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Geodesics of a principal bundle with Kaluza—Klein metric

PÉTER T. NAGY

Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

Let be given a riemannian space-time manifold M, a principal fiber bundle $\pi\colon P\to M$ over M and a connection form φ on P. The structure group G of the bundle $\{P, \pi, M\}$ acting on P from right and the connection foım φ on P can be interpreted as internal symmetry group and potential of a given Yang—Mills field (cf. [2]). We suppose that the group G is equipped with a left invariant riemannian metric determined by a scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G. We call the riemannian metric defined by

$$\langle X, Y\rangle_P := \langle \pi_* X, \pi_* Y\rangle_M + \langle \varphi(X), \varphi(Y)\rangle_{\mathfrak{g}}, \quad (X, Y\in TP) \tag{1}$$

the Kaluza—Klein metric on P.

Usually, the bundle space metric $\langle\cdot,\cdot\rangle_P$ in Yang—Mills theory is defined with help of a biinvariant group metric. Now we generalize the construction of bundle metric using an arbitrary left invariant group metric instead of the biinvariant one. We shall investigate the geodesics in the bundle space P which can be interpreted as trajectories of "combined" classical motions of test particles in a given external Yang—Mills field φ and gravitational field $\langle\cdot,\cdot\rangle_M$.

The equation of geodesics in a principal bundle has been studied in [2], [3] under the assumption that the scalar product $\langle .,.\rangle_{\mathfrak{g}}$ is given by the Killing—Cartan form. For the description of geodesics in P with respect to the metric (1) we shall apply our results on geodesics in a riemannian submerison [5]. We shall use V. I. Arnold's approach of investigation of geodesics on a Lie group with left invariant metric interpreted as motion of a generalized rigid body [1].

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2. Geodesics on a Lie group with left invariant metric

Let be given a scalar product $\langle .,. \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of the Lie group G. $\mathfrak{g}$ can be considered as the Lie algebra of left invariant vectorfield on G, thus $\langle .,. \rangle_{\mathfrak{g}}$ defines a left invariant riemannian metric on G. A scalar product on $T_gG (g \in G)$ defines a linear operator maping the tangent space T_gG onto its dual space T_g^*G. This maping gives a left invariant tensorfield I_g: $T_gG \to T_g^*G$ satisfying

$$(I_g(v), w) = \langle v, w \rangle_{\mathfrak{g}}, \quad v, w \in T_gG.$$

Proposition 1 (V. I. Arnold). *The curve $g(t)$ on the Lie group G is a geodesic with respect to the left invariant riemannian metric $\langle .,. \rangle_{\mathfrak{g}}$ if and only if the covector*

$$m(t) := R^*_{g(t)} \circ I_{g(t)} \dot{g}(t) \in \mathfrak{g}^* \tag{2}$$

is constant, where R_g: $G \to G$ is the right translation: $R_g(h) = hg$.

Proof. Let $e_1(g), \ldots, e_k(g) \in \mathfrak{g}$ be a left invariant orthonormal frame on G and $\omega^1, \ldots, \omega^k$ its dual coframe, that is $\omega^\lambda(e_\mu) = \delta^\lambda_\mu$ is satisfied (here and below λ, μ, ν run through the indices $1, \ldots, k$). If $c^\lambda_{\mu\nu}$ are the structure constants, that is

$$d\omega^\lambda = -c^\lambda_{\mu\nu} \omega^\mu \wedge \omega^\nu$$

the riemannian connection form $\varphi = \varphi^\lambda_\mu e_\lambda \otimes \omega^\mu$ can be expressed by

$$\varphi^\lambda_\mu = \frac{1}{2}(c^\lambda_{\nu\mu} - c^\mu_{\nu\lambda} + c^\nu_{\lambda\mu})\omega^\nu.$$

Indeed, $\varphi^\lambda_\mu = -\varphi^\mu_\lambda$ and $d\omega^\lambda = -\varphi^\lambda_\mu \wedge \omega^\mu$ are satisfied, and these properties determine the riemannian connection form uniquely. We get the equation of geodesics

$$\nabla_t \dot{g} = \nabla_t(\dot{g}^\lambda e_\lambda) = \left(\frac{d\dot{g}^\lambda}{dt} + c^\nu_{\lambda\mu} \dot{g}^\mu \dot{g}^\nu\right) e_\lambda.$$

The components of the tensor I_g are δ^λ_μ by the choice of frames, hence the equation of geodesics can be written as

$$\frac{d}{dt}(L^*_g \circ I_g \dot{g}) = \mathrm{ad}^*_v \circ L^*_g \circ I_g \dot{g}, \quad v = L_{g^{-1}*} \dot{g},$$

where L_g: $G \to G$ is the left translation and ad^*_v: $\mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of $\mathfrak{g}$ that is $(\mathrm{ad}^*_v \zeta, w) = I([v, w])$, for any $w \in \mathfrak{g}$, $\zeta \in \mathfrak{g}^*$.

On the other hand we get using the fact that

$$L^*_g \circ I_g \dot{g} = L^*_g \circ R^*_{g^{-1}} m(t) = \mathrm{Ad}^*_g m(t)$$

the following equation

$$\frac{d}{dt}(L_g^* \circ I_g \dot{g}) = \frac{d}{dt}(\mathrm{Ad}_g^*)m + \mathrm{Ad}_g^* \frac{dm}{dt}. \tag{3}$$

We have

$$\frac{d}{dt}(\mathrm{Ad}_{g(t)})_{t_0} = \mathrm{Ad}_{g(t_0)} \frac{d}{dt}(\mathrm{Ad}_{g^{-1}(t_0)g(t)})_{t_0} = \mathrm{Ad}_{g(t_0)} \circ \mathrm{ad}_v,$$

where $v = L_{g^{-1}*}\dot{g} \in \mathfrak{g}$, thus we get

$$\frac{d}{dt}(\mathrm{Ad}_g^*) = \mathrm{ad}_v^* \circ \mathrm{Ad}_g^*$$

and consequently

$$\frac{d}{dt}(L_g^* \circ I_g \dot{g}) = \mathrm{ad}_v^* \circ \mathrm{Ad}_g^* \circ R_{g^{-1}}^* \circ I_g \dot{g} + \mathrm{Ad}_g^* \frac{dm}{dt} = \mathrm{ad}_v^* \circ L_g^* \circ I_g \dot{g} + \mathrm{Ad}_g^* \frac{dm}{dt}.$$

Comparing with (3) and using the fact that the operator Ad_g^* is invertible, the Proposition follows.

3. The Kaluza—Klein metric

We consider now the riemannian scalar product (1) on the total space P of the principal fiber bundle $\{P, \pi, M\}$. The riemannian manifolds P, M and the projection $\pi\colon P \to M$ yield a riemannian submersion, that is the map $\pi_*\colon TP \to TM$ preserves the length of horizontal vectors.

The principal fiber bundle $\{O_M(P), p, P\}$ of adapted frames on P of the submersion $\pi\colon P \to M$ is defined by the following: $\{O_M(P), p, P\}$ is a subbundle of the orthonormal frame bundle $\{O(P), p, P\}$ over P consisting of frames $(x; e_1, \ldots, \ldots, e_{n+k}) \in O(P)$ such that the vectors $e_1, \ldots, e_n$ are horizontal (i.e. orthogonal to the fiber $\pi^{-1}(y)$, where $y = \pi(x)$) and the vectors $e_{n+1}, \ldots, e_{n+k}$ are vertical (i.e. tangent to the fiber $\pi^{-1}(y)$). The structure group of the bundle of adapted frames is the product of orthogonal groups $O(n) \times O(k)$.

If ω and θ denote the $\mathbf{R}^{n+k}$-valued canonical form and the orthogonal Lie algebra $\mathfrak{o}(n+k)$-valued Riemannian connection form on $O(P)$, their components $\omega^a, \omega^\alpha, \theta_b^a, \theta_\beta^a, \theta_b^\alpha, \theta_\beta^\alpha$ satisfy the structure equations

$$d\omega^a = -\theta_c^a \wedge \omega^c - \theta_\gamma^a \wedge \omega^\gamma, \tag{4a}$$

$$d\omega^\alpha = -\theta_c^\alpha \wedge \omega^c - \theta_\gamma^\alpha \wedge \omega^\gamma, \tag{4b}$$

where the indices run the values: $a, b, c = 1, \ldots, n$; $\alpha, \beta, \gamma = n+1, \ldots, n+k$.

The form with the components θ_b^a, θ_β^α is a connection form on the adapted frame bundle $\{O_M(P), p, P\}$ resulted by the projection $\mathfrak{o}(n+k) \to \mathfrak{o}(n) \oplus \mathfrak{o}(k)$ of the values of the connection form θ. The difference form with components θ^a_β, θ^α_b is a tensorial form, thus they can be written as linear combinations of the components of the canonical form

(5) $$\theta^a_\beta = \frac{1}{2} A^a_{\beta c}\omega^c + \frac{1}{2} T^a_{\beta\gamma}\omega^\gamma \quad \theta^\alpha_b = -\theta^b_\alpha,$$

where the tensors $A = A^a_{\beta c} e_a \otimes \omega^\beta \otimes \omega^c$ and $T = T^a_{\beta\gamma} e_a \otimes \omega^\beta \otimes \omega^\gamma$ are the fundamental invariants of the submersion (in detail see in [1]).

Let be given an orthonormal frame $E_1, \ldots, E_k$ in the Lie algebra $\mathfrak{g}$. It defines a global orthonormal vertical frame field $\tilde{e}_{n+1}, \ldots, \tilde{e}_{n+k}$ on P such that $\varphi(\tilde{e}_{n+\lambda}) = E_\lambda$. Thus the adapted frame bundle $\{O(P), p, P\}$ can be reduced to the subbundle $\{\tilde{O}_M(P), p, P\}$ consisting of the frames $\{e_1, \ldots, e_n, \tilde{e}_{n+1}, \ldots, \tilde{e}_{n+k}\}$, where $e_1, \ldots, e_n$ are orthonormal horizontal tangent vectors. If we consider the $\mathbf{R}^{n+k}$-valued canonical form ω on this subbundle $\tilde{O}_M(P)$ and if we use the identification $\mathbf{R}^{n+k} = \mathbf{R}^n \oplus \mathfrak{g}$, the components $\omega^{n+1}, \ldots, \omega^{n+k}$ of the canonical form ω are corresponding to the components $\varphi^1, \ldots, \varphi^k$ of the connection form φ with respect to the frame $E_1, \ldots, E_k$ in $\mathfrak{g}$. It follows that the forms $\omega^{n+1}, \ldots, \omega^{n+k}$ satisfy the structure equations of the connection φ:

(6) $$d\omega^{\lambda+n} = -\frac{1}{2} c^\lambda_{\mu\nu}\omega^{n+\mu} + \frac{1}{2} \Omega^\lambda_{bc}\omega^b \wedge \omega^c.$$

As compared (6) with (4b) and (5) we get

(7) $$A^{n+\lambda}_{bc} = \Omega^\lambda_{bc}.$$

Since the parallel translation with help of the connection on the bundle $\{P, \pi, M\}$ commutes with the right action of G on P, it follows that the translation of the fibers is isometric. Indeed, if $y(t)$ is a curve in M, $v \in T_{x_0}P$ is a vertical tangent vector to the $\pi^{-1}(y(t_0))$ and $\varphi(v) = w \in \mathfrak{g}$, then $v = d/dt(R \exp tw)x_0|_0$. If τ_{t_0,t_1} denotes the translation $\pi^{-1}(y(t_0)) \to \pi^{-1}(y(t_1))$ along $y(t)$ we have

$$(\tau_{t_0,t_1})_* v = \frac{d}{dt}(R_{\exp tw} x_1)_0,$$

where $x(t)$ is a horizontal lift of the curve $y(t)$ such that $x(t_0) = x_0$, $x(t_1) = x_1$. It follows

$$\|v\|_P = \|\varphi(v)\|_{\mathfrak{g}} = \|w\|_{\mathfrak{g}} = \|(\tau_{t_0,t_1})_* v\|_P,$$

thus we get the isometry property. It follows that the normal variation of the metric tensor of the fibers vanishes i.e. the fibers are totally geodesic submanifolds:

(8) $$T^\alpha_{\beta\gamma} \equiv 0.$$

We summarize our results (7), (8).

Theorem 1. *The Kaluza—Klein metric on P defines a riemannian submersion on the bundle* $\{P, \pi, M\}$. *The fibers of the bundle arre totally geodesic submanifolds. The fundamental tensors of the submersion satisfy*

$$\langle A(Z)Y, U\rangle_P = \langle \Omega(U, Y), \varphi(Z)\rangle_{\mathfrak{g}},$$
$$T(Z, U) = 0$$

for any tangent vectors $Y, Z, U \in T_x P$, $x \in P$.

4. Geodesics on P

We shall apply our recent result ([5], p. 353).

Theorem. *If* $\{P, \pi, M\}$ *is a riemannian submersion with totally geodesic fibers the curve* $x(\sigma)$ *is a geodesic of P if and only if the following conditions are satisfied*

(i) *the first vector of curvature of the projection curve* $y(\sigma)=\pi\circ x(\sigma)$ *is*

$$\tilde{\nabla}_\sigma y' = -\pi_* A(x')x',$$

where σ *is the arc-length parameter of y, prime denotes the derivative by* σ *and* $\tilde{\nabla}$ *is the riemannian covariant derivative on M,*

(ii) *the development* $z(\sigma)=\tau_{\sigma,\sigma_0} x(\sigma)$ *of* $x(\sigma)$ *in the fiber* $\pi^{-1}(y(\sigma_0))$ *is a geodesic, where the map* $\tau_{\sigma,\sigma_0}\colon \pi^{-1}(y(\sigma))\to\pi^{-1}(y(\sigma_0))$ *is the parallel translation of the fibers along* $y(\sigma)$ *defined by the horizontal subspaces of TP.*

If $\{P, \pi, M\}$ is a principal fiber bundle with structure group G and the riemannian submersion on this bundle is defined by the Kaluza—Klein metric (1) we get the following description of geodesics.

Theorem 2. *The curve* $x(\sigma)$ *is a geodesic of P if and only if*

(i) *the curve* $z(\sigma)=\tau_{\sigma,\sigma_0}x(\sigma)$ *is a geodesic on the Lie group* $\pi^{-1}(y(\sigma_0))\cong G$ *with respect to the induced fiber metric,*

(ii) *the first vector of curvature of the projection curve* $y(\sigma)=\pi\circ x(\sigma)$ *satisfies for any tangent vectorfield U of M along* $y(\sigma)$

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \big(m, \Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})\big),$$

where $\bar{y}(\sigma)$ *is a horizontal lift of the curve* $y(\sigma)$, $\bar{Y}(\sigma)$ *and* $\bar{U}(\sigma)$ *are horizontal lifts of the vectorfields* $y'(\sigma)$ *and* $U(\sigma)$ *and* $m\in\mathfrak{g}^*$ *is a constant covector corresponding to the geodesic* $z(\sigma)$ *on* $\pi^{-1}(y(\sigma_0))\cong G$

$$m(\sigma) = R^*_{g(\sigma)}\circ I_{g(\sigma)} z'$$

(cf. Proposition 1, $z(\sigma)=\bar{y}(\sigma)g(\sigma)$*).*

Proof. We have to prove the property (ii). We know that

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = -\langle\pi_* A(x')x', U\rangle_M = -\langle A(x')x', \bar{U}\rangle_P.$$

By Theorem 1 we get

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle\varphi(x'), \ \Omega_x(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}}.$$

If we identify the fibers $\pi^{-1}(y(\sigma))$ with the group G such that the unit of the group G is corresponding to the section $\bar{y}(\sigma)$ we can write

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle\varphi(z'), \ \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}},$$

where $x(\sigma)\in\pi^{-1}(y(\sigma))$ corresponds to the pair $\{y(\sigma), z(\sigma)\}$, $z(\sigma)\in\pi^{-1}(y(\sigma_0))$. But $\varphi(z')=(L_{g^{-1}})_*\ z'$ and the scalar product $\langle.,.\rangle_{\mathfrak{g}}$ is left invariant thus we have

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle(L_{g-1})_* z', \ \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} = \langle z', (L_g)_* \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} =$$
$$= \langle z', (R_g)_* \circ \mathrm{Ad}_g \Omega_z(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}}.$$

Since $\mathrm{Ad}_g\ \Omega_z(\bar{Y}, \bar{U}) = \Omega_{zg^{-1}}(\bar{Y}, \bar{U}) = \Omega_{\bar{y}}(\bar{Y}, \bar{U})$ we get

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \langle z', (R_g)_*(\Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})\rangle_{\mathfrak{g}} = \big(R^*_{g(\sigma)} \circ I_{g(0)}\, z', \Omega_{\bar{y}(g)}(\bar{Y}, \bar{U})\big)$$

Thus the theorem is proved.

5. Geodesics on the orthonormal frame bundle

It is well known that the configuration space of a moving rigid body in the euclidean space E^3 is the orthonormal frame bundle $O(E^3)$ over E^3. This bundle is a trivial one that is the decomposition $O(E^3)=E^3\times O(3)$ is canonically defined and the action function of a free motion (the kinetic energy function) is the sum of the kinetic energy function of a moving masspoint in E^3 and of a rotating body. It means by the "Least Action Principle" that the trajectories of combined advancing and rotating motions will be geodesics on $O(E^3)$ with respect to the Kaluza—Klein metric corresponding to the trivial connection on the bundle $\{O(E^3), \pi, E^3\}$.

Analogously we can consider the orthonormal frame bundle $\{O(M), \pi, M\}$ over a riemannian manifold M as configuration space of "infinitesimal" rigid bodies in M. A scalar product $\langle.,.\rangle_{\mathfrak{o}(n)}$ on the orthogonal Lie algebra $\mathfrak{o}(n)$ corresponds to the kinetic energy function of a rotation, the Klauza—Klein metric (1) on $O(M)$ defined by the riemannian scalar product $\langle.,.\rangle_M$ on M, the riemannian connection form φ on $O(M)$ and the scalar product $\langle.,.\rangle_{\mathfrak{o}(n)}$ gives an action integral

$$\int \{\langle\pi_*\dot{x}, \pi_*\dot{x}\rangle_M + \langle\varphi(\dot{x}), \varphi(\dot{x})\rangle_{\mathfrak{o}(n)}\}\, dt$$

on the configuration space $O(M)$, the geodesics with respect to this metric will describe the inertial motion of an infinitesimal rigid body, corresponding to the least action principle. (The mass of the rigid body is supposed to be equal to 1.)

At the same time a detailed investigation of geodesics on $O(M)$ can serve as an application of the previously discussed equation of the classical motion of a Yang—Mills particle.

If $x(t)$ is a curve in $O(M)$ describing a combined motion of an infinitesimal rigid body (or particle) in the riemannian space M we call the projection curve $y(t)=\pi\circ x(t)$ the trajectory of the advancing motion and the curve $z(t)=\tau_{t,t_0}\circ x(t)$ on the fiber $\pi^{-1}(y(t_0))$ the trajectory of the rotation.

We know by Theorem 2 that the trajectory of the rotation of an inertial motion is a geodesic on the fiber $\pi^{-1}(y(t_0))$ and the trajectory of the advancing motion satisfies the differential equation

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = (m, \Omega_{\bar{y}(\sigma)}(\bar{Y}, \bar{U})),$$

where m is a constant $\in\mathfrak{o}(n)^*$ defined by the initial values of the geodesic $x(t)$ and $\dot{x}(t)$. It means that the curve $y(t)$ is a trajectory of the motion of a masspoint in M under the action of the force $(m, \Omega_{\bar{y}}(\bar{Y}, \bar{U}))$. In our case $P=O(M)$ the Lie algebra valued curvature form Ω defines a tensorfield R on M called the curvature tensor and the constant $m\in\mathfrak{o}(n)^*$ defines a covariant constant tensorfield M_σ along the curve $y(\sigma)$ satisfying

$$R(X, Y)U = z\cdot(2\Omega(\bar{X}, \bar{Y})(z^{-1}U)) \quad \text{for} \quad X, Y, U\in T_yM$$

and

$$M_\sigma X = z\cdot(m^*(z^{-1}X)) \quad \text{for} \quad X\in T_{y(\sigma)}M,$$

where $z\in\pi^{-1}(y)$ is identified with the map $\mathbf{R}^n\to T_yM$ defined by the frame z and m^* denotes the vector from the Lie algebra $\mathfrak{o}(n)$ corresponding to the covector $m\in\mathfrak{o}(n)^*$ determined by the Killing—Cartan scalar product on $\mathfrak{o}(n)$, that is

$$m(v) = \text{Trace } m^*v \quad \text{for} \quad v\in\mathfrak{o}(n).$$

Thus we get the following

Theorem 3. *The trajectory $y(\sigma)$ of the advancing motion of an infinitesimal rigid body in a riemannian space M satisfies the equation*

$$\langle\tilde{\nabla}_\sigma y', U\rangle_M = \frac{1}{2}\text{Trace}\,(M_\sigma\circ R(y', U)) \quad \textit{for} \quad U\in T_yM,$$

where $\tilde{\nabla}_\sigma M_\sigma = 0$.

If the riemannian space M is of constant curvature k then the components R^a_{bcd} of the curvature tensor are

$$R^a_{bcd} = k\cdot(\delta^a_c g_{bd} - \delta^a_d g_{bc}\cdot)$$

Hence we can write

$$\frac{1}{2}\operatorname{Trace}\left(M_\sigma \circ R(y', U)\right) = \frac{1}{2} k \cdot M_a^b(\delta_c^a g_{bd} - \delta_d^a g_{bc}) y'^c U^d =$$

$$= \frac{1}{2} k \cdot (M_c^b y'^c g_{bd} U^d - M_d^b g_{bc} y'^c U^d),$$

where M_a^b are the components of the tensor M_σ. Since the tensor M_σ is corresponding to the vector $m^* \in \mathfrak{o}(n)$ it is antisymmetric that is

$$\langle M_\sigma U, V\rangle = -\langle U, M_\sigma V\rangle \quad \text{for} \quad U, V \in TM,$$

or equivalently

$$M_c^b g_{bd} = -M_d^b g_{bc}.$$

Thus we get

$$\frac{1}{2}\operatorname{Trace}\left(M_\sigma \circ R(y', U)\right) = k \cdot \langle M_\sigma y', U\rangle_M.$$

It follows

Theorem 4. *The trajectory of the advancing motion of an infinitesimal rigid body in a riemannian space of constant curvature k satisfies the equation*

$$\tilde{\nabla}_\sigma y' = k \cdot M_\sigma y',$$

where $\tilde{\nabla}_\sigma M_\sigma = 0$.

If $\dim M = 3$, the action of antisymmetric tensors on the tangent space can be written in the form of cross product

$$M_\sigma y' = \mu_\sigma \times y',$$

where μ_σ is a uniquely determined tangent vectorfield along $y(\sigma)$. Since $\tilde{\nabla}_\sigma M_\sigma = 0$ we have $\tilde{\nabla}_\sigma \mu_\sigma = 0$. Thus we get

Corollary. *The trajectory of the advancing motion of an infinitesimal rigid body in a riemannian 3-space of constant curvature satisfies the equation*

$$\tilde{\nabla}_\sigma y' = k \cdot \mu_\sigma \times y',$$

where the vectorfield μ_σ *along* $y(\sigma)$ *is covariant constant.*

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BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY