac{|B(s)|}{(s(1-s))^{1-\nu}},$$

where $B(s)$, $0 \leqq s \leqq 1$, is a Brownian bridge defined on [0, 1]. When $\nu = 1/2$, with additional normalizing constants applied, the limiting distribution of the left side of (4) is an extreme value distribution. For details the reader is referred to O'REILLY [9], EICKER [6], JAESCHKE [7], the discussion in MASON [8], or to the exhaustive study in M. CSÖRGŐ, S. CSÖRGŐ, HORVÁTH, and MASON (Cs—Cs—H—M) [4].

When $\nu = 0$, the limiting Poisson behaviour of the left side of (3) can break down in another way, if the supremum is not taken over the entire interval [0, 1]. RÉNYI [11] (also see M. CSÖRGŐ [3]) showed that for any fixed $0 < a < 1$, as $n \to \infty$,

$$\left(\frac{a}{1-a}\right)^{1/2} \sup_{a \leqq s \leqq 1} n^{1/2} \frac{\{G_n(s) - s\}}{s} \xrightarrow{\mathcal{D}} \sup_{0 \leqq t \leqq 1} W(t), \tag{5}$$

and

$$\left(\frac{a}{1-a}\right)^{1/2} \sup_{a \leqq s \leqq 1} n^{1/2} \frac{|G_n(s) - s|}{s} \xrightarrow{\mathcal{D}} \sup_{0 \leqq t \leqq 1} |W(t)|, \tag{6}$$

where $W(t)$, $0 \leqq t \leqq 1$, denotes a standard Brownian motion defined on [0, 1]. CSÁKI [2] demonstrated that (5) remains true if a is replaced by any sequence of positive constants a_n such that as $n \to \infty$,

$$0 < a_n < 1, \quad a_n \to 0, \quad \text{and} \quad na_n \to \infty. \tag{7}$$

This suggests that if the supremum on the left side of (3) is restricted to $[a_n, 1]$, where the sequence a_n satisfies condition (7), when appropriately normalized, its limiting distribution should be the same as that of the supremum of a certain Gaussian process; and the same should be true if the supremum on the left side of (4) is restricted to an interval of the form $[a_n, 1-a_n]$. In the next section, we shall show that this is indeed the case. Such statistics will be called *generalized Rényi statistics.*

The main tool which we shall use to establish our results will be a new Brownian bridge approximation to the uniform empirical and quantile processes recently obtained by Cs—Cs—H—M [4]. We shall now describe some of its basic features.

In Cs—Cs—H—M [4] a probability space $(\Omega, \mathscr{A}, P)$ is constructed carrying a sequence $U_1, U_2, \ldots,$ of independent Uniform (0, 1) random variables and a sequence of Brownian bridges $B_n(s)$, $0 \leqq s \leqq 1$, $n=1, 2, \ldots,$ such that for the uniform empirical process

$$\alpha_n(s) = n^{1/2}\{G_n(s)-s\}, \quad 0 \leqq s \leqq 1,$$

and the uniform quantile process

$$\beta_n(s) = n^{1/2}\{s-U_n(s)\}, \quad 0 \leqq s \leqq 1,$$

where

$$U_n(s) = \begin{cases} U_{k,n}, & \text{if} \quad (k-1)/n < s \leqq k/n, \quad k = 1, \ldots, n \\ U_{1,n}, & \text{if} \quad s = 0, \end{cases}$$

we have

$$\sup_{1/n \leqq s \leqq 1} \frac{|\alpha_n(s)-B_n(s)|}{s^{1/2-\delta_1}} = O_P(n^{-\delta_1}), \tag{8}$$

$$\sup_{0 \leqq s \leqq 1-1/n} \frac{|\alpha_n(s)-B_n(s)|}{(1-s)^{1/2-\delta_1}} = O_P(n^{-\delta_1}), \tag{9}$$

$$\sup_{1/(n+1) \leqq s \leqq 1} \frac{|\beta_n(s)-B_n(s)|}{s^{1/2-\delta_2}} = O_P(n^{-\delta_2}), \tag{10}$$

and

$$\sup_{0 \leqq s \leqq 1-1/(n+1)} \frac{|\beta_n(s)-B_n(s)|}{(1-s)^{1/2-\delta_2}} = O_P(n^{-\delta_2}), \tag{11}$$

where δ_1 and δ_2 are any fixed numbers such that $0 \leqq \delta_1 < 1/4$ and $0 \leqq \delta_2 < 1/2$. Statements (8) and (9) are contained in Corollary 2.1, while statements (10) and (11) follow from Theorem 2.1 of the above paper. We shall also need the fact that statements (8) and (9) remain true on this probability space for any $0 < \delta_1 \leqq 1/2$ with the supremum taken over [0, 1] and the $O_P(n^{-\delta_1})$ replaced by $o_P(1)$. This follows from the general results on *q-metric convergence* in Cs—Cs—H—M [4]. *In the proofs of the next section it will be assumed without comment that we are on the probability space just described.*

2. The main results

For any $0 \leqq \nu < 1/2$, let

$$X_\nu := \sup_{0 \leqq t \leqq 1} \frac{W(t)}{t^\nu},$$

and

$$Y_\nu := \sup_{0 \leqq t \leqq 1} \frac{|W(t)|}{t^\nu}.$$

Since $0 \leqq \nu < 1/2$, a simple application of the law of the iterated logarithm for Brownian motion shows that X_ν and Y_ν are almost surely finite.

When $\nu = 0$, our first theorem contains the results of RÉNYI [11] and CSÁKI [2] quoted in the Introduction.

Theorem 1. *Let a_n be any sequence of positive contants such that for some $0 < \beta < 1$ we have $0 < a_n \leqq \beta$ for all large enough n, and $na_n \to \infty$. Then for every $0 \leqq \nu < 1/2$, as $n \to \infty$,*

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} \alpha_n(s)/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathscr{D}} X_\nu; \tag{12}$$

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} \alpha_n(s)/((1-s)^{1-\nu}s^\nu) \xrightarrow{\mathscr{D}} X_\nu; \tag{13}$$

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} |\alpha_n(s)|/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathscr{D}} Y_\nu; \tag{14}$$

and

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} |\alpha_n(s)|/((1-s)^{1-\nu}s^\nu) \xrightarrow{\mathscr{D}} Y_\nu. \tag{15}$$

Proof. First consider (12) and (14). Choose any $0 \leqq \nu < 1/2$. Observe that for all n sufficiently large

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/(s^{1-\nu}(1-s)^\nu) \leqq$$
$$\leqq (1-\beta)^{-1/2} a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq \beta} |\alpha_n(s) - B_n(s)|/s^{1-\nu} +$$
$$+ \beta^{-1/2}(1-\beta)^{-1/2+\nu} \sup_{\beta \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/(1-s)^\nu := \Delta_{1,n} + \Delta_{2,n}.$$

Applying the version of statement (9) with the choice $\delta_1 = 1/2 - \nu$, where the supremum is taken over [0, 1], we see that

$$\Delta_{2,n} = o_P(1). \tag{16}$$

Also notice that for $0 < \delta_1 < 1/4$, not necessarily the same δ_1 as above,

$$\Delta_{1,n} \leqq (1-\beta)^{-1/2} a_n^{-\delta_1} \sup_{1/n \leqq s \leqq 1} |\alpha_n(s) - B_n(s)|/s^{1/2-\delta_1}. \tag{17}$$

Now applying (8) we see that the right side of inequality (17) equals

(18) $$(1-\beta)^{-1/2}a_n^{-\delta_1}O_P(n^{-\delta_1}) = O_P((na_n)^{-\delta_1}),$$

which by the assumption that $na_n \to \infty$ as $n \to \infty$ equals $o_P(1)$.

Since for each $n \geqq 1$ such that $0 < a_n < 1$ the process

$$\left\{\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} B_n(s)/(s^{1-\nu}(1-s)^\nu) \colon a_n \leqq s \leqq 1\right\}$$

is equal in distribution to the process

$$\left\{\left(\frac{a_n}{1-a_n}\right)^{-\nu} \left(\frac{t}{1-t}\right)^\nu W\left(\left(\frac{a_n}{1-a_n}\right)\left(\frac{1-t}{t}\right)\right) \colon a_n \leqq t \leqq 1\right\},$$

we have for each $-\infty < x < \infty$ that

$$P\left\{\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} B_n(s)/(s^{1-\nu}(1-s)^\nu) \leqq x\right\}$$

$$= P\left\{\left(\frac{a_n}{1-a_n}\right)^{-\nu} \sup_{a_n \leqq t \leqq 1} \left(\frac{t}{1-t}\right)^\nu W\left(\left(\frac{a_n}{1-a_n}\right)\left(\frac{1-t}{t}\right)\right) \leqq x\right\} =$$

$$= P\left\{\sup_{0 \leqq t \leqq 1} W(t)/t^\nu \leqq x\right\}.$$

Obviously the same statement holds with $B_n(s)$ and $W(t)$ replaced by $|B_n(s)|$ and $|W(t)|$ respectively. Thus on account of (16), (17), and (18) we have (12) and (14). Assertions (13) and (15) follow from (12) and (14) respectively by symmetry considerations. This completes the proof of Theorem 1.

Theorem 2. *Let a_n be any sequence of positive constants such that $na_n \to \infty$ and $a_n \to 0$ as $n \to \infty$. Then for every $0 \leqq \nu < 1/2$, as $n \to \infty$,*

(19) $$S_{n,\nu} := a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} \alpha_n(s)/s^{1-\nu} \xrightarrow{\mathcal{D}} X_\nu;$$

(20) $$S'_{n,\nu} := a_n^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} \alpha_n(s)/(1-s)^{1-\nu} \xrightarrow{\mathcal{D}} X_\nu;$$

(21) $$T_{n,\nu} := a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq 1} |\alpha_n(s)|/s^{1-\nu} \xrightarrow{\mathcal{D}} Y_\nu;$$

and

(22) $$T'_{n,\nu} := a_n^{1/2-\nu} \sup_{0 \leqq s \leqq 1-a_n} |\alpha_n(s)|/(1-s)^{1-\nu} \xrightarrow{\mathcal{D}} Y_\nu.$$

Moreover, the random variables $S_{n,\nu}$ and $S'_{n,\nu}$, respectively the random variables $T_{n,\nu}$ and $T'_{n,\nu}$, are asymptotically independent.

Proof. Choose any $0 \leqq \nu < 1/2$. Let b_n denote any sequence of positive constants such that (i) $nb_n \to \infty$, (ii) $b_n \to 0$, and (iii) $a_n/b_n \to 0$ as $n \to \infty$. Write

$$S_{n,\nu}(a_n, b_n) := a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq b_n} \alpha_n(s)/s^{1-\nu}$$

and

$$S'_{n,\nu}(a_n, b_n) := a_n^{1/2-\nu} \sup_{1-b_n \leqq s \leqq 1-a_n} \alpha_n(s)/(1-s)^{1-\nu}.$$

Notice that for all n sufficiently large

$$|S_{n,\nu} - S_{n,\nu}(a_n, b_n)| \leqq \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{b_n \leqq s \leqq 1} |\alpha_n(s)|/\big(s^{1-\nu}(1-s)^{\nu}\big). \tag{23}$$

Applying (14) and (iii), we see that the right side of inequality (23) equals

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \left(\frac{b_n}{1-b_n}\right)^{-1/2+\nu} O_P(1) = o_P(1).$$

Thus we have

$$|S_{n,\nu} - S_{n,\nu}(a_n, b_n)| = o_P(1). \tag{24}$$

In the same way we have

$$|S'_{n,\nu} - S'_{n,\nu}(a_n, b_n)| = o_P(1). \tag{25}$$

Hence to prove (19) and (20) it is sufficient to show that, as $n \to \infty$,

$$S_{n,\nu}(a_n, b_n) \xrightarrow{\mathscr{D}} X_\nu, \tag{26}$$

and

$$S'_{n,\nu}(a_n, b_n) \xrightarrow{\mathscr{D}} X_\nu. \tag{27}$$

Clearly

$$\sup_{a_n \leqq s \leqq b_n} \left| a_n^{1/2-\nu} \alpha_n(s)/s^{1-\nu} - \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \alpha_n(s)/\big(s^{1-\nu}(1-s)^{\nu}\big) \right| \leqq$$

$$\leqq \sup_{a_n \leqq s \leqq b_n} |(1-a_n)^{1/2-\nu}(1-s)^{\nu} - 1| \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} |\alpha_n(s)|/\big(s^{1-\nu}(1-s)^{\nu}\big),$$

which obviously equals

$$o(1) \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq b_n} |\alpha_n(s)|/\big(s^{1-\nu}(1-s)^{\nu}\big).$$

Statement (14) implies that this last expression equals $o(1)O_P(1) = o_P(1)$.

Notice that it was shown above that

$$\left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{b_n \leqq s \leqq 1} |\alpha_n(s)|/\big(s^{1-\nu}(1-s)^{\nu}\big) = o_P(1). \tag{28}$$

From (28) and (12) we have that

$$(29) \qquad \left(\frac{a_n}{1-a_n}\right)^{1/2-\nu} \sup_{a_n \leqq s \leqq b_n} \alpha_n(s)/(s^{1-\nu}(1-s)^\nu) \xrightarrow{\mathscr{D}} X_\nu,$$

which by the preceding arguments implies (26). Since

$$S'_{n,\nu}(a_n, b_n) \overset{\mathscr{D}}{=} S_{n,\nu}(a_n, b_n),$$

assertion (27) follows from (26). Hence we have established (19) and (20). Statements (21) and (22) follow by almost the same argument as that just given.

We shall now demonstrate that the random variables $S_{n,\nu}$ and $S'_{n,\nu}$ are asymptotically independent. On account of (24) and (25) it suffices to show that the random variables $S_{n,\nu}(a_n, b_n)$ and $S'_{n,\nu}(a_n, b_n)$ are asymptotically independent.

Choose any sequence of positive integers $1 \leqq k_n \leqq n$ such that $k_n \to \infty$, $k_n/n \to 0$ and $nb_n/k_n \to 1/2$ as $n \to \infty$. Observe that the function

$$V_{n,\nu} := S_{n,\nu}(a_n, b_n) I(U_{k_n,n} > b_n)$$

$(I(x>y)=1$ or 0 according as $x>y$ or $x \leqq y)$ is almost surely a function only of the lower extreme order statistics $U_{1,n}, \dots, U_{k_n,n}$ and the random variable

$$V'_{n,\nu} := S'_{n,\nu}(a_n, b_n) I(1-b_n > U_{n-k_n,n})$$

is almost surely a function only of the upper extreme order statistics $U_{n-k_n,n}, \dots,$ $\dots, U_{n,n}$. Since $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$, we conclude by Satz 4 of Rossberg [12] that the random variables $V_{n,\nu}$ and $V'_{n,\nu}$ are asymptotically independent. Also an elementary argument shows that, as $n \to \infty$,

$$n U_{k_n,n}/k_n \xrightarrow{P} 1$$

and

$$n(1-U_{n-k_n,n})/k_n \xrightarrow{P} 1.$$

(See page 18 of Balkema and de Haan [1].) Thus by our choice of k_n, we have, as $n \to \infty$,

$$P\{V_{n,\nu} = S_{n,\nu}(a_n, b_n) \text{ and } V'_{n,\nu} = S'_{n,\nu}(a_n, b_n)\} \to 1,$$

which implies that the random variables $S_{n,\nu}(a_n, b_n)$ and $S'_{n,\nu}(a_n, b_n)$ are asymptotically independent. Subsequently the same is true for $S_{n,\nu}$ and $S'_{n,\nu}$. The proof of the assertion that $T_{n,\nu}$ and $T'_{n,\nu}$ are asymptotically independent is along the same lines, so the details are omitted. The proof of Theorem 2 is now complete.

The following theorem should be compared to the result stated in (4) in the Introduction. We see that the generalized Rényi statistic version of the statistic on the left side of (4) given in (31) below also exhibts asymptotic independence behavior due to the asymptotic independence of the suprema of the weighted empirical process in the upper and lower regions of (0, 1), except that now this behavior is Gaussian instead of Poisson.

Theorem 3. *Let a_n be any sequence of positive constants such that $na_n \to \infty$ and $a_n \to 0$ as $n \to \infty$. Then for every $0 \leqq \nu < 1/2$, as $n \to \infty$,*

$$a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq 1-a_n} \alpha_n(s)/(s(1-s))^{1-\nu} \xrightarrow{\mathcal{D}} X_\nu \vee X'_\nu \tag{30}$$

and

$$a_n^{1/2-\nu} \sup_{a_n \leqq s \leqq 1-a_n} |\alpha_n(s)|/(s(1-s))^{1-\nu} \xrightarrow{\mathcal{D}} Y_\nu \vee Y'_\nu, \tag{31}$$

where $X'_\nu \stackrel{\mathcal{D}}{=} X_\nu$, respectively $Y'_\nu \stackrel{\mathcal{D}}{=} Y_\nu$, and X_ν and X'_ν, respectively Y_ν and Y'_ν, are independent random variables.

Proof. Choose any $0 \leqq \nu < 1/2$. Let b_n denote any sequence of positive constants satisfying conditions (i), (ii), and (iii) as stated in the proof of Theorem 2. Observe that

$$a_n^{1/2-\nu} \sup_{b_n \leqq s \leqq 1-b_n} |\alpha_n(s)|/(s(1-s))^{1-\nu} \leqq$$

$$2^{-1+\nu} a_n^{1/2-\nu} \sup_{b_n \leqq s \leqq 1/2} |\alpha_n(s)|/s^{1-\nu} + 2^{-1+\nu} a_n^{1/2-\nu} \sup_{1/2 \leqq s \leqq 1-b_n} |\alpha_n(s)|/(1-s)^{1-\nu},$$

which by Theorem 2 and (iii) equals

$$O_P((a_n/b_n)^{1/2-\nu}) = o_P(1).$$

Notice that

$$\Big|\sup_{a_n \leqq s \leqq b_n} \alpha_n(s)/(s(1-s))^{1-\nu} - S_{n,\nu}(a_n, b_n)\Big| \leqq \sup_{a_n \leqq s \leqq b_n} |(1-s)^{-1+\nu} - 1|\,|\alpha_n(s)|/s^{1-\nu},$$

which by (21) and (ii) equals $o(1)O_P(1) = o_P(1)$. Similarly we have

$$\Big|\sup_{1-b_n \leqq s \leqq 1-a_n} \alpha_n(s)/(s(1-s))^{1-\nu} - S'_{n,\nu}(a_n, b_n)\Big| = o_P(1).$$

Therefore by (26), (27) and the asymptotic independence of $S_{n,\nu}(a_n, b_n)$ and $S'_{n,\nu}(a_n, b_n)$ established in the proof of Theorem 2, we have (30). Assertion (31) follows by essentially the same argument. Thus Theorem 3 is proven.

With very slight modification of the proofs of the foregoing theorems it can be shown that the statements of Theorems 2 and 3 remain true with α_n replaced by β_n. The statements of Theorem 1 with β_n substituted for α_n also remain true if the suprema in (12) and (14) are taken over the interval $[a_n, 1-1/(n+1)]$ and the suprema in (13) and (15) are taken over the interval $[1/(n+1), 1-a_n]$.

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