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On the comparison of multiplier processes in Banach spaces

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday, in friendship and high esteem

1. Introduction. This paper continues our previous investigations (cf. [3—5; 7]) on the comparison of (commutative) approximation processes in Banach spaces. Whereas the results of [3—5] were based upon rather restrictive global divisibility conditions, a local divisibility property was employed in [7] to estimate a given process in terms of the particular one of best approximation. The present paper now yields results on the general comparison of different processes, which in particular include classical inverse approximation theorems in the applications.

Since this paper, though essentially self-contained, may indeed be considered as a sequel to [7], we may be very brief concerning motivation for the approach and results. In fact, in [7] we followed the multiplier approach of [3—5] and employed global criteria for multipliers, based upon (radial) Riesz summability and corresponding global $BV_{j+1}[0, \infty]$-classes of functions. Section 2 now indicates how these concepts may be localized in order to formulate counterparts to those local conditions, important in the classical context of trigonometric analysis.

In Section 3 these localized concepts are then used to derive the general comparison Theorem 3.8. Here we are heavily influenced by work of H. S. Shapiro concerned with local divisibility within the Wiener ring of Fourier—Stieltjes transforms (see [13, Chapter 9], also Remark 3.11 for more detailed information). Indeed, standard "partition of unity" arguments are now available, even in the present abstract setting (cf. [2; 10; 12; 17] for similar arguments in the context of Besov spaces).

In Section 4 some first illustrating applications are given, emphasizing the unifying approach to the subject. In fact, we essentially confine ourselves to those concrete problems, already treated in [7], in order to point out the additional results now available.

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2. Local multipliers in Banach spaces. For a complex Hilbert space H let E be a (countably additive, selfadjoint, bounded linear) spectral measure in $\mathbf{R}^n$, the Euclidean n-space ($n\in\mathbf{N}$, the set of natural numbers) with inner product $\langle x, y\rangle := \sum_{k=1}^{n} x_k y_k$ and norm $|x|:=\langle x, x\rangle^{1/2}$. If $L^\infty(\mathbf{R}^n, E)$ is the space of complex-valued, E-essentially bounded functions, then for each $\tau\in L^\infty(\mathbf{R}^n, E)$ the integral

$$T(\tau) := \int_{\mathbf{R}^n} \tau(x)\, dE(x)$$

is a bounded linear operator of H into itself (for basic properties and further details see [9, pp. 900, 1930, 2186]).

For a given orthonormal structure (H, E) let X be a complex Banach space with norm $\|\cdot\|$ such that H and X are continuously embedded in some linear Hausdorff space (this hypothesis should be added in [5], see [17, p. 116]) and such that $H\cap X$ is dense in H and X, i.e.,

$$\overline{H\cap X}^{\|\cdot\|_H} = H, \quad \overline{H\cap X}^{\|\cdot\|} = X. \tag{2.1}$$

Then (cf. [5]) $\tau\in L^\infty(\mathbf{R}^n, E)$ is called a multiplier on X if for each $f\in H\cap X$

$$T(\tau)f := \int_{\mathbf{R}^n} \tau(x)dE(x)f\in H\cap X, \quad \|T(\tau)f\| \leqq C\|f\| \tag{2.2}$$

(here and in the following C denotes a constant which may have different values at each occurrence). In view of (2.1, 2) the closure of $T(\tau)$ (represented by the same symbol) belongs to $[X]$, the space of bounded linear operators of X into itself. The set of all multipliers τ on X is denoted by $M=M(X)$, the corresponding set of multiplier operators $T(\tau)$ by $[X]_M$. With the natural vector operations, pointwise multiplication, and norm

$$\|\tau\|_M := \|T(\tau)\|_{[X]} := \sup\{\|T(\tau)f\|: f\in H\cap X, \ \|f\| \leqq 1\}$$

M is a commutative Banach algebra with unit, isometrically isomorphic (under T) to the subspace $[X]_M\subset[X]$.

To deal with multipliers, let us consider the Riesz factor ($u\in[0, \infty)$, $t\in(0, \infty)$, $j\in\mathbf{P}:=\mathbf{N}\cup\{0\}$)

$$r_{j,t}(u) := \begin{cases} (1-u/t)^j, & 0 \leqq u \leqq t \\ 0, & u > t. \end{cases}$$

In the following $\mathscr{J}$ denotes an arbitrary index set. Moreover, $\alpha\circ\beta$ is the composition (in case it is defined) of the functions α and β: $(\alpha\circ\beta)(x):=\alpha(\beta(x))$, and α^{-1} the inverse function.

Definition 2.1. Let X be a Banach space satisfying (2.1) (with respect to a given orthonormal structure (H, E)) and consider a family $\psi := \{\psi_\varrho : \varrho \in \mathcal{J}\}$ of functions $\psi_\varrho(x)$, defined on $\mathbf{R}^n$ with values in $[0, \infty)$. If $T(r_{j,t} \circ \psi_p)$ is X-measurable in $t>0$ (this condition should be added in [7], see [5]) and if for some $j \in \mathbf{P}$ the Riesz summability condition

$$(2.3) \qquad r_{j,t} \circ \psi_\varrho \in M \quad \text{with} \quad \|r_{j,t} \circ \psi_\varrho\|_M \leqq C < \infty$$

holds true, uniformly for $t>0$, $\varrho \in \mathcal{J}$, then X is called *R^j_ψ-bounded.*

Local multiplier criteria may then be derived in terms of the following classes of functions (see [11]).

Definition 2.2. For $0 \leqq a < b \leqq \infty$ and $j \in \mathbf{P}$ *the space* $BV_{j+1}[a, b]$ is defined as the set of all complex-valued functions τ which are j-times differentiable on (a, b) such that $\tau^{(j)}$ is of bounded variation on each compact subinterval of (a, b) and

$$\int_{a+}^{b-} u^j \, |d\tau^{(j)}(u)| < \infty.$$

Obviously, $BV_{j+1}[c, d] \subset BV_{j+1}[a; b]$ for $0 \leqq c \leqq a < b \leqq d \leqq \infty$. Moreover, $BV_{j+1}[a, b]$ is a Banach space under the norm

$$\|\tau\|_{BV_{j+1}[a,b]} := \frac{1}{j!} \int_a^b u^j \, |d\tau^{(j)}(u)| + \sum_{k=0}^{j} \frac{1}{k!} \Big| \lim_{u \to b-} u^k \tau^{(k)}(u) \Big|.$$

Theorem 2.3. *Let X be R^j_ψ-bounded and τ a complex-valued function, defined on $[0, \infty)$, such that $\tau \in BV_{j+1}[a, b]$ for some $0 \leqq a < b \leqq \infty$. Then for each $\sigma_\varrho \in M(X)$ satisfying*

$$(2.4) \qquad \sigma_\varrho(x) = 0 \quad \textit{for} \quad x \in \mathbf{R}^n \quad \textit{with} \quad \psi_\varrho(x) \notin [a, b]$$

one has $\sigma_\varrho(\tau \circ \psi_\varrho) \in M(X)$. In fact,

$$(2.5) \qquad \|\sigma_\varrho(\tau \circ \psi_\varrho)\|_M \leqq C \|\sigma_\varrho\|_M \|\tau\|_{BV_{j+1}[a,b]}.$$

For a proof of this theorem as well as for further details concerning these localized concepts of BV_{j+1}-classes and multipliers see [11].

Remark 2.4. For $a=0$, $b=\infty$ condition (2.4) is empty so that the unit $\sigma_\varrho(x)=1$ for all $x \in \mathbf{R}^n$ is admissible. Thus Theorem 2.3 also includes our previous multiplier criterion $BV_{j+1}[0, \infty] \circ \psi \subset M(X)$, in particular, for every $\tau \in BV_{j+1}[0, \infty]$ (cf. Remark 2.8)

$$(2.6) \qquad \big\|T\big(\tau \circ (t\psi_\varrho)\big)\big\|_{[X]} =: \|\tau \circ (t\psi_\varrho)\|_M \leqq C \|\tau\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$, $\varrho \in \mathcal{J}$ (cf. [5; 15], also for fractional extensions).

To formulate some further results concerning $BV_{j+1}[a, b]$ (see [11; 16] for detailed proofs), let $C_{00}^{\infty}[0, \infty)$ be the set of realvalued functions on $[0, \infty)$, arbitrarily often differentiable with compact support (in notation: supp).

Proposition 2.5. *One has* $C_{00}^{\infty}[0, \infty) \subset BV_{j+1}[0, \infty]$. *Moreover, for* $\lambda \in C_{00}^{\infty}[0, \infty)$ *the family* $\{\lambda(tu)\colon t \in (0, \infty)\}$ *is continuous in t with respect to the topology of* $BV_{j+1}[0, \infty]$, *thus*

$$\lim_{s \to t} \|\lambda(su) - \lambda(tu)\|_{BV_{j+1}[0,\infty]} = 0.$$

As an immediate consequence of (2.6) we conclude that for $\lambda \in C_{00}^{\infty}[0, \infty)$ and ψ, subject to (2.3), the family $\{\lambda(t\psi_\varrho(x))\colon t \in (0, \infty)\}$ is continuous in t with respect to the topology of M, thus for each $t \in (0, \infty)$

$$\lim_{s \to t} \left\| T(\lambda \circ (s\psi_\varrho)) - T(\lambda \circ (t\psi_\varrho)) \right\|_{[X]} = 0, \tag{2.7}$$

uniformly for $\varrho \in \mathcal{J}$.

Theorem 2.6. *Consider families* $\{a_\varrho\}$, $\{b_\varrho\}$ *of numbers with* $0 \leqq a_\varrho < b_\varrho \leqq \infty$ *for each* $\varrho \in \mathcal{J}$. *Suppose that the functions* $\tau_\varrho \in BV_{j+1}[a_\varrho, b_\varrho]$ *satisfy (cf. Definition 3.7)*

$$\sup_{\varrho \in \mathcal{J}} \|\tau_\varrho\|_{BV_{j+1}[a_\varrho, b_\varrho]} < \infty, \tag{2.8}$$

$$\inf \{|\tau_\varrho(u)|\colon a_\varrho < u < b_\varrho,\ \varrho \in \mathcal{J}\} > 0. \tag{2.9}$$

Then $1/\tau_\varrho \in BV_{j+1}[a_\varrho, b_\varrho]$, *uniformly for* $\varrho \in \mathcal{J}$.

Remark 2.7. To illustrate condition (2.8), let $D^{(j)}$, $j \in \mathbf{N}$, be the set of real-valued, continuous, strictly increasing functions η on $[0, \infty)$ with $\eta(0) = 0$, $\lim_{u \to \infty} \eta(u) = \infty$ which are $(j+1)$-times differentiable on $(0, \infty)$ such that

$$\text{(2.10)} \quad \begin{array}{ll} \text{(i)} & u^k |\eta^{(k+1)}(u)| \leqq C\eta'(u) \quad (0 \leqq k \leqq j,\ u > 0), \\ \text{(ii)} & \lim_{u \to 0+} u\eta'(u) = 0. \end{array}$$

If $\varphi(\varrho)$ is a (real-valued) positive function on $\mathcal{J}$, then for every $\tau \in BV_{j+1}[a, b]$ and $\eta \in D^{(j)}$ the functions $\tau_{\varphi(\varrho)\eta}(u) := \tau(\varphi(\varrho)\eta(u))$ (of Hardy-type) belong to $BV_{j+1}[a_\varrho, b_\varrho]$ with $a_\varrho = \eta^{-1}(a/\varphi(\varrho))$, $b_\varrho = \eta^{-1}(b/\varphi(\varrho))$, and one has, uniformly for $\varrho \in \mathcal{J}$,

$$\|\tau_{\varphi(\varrho)\eta}\|_{BV_{j+1}[a_\varrho, b_\varrho]} \leqq C \|\tau\|_{BV_{j+1}[a, b]}. \tag{2.11}$$

Remark 2.8. Obviously, $r_{j,1} \in BV_{j+1}[0, \infty]$, and therefore by (2.11) (take $\eta(u) = u$) $r_{j,t} \in BV_{j+1}[0, \infty]$, uniformly for $t > 0$. Again by (2.11) it then follows that for every $\eta \in D^{(j)}$ and positive function $\varphi(\varrho)$ on $\mathcal{J}$

$$\|(r_{j,t})_{\varphi(\varrho)\eta}\|_{BV_{j+1}[0,\infty]} \leqq C \|r_{j,1}\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$, $\varrho\in\mathcal{J}$. In view of Remark 2.4 this implies that if X is R_ψ^j-bounded, then X is also $R_{\tilde{\psi}}^j$-bounded with $\tilde{\psi}_\varrho=\varphi(\varrho)(\eta\circ\psi_\varrho)$.

3. General comparison theorems. Throughout X denotes an R_ψ^j-bounded Banach space.

Definition 3.1. A family $\{\tau_\varrho\}_{\varrho\in\mathcal{J}}$ of uniformly bounded multipliers is called *locally divisible (at the origin) of order* ψ if (cf. [7]) there exist some $\delta>0$ and a family $\{\theta_\varrho\}_{\varrho\in\mathcal{J}}$ of uniformly bounded multipliers such that

$$\tau_\varrho(x)=\psi_\varrho(x)\theta_\varrho(x) \quad \text{in case } \psi_\varrho(x)\leqq\delta. \tag{3.1}$$

If (3.1) holds true for all $x\in\mathbf{R}^n$, $\varrho\in\mathcal{J}$, then the family $\{\tau_\varrho\}$ is said to be *globally divisible*.

Proposition 3.2. *Local divisibility implies the global one of the same order.*

Proof. We proceed as in [7]. Let $\{\tau_\varrho\}$ satisfy (3.1) and $\lambda\in C_{00}^\infty[0,\infty)$ be such that $\lambda(t)=1$ for $0\leqq t\leqq\delta/2$ and $=0$ for $t\geqq\delta$. Since $1-\lambda(t)=0$ for $0\leqq t\leqq\delta/2$, the function $\sigma(t):=(1-\lambda(t))/t$ belongs to $BV_{j+1}[0,\infty]$. Thus $\{\sigma\circ\psi_\varrho\}$, $\{\lambda\circ\psi_\varrho\}\subset M$, uniformly for $\varrho\in\mathcal{J}$ (cf. (2.6)). Moreover, on $\mathbf{R}^n$

$$1-\lambda\circ\psi_\varrho=\psi_\varrho(\sigma\circ\psi_\varrho),\quad \tau_\varrho(\lambda\circ\psi_\varrho)=\psi_\varrho\theta_\varrho(\lambda\circ\psi_\varrho),$$

and therefore $\tau_\varrho=\tau_\varrho(\lambda\circ\psi_\varrho)+\tau_\varrho(1-\lambda\circ\psi_\varrho)=\psi_\varrho[\theta_\varrho(\lambda\circ\psi_\varrho)+\tau_\varrho(\sigma\circ\psi_\varrho)]$. Hence the assertion follows since the terms in $[\ldots]$ are bounded in M, uniformly for $\varrho\in\mathcal{J}$.

Remark 3.3. Let τ be a function on $[0,\infty)$ satisfying $\{\tau\circ\psi_\varrho\}\subset M$, uniformly for $\varrho\in\mathcal{J}$. Let $\eta\in D^{(j)}$ be such that $\tau/\eta\in BV_{j+1}[0,\delta]$ for some $\delta>0$. If λ is given as in the previous proof, then again $\lambda\circ\psi_\varrho\in M$ and $(\lambda\circ\psi_\varrho)(x)=0$ for all $x\in\mathbf{R}^n$ with $\psi_\varrho(x)>\delta$. Therefore $\theta_\varrho:=(\lambda\circ\psi_\varrho)((\tau/\eta)\circ\psi_\varrho)\in M$ by Theorem 2.3 with

$$\|\theta_\varrho\|_M\leqq C\|\lambda\circ\psi_\varrho\|_M\|\tau/\eta\|_{BV_{j+1}[0,\delta]}\leqq C\|\lambda\|_{BV_{j+1}[0,\infty]}\|\tau/\eta\|_{BV_{j+1}[0,\delta]},$$

uniformly for $\varrho\in\mathcal{J}$. But if $(\eta\circ\psi_\varrho)(x)\leqq\eta(\delta/2)(=:\tilde{\delta})$, then

$$(\eta\circ\psi_\varrho)(x)\theta_\varrho(x)=\lambda(\psi_\varrho(x))\tau(\psi_\varrho(x))=(\tau\circ\psi_\varrho)(x)$$

so that the family $\{\tau\circ\psi_\varrho\}$ is locally divisible of order $\eta\circ\psi$ (cf. Remark 2.8). Thus, local BV_{j+1}-conditions (at the origin) ensure corresponding local (and therefore global) divisibility properties.

For $q>1$ let $p\in C_{00}^\infty[0,\infty)$ be such that (partition of unity)

$$0\leqq p(u)\leqq 1,\quad \operatorname{supp}(p)\subset[1,q],\quad \int_0^\infty p(u)\frac{du}{u}=1. \tag{3.2}$$

Since $\int_0^\infty p(us)u^{-1}du=1$ for every $s>0$, one has for the function

$$v(0):=1,\quad v(s):=\int_1^\infty p(us)\frac{du}{u}=\int_s^\infty p(u)\frac{du}{u} \tag{3.3}$$

that $v\in C_{00}^\infty[0,\infty)$ with $v(s)=1$ for $0\leqq s\leqq 1$ and $v(s)=0$ for $s\geqq q$.

Lemma 3.4. *For* $s, t\in[0,\infty)$ *there hold true the identities*

$$1-v(ts)=\int_0^t p(us)\frac{du}{u}, \tag{3.4}$$

$$p(s)=p(s)(1-v(qs)), \tag{3.5}$$

$$1-v(s)=\int_1^\infty [1-v(us)-p(us)]\frac{du}{u^2}. \tag{3.6}$$

Proof. (3.4,5) are immediate consequences of the definitions. Moreover,

$$\int_1^\infty [1-v(us)]\frac{du}{u^2}=\int_1^\infty \Big[\Big(\int_0^1+\int_1^u\Big)p(rs)\frac{dr}{r}\Big]\frac{du}{u^2}=$$

$$=\int_0^1 p(rs)\frac{dr}{r}+\int_1^\infty \Big[\int_r^\infty \frac{du}{u^2}\Big]p(rs)\frac{dr}{r}=1-v(s)+\int_1^\infty p(rs)\frac{dr}{r^2}.$$

Consider the operators $T(p\circ(t\psi_\varrho))$, $T(v\circ(t\psi_\varrho))$ which belong to $[X]_M$, uniformly for $t>0$, $\varrho\in\mathcal{J}$ (cf. (2.6)). By (2.7) terms like $T(p\circ(t\psi_\varrho))f$ are continuous in t with respect to the topology of X so that the following integrals are well-defined (in X).

Proposition 3.5. (a): *For each* $f\in X$, $t>0$, $\varrho\in\mathcal{J}$

$$\|T(p\circ(t\psi_\varrho))f\|\leqq C\|f-T(v\circ(qt\psi_\varrho))f\|, \tag{3.7}$$

$$\|f-T(v\circ\psi_\varrho)f\|\leqq\int_1^\infty [\|f-T(v\circ(u\psi_\varrho))f\|+\|T(p\circ(u\psi_\varrho))f\|]\frac{du}{u^2}, \tag{3.8}$$

$$\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2}\leqq C\int_1^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2}. \tag{3.9}$$

(b): *If for each* $f\in X$, $\varrho\in\mathcal{J}$ *(theorem of Weierstrass-type)*

$$\lim_{t\to 0+}\|T(v\circ(t\psi_\varrho))f-f\|=0, \tag{3.10}$$

then one has additionally

$$\|f-T(v\circ(t\psi_\varrho))f\|\leqq\int_0^t \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u}. \tag{3.11}$$

Proof. In view of (2.6) assertion (3.7) is an immediate consequence of (3.5), whereas (3.8) follows by (3.6). Furthermore, (3.7) delivers

$$\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} \leqq C\int_1^\infty \|f-T(v\circ(qu\psi_\varrho))f\|\frac{du}{u^2} =$$

$$= qC\int_q^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2} \leqq qC\int_1^\infty \|f-T(v\circ(u\psi_\varrho))f\|\frac{du}{u^2},$$

thus (3.9). Concerning (3.11), the identity (3.4) implies that for $f\in X$, $0<\varepsilon<t$

$$f-T(v\circ(t\psi_\varrho))f = \int_\varepsilon^t T(p\circ(u\psi_\varrho))f\frac{du}{u}+[f-T(v\circ(\varepsilon\psi_\varrho))f].$$

In view of (3.10) this yields the assertion upon letting $\varepsilon\to 0+$.

The following result is to be compared with the Steckin-type estimate of [7] which now appears as an auxiliary result towards Theorem 3.8.

Theorem 3.6. *If $\{\tau_\varrho\}_{\varrho\in\mathcal{J}}$ is locally divisible of order ψ, then for each $f\in X$, $\varrho\in\mathcal{J}$*

$$\|T(\tau_\varrho)f\| \leqq C\int_1^\infty \|T(v\circ(u\psi_\varrho))f-f\|\frac{du}{u^2}. \tag{3.12}$$

Moreover, if (3.10) *holds true, then*

$$\|T(\tau_\varrho)f\| \leqq C\int_0^\infty \|T(p\circ(u\psi_\varrho))f\|\min\left\{1,\frac{1}{u}\right\}\frac{du}{u}. \tag{3.13}$$

Proof. Since $\chi(u):=up(u)=uv(u/q)p(u)$, $uv(u/q)\in C_{00}^\infty[0,\infty)\subset BV_{j+1}[0,\infty]$, one has the estimate (cf. (2.6))

$$\|T(\chi\circ(t\psi_\varrho))f\| \leqq C\|T(p\circ(t\psi_\varrho))f\| \tag{3.14}$$

as well as the identity (cf. (3.3))

$$\psi_\varrho(v\circ\psi_\varrho) = \int_1^\infty \psi_\varrho(p\circ(u\psi_\varrho))\frac{du}{u} = \int_1^\infty \chi\circ(u\psi_\varrho)\frac{du}{u^2},$$

the latter integral being absolutely convergent with respect to the topology of M (cf. (2.6, 7)). Consequently, since by Proposition 3.2 the family $\{\tau_\varrho\}$ is also globally divisible of order ψ, say $\tau_\varrho=\psi_\varrho\theta_\varrho$, one has the representation

$$\tau_\varrho(v\circ\psi_\varrho) = \Theta_\varrho\int_1^\infty \chi\circ(u\psi_\varrho)\frac{du}{u^2},$$

and therefore by (3.14)

$$\|T(\tau_\varrho)f\| \leqq \|T(\tau_\varrho)T(v\circ\psi_\varrho)f\| + \|T(\tau_\varrho)[f-T(v\circ\psi_\varrho)f]\| \leqq$$

$$\leqq C\Big[\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} + \|f-T(v\circ\psi_\varrho)f\|\Big]. \tag{3.15}$$

Thus (3.12) follows by (3.8, 9). Finally, (3.13) is a consequence of (3.10, 11, 15) since

$$\|T(\tau_\varrho)f\| \leqq C\Big[\int_1^\infty \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u^2} + \int_0^1 \|T(p\circ(u\psi_\varrho))f\|\frac{du}{u}\Big] =$$

$$= C\int_0^\infty \|T(p\circ(u\psi_\varrho))f\| \min\Big\{1, \frac{1}{u}\Big\}\frac{du}{u}.$$

To formulate the main result, let $\mathscr{A}$ be the set of functions α, continuously differentiable and positive on $(0, \infty)$ such that

$$\lim_{t\to 0+}\alpha(t) = 0, \quad \lim_{t\to\infty}\alpha(t) = \infty, \quad \alpha'(t) > 0 \quad (t > 0).$$

Obviously, $D^{(j)}\subset\mathscr{A}$ for every $j\in\mathbf{P}$. Moreover, if $\alpha, \beta\in\mathscr{A}$, then $\alpha\circ\beta, \alpha^{-1}\in\mathscr{A}$, too.

Definition 3.7. Let $\alpha\in\mathscr{A}$ and β be any function with $\alpha(t)<\beta(t)$ for each $t>0$. A family $\{\sigma_t\}_{t>0}$ with $\sigma_t\in BV_{j+1}[\alpha(t), \beta(t)]$ is said to satisfy the *Tauberian condition of type* (α, β) if

$$\sup_{t>0}\|\sigma_t\|_{BV_{j+1}[\alpha(t),\beta(t)]} < \infty, \tag{3.16}$$

$$\inf\{|\sigma_t(u)| : \alpha(t) < u < \beta(t),\ t > 0\} > 0. \tag{3.17}$$

In view of Theorem 2.6 conditions (3.16, 17) are chosen in such a way that $1/\sigma_t\in BV_{j+1}[\alpha(t), \beta(t)]$, uniformly for $t>0$.

Theorem 3.8. *Let $\gamma:=\{\gamma_\varrho\}_{\varrho\in\mathscr{I}}\subset\mathscr{A}$ be such that X is (R^j_ψ-and) $R^j_{\gamma\circ\psi}$-bounded. Suppose that the family $\{\tau_\varrho\}_{\varrho\in\mathscr{I}}$ is locally divisible of order ψ and that the family $\{\sigma_t\}_{t>0}$ satisfies the Tauberian condition of type (α, β) such that for some $q>1$*

$$\sup_{\varrho\in\mathscr{I}}\gamma_\varrho(q\delta_\varrho(t)) \leqq \beta(t) \quad (t > 0), \tag{3.18}$$

where $\delta_\varrho:=\gamma_\varrho^{-1}\circ\alpha$. Let $\sigma_t\circ\gamma_\varrho\circ\psi_\varrho$ belong to $M(X)$ such that $\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\|$ is measurable in t. If $\beta(t)=\infty$ for all $t>0$ (i.e., (3.18) is trivial), then one has the comparison estimate $(f\in X,\ \varrho\in\mathscr{I})$

$$\|T(\tau_\varrho)f\| \leqq C\int_0^{\delta_\varrho^{-1}(1)} \|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|\,\delta'_\varrho(u)\,du, \tag{3.19}$$

whereas in the general case $\beta(t)\leqq\infty$ *the additional assumption* (3.10) *implies*

$$\|T(\tau_\varrho)f\|\leqq C\int_0^\infty \|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|\min\{1,1/\delta_\varrho(u)\}\delta_\varrho'(u)\,du. \tag{3.20}$$

Proof. First of all, $\alpha,\gamma_\varrho\in\mathscr{A}$ imply $\gamma_\varrho^{-1},\delta_\varrho\in\mathscr{A}$ for each $\varrho\in\mathscr{J}$. Substituting $u=1/\delta_\varrho(t)$ it follows by (3.12, 13) that

$$\|T(\tau_\varrho)f\|\leqq C\int_0^{\delta_\varrho^{-1}(1)}\left\|T\left(v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f-f\right\|\delta_\varrho'(t)\,dt, \tag{3.21}$$

$$\|T(\tau_\varrho)f\|\leqq C\int_0^\infty\left\|T\left(p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\min\{1,1/\delta_\varrho(t)\}\delta_\varrho'(t)\,dt, \tag{3.22}$$

respectively. Let us first consider the case $\beta(t)=\infty$ for all $t>0$. Since X is R_ψ^j-bounded, the multipliers $1-v(\psi_\varrho(x)/\delta_\varrho(t))$ belong to M, uniformly for $t>0$, $\varrho\in\mathscr{J}$ (cf. (2.6)), and vanish (cf. (3.3)) for $(\gamma_\varrho\circ\psi_\varrho)(x)\leqq\alpha(t)$. Thus Theorem 2.3, 6 yield

$$\mu_{t,\varrho}:=\left[1-v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right]/\sigma_t\circ\gamma_\varrho\circ\psi_\varrho\in M,$$

$$\|\mu_{t,\varrho}\|_M\leqq C\left\|1-v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right\|_M\|1/\sigma_t\|_{BV_{j+1}[\alpha(t),\infty]}\leqq C,$$

since X is $R_{\gamma\circ\psi}^j$-bounded, too. Hence

$$\left\|f-T\left(v\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\leqq C\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\| \tag{3.23}$$

which establishes (3.19) in view of (3.21). To prove (3.20), one has by (3.2, 18) that $p(\psi_\varrho(x)/\delta_\varrho(t))=0$ for $(\gamma_\varrho\circ\psi_\varrho)(x)\notin[\alpha(t),\beta(t)]$. Again Theorem 2.3, 6 yield

$$\mu_{t,\varrho}:=p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}/\sigma_t\circ\gamma_\varrho\circ\psi_\varrho\in M$$

with $\|\mu_{t,\varrho}\|_M\leqq C$. Hence

$$\left\|T\left(p\circ\frac{\psi_\varrho}{\delta_\varrho(t)}\right)f\right\|\leqq C\|T(\sigma_t\circ\gamma_\varrho\circ\psi_\varrho)f\|,$$

giving (3.20) in view of (3.22).

Remark 3.9. Concerning the measurability of $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ with respect to u, assumed in Theorem 3.8, the proof indeed proceeds via the integrals on the right-hand side of (3.21, 22) (well-defined in view of (2.7)) plus a pointwise estimate of the integrands (cf. (3.23)). So, if the measurability of $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ cannot be assured in advance, one may replace the majorant $\|T(\sigma_u\circ\gamma_\varrho\circ\psi_\varrho)f\|$ in the pointwise estimate (3.23) by some measurable one, e.g., by the monotone majorant $\sup\{\|T(\sigma_r\circ\gamma_\varrho\circ\psi_\varrho)f\|:\ r\geqq u\}$ (cf. [13, p. 219 ff], in particular the notion of a σ-modulus, which indeed generalizes the classical modulus of continuity (4.10)).

Remark 3.10. Obviously, (3.18) is satisfied if γ_ϱ is a homogeneous function (of some fixed positive degree) and $\tilde{q}\alpha(t)\leqq\beta(t)$ for some $\tilde{q}>1$. On the other hand, if, e.g., $\psi_\varrho(x)=\varrho^{-1}\log(1+|x|)$ and $\gamma_\varrho(t)=e^{\varrho t}-1$ so that $(\gamma_\varrho\circ\psi_\varrho)(x)=|x|$ (cf. (4.1)), then (3.18) reduces to $(1+\alpha(t))^q\leqq 1+\beta(t)$.

Remark 3.11. As already mentioned, the results of this section are extensions of corresponding ones known in the concrete situation of the (trigonometric) Fourier spectral measure (cf. Section 4). More specifically, the estimate (3.19) of Theorem 3.8 is to be compared with [1, Corollary 2.4], whereas (3.20) is related to [13, Theorem 9.4.4.5]. Of course, the present methods of proof need different tools (cf. Section 2), due to the abstract setting. Let us mention that one may now also formulate a counterpart to [13, Theorem 9.4.4.4], based upon local divisibility (at the origin) of two families of multipliers.

Without going into details, let us finally mention that, even in the present abstract frame, the sharpness of the estimates obtained may again be discussed along the lines outlined in [7] (see also the literature cited there).

4. Applications. Let us recall that the approach of Section 2 to a multiplier theory in abstract spaces subsumes many classical orthogonal expansions in the applications. Since this is already worked out in our previous papers (cf. [5; 15] and the literature cited there), we may here concentrate ourselves to a very important special situation, the (trigonometric) Fourier spectral measure over $\mathbf{R}^n$.

To this end, let X be one of the spaces $L^p(\mathbf{R}^n)$, $1\leqq p<\infty$, of functions f, pth power (Lebesgue) integrable over $\mathbf{R}^n$ with (finite) norm

$$\|f\|_p := \Big((2\pi)^{-n/2}\int_{\mathbf{R}^n}|f(x)|^p\,dx\Big)^{1/p}.$$

Let $\mathscr{F}$ be the Fourier—Plancherel transform on L^2 and $\mathscr{F}^{-1}$ the inverse transform. For a Borel measurable set $B\subset\mathbf{R}^n$ let $\mathscr{P}_B$ be the multiplication projection

$$(\mathscr{P}_B f)(x):=f(x)$$

for $x\in B$ and $=0$ for $x\notin B$. Then $E(B):=\mathscr{F}^{-1}\mathscr{P}_B\mathscr{F}$ is a spectral measure for $H=L^2$ (cf. [9, p. 1989]). Furthermore, for the spaces X mentioned above condition (2.1) is satisfied, and (2.2) coincides with the classical definition of Fourier multipliers $\tau\in M_p(\mathbf{R}^n):=M(L^p(\mathbf{R}^n))$ (cf. [14, p. 94]).

Concerning the Riesz summability condition (2.3) it is a classical result (cf. [14, p. 114]) that

$$\text{(4.1)}\qquad \psi_\varrho(x)=|x|,\quad \mathscr{J}=\{1\}\Rightarrow(2.3)\quad\text{for}\quad j>(n-1)|1/p-1/2|.$$

Other admissible choices of ψ_ϱ used in the following are based upon the fact that (in the Fourier spectral case) any surjective affine transformation A from $\mathbf{R}^n$ to $\mathbf{R}^m$ induces an isometry from $M_p(\mathbf{R}^m)$ to $M_p(\mathbf{R}^n)$ via $\sigma(Ax)$, $x\in\mathbf{R}^n$, $\sigma\in M_p(\mathbf{R}^m)$ (cf. [2, p. 15]). For example, take $m=1$, $0\neq\varrho\in\mathbf{R}^n$, and $Ax=\langle\varrho, x\rangle$. Then it follows by (4.1) (on $\mathbf{R}^1$) that

$$\psi_\varrho(x)=|\langle\varrho, x\rangle|,\quad \mathscr{J}=\mathbf{R}^n\setminus\{0\}\Rightarrow(2.3)\quad\text{for}\quad j>0. \tag{4.2}$$

Note that in all these cases condition (3.10) is satisfied (theorem of Weierstrass-type).

In the following we revisit those applications, already mentioned in [7], and point out what kind of additional results are now available via the (localized) concepts of the previous sections.

4.1. Abel—Cartwright means. Let $\eta\in D^{(j)}$ for some $j>(n-1)|1/p-1/2|$ and $\varphi(t)>0$ for $t>0$. Consider the Abel—Cartwright means $W(\varphi(t)\eta)$, corresponding to the multiplier $w(\varphi(t)\eta(|x|))$, $w(u):=e^{-u}$. Since $w\in BV_{j+1}[0,\infty]$ for every $j\in\mathbf{P}$, the operators $W(\varphi(t)\eta)$ are well-defined in $[L^p(\mathbf{R}^n)]$ (cf. (2.6, 11), (4.1)). The results of Section 3 may now be used to compare means of different orders η.

Corollary 4.1. *Let* $j>(n-1)|1/p-1/2|$ *and* $\eta_k\in D^{(j)}$, $k=1,2$. *Then for every* $f\in L^p(\mathbf{R}^n)$, $t>0$

$$\|W(\varphi(t)\eta_1)f-f\|_p\leqq C\varphi(t)\int\limits_0^{1/\varphi(t)}\left\|W\left(\frac{\eta_2}{\eta_2(\eta_1^{-1}(u))}\right)f-f\right\|_p du. \tag{4.3}$$

Proof. Let $t\in\mathscr{J}=(0,\infty)$, $\tau_t(x)=1-w(\varphi(t)\eta_1(|x|))$. Since

$$(1-e^{-u})/u\in BV_{j+1}[0,\infty],$$

it follows that τ_t is globally divisible of order $\varphi(t)\eta_1(|x|)$ (cf. (2.6, 11), Remark 2.8, (4.1)). Setting $\sigma_s(u)=1-w(\eta_2(u)/\eta_2(\eta_1^{-1}(s)))$, one has $\sigma_s\in BV_{j+1}[0,\infty]$, uniformly for $s>0$ (cf. (2.11)), and $\sigma_s(u)\geqq1-e^{-1}$ for $u\geqq\eta_1^{-1}(s)$. Hence it follows that $\{\sigma_s\}_{s>0}$ satisfies the Tauberian condition with $\alpha(s)=\eta_1^{-1}(s)$, $\beta(s)=\infty$. Moreover, for $\gamma_t(u)=$ $=\eta_1^{-1}(u/\varphi(t))$, thus $\delta_t(u)=\varphi(t)u$, one has $\sigma_s(\gamma_t(\varphi(t)\eta_1(|x|)))=\sigma_s(|x|)\in M$ (cf. (2.6), (4.1)). Therefore (3.19) implies (4.3) (note that the integrand depends continuously upon u, analogously to Proposition 2.5, (2.7)).

In particular, $\eta_\gamma(u)=\varphi(u)=u^\gamma$, $\gamma>0$, yields the standard Abel—Cartwright means $W_\gamma(t)(:=W(t^\gamma\eta_\gamma))$ which subsume for $\gamma=1$ the Abel—Poisson and for $\gamma=2$ the Gauss—Weierstrass means (cf. (4.12)). Corollary 4.1 then reduces to

Corollary 4.2. *For every* $\gamma,\delta>0$ *one has*

$$\|W_\gamma(t)f-f\|_p\leqq Ct^\gamma\int\limits_t^\infty\|W_\delta(u)f-f\|_p u^{-\gamma-1}du. \tag{4.4}$$

Since $(1-\exp\{-u^\gamma\})/(1-\exp\{-u^\delta\})\in BV_{j+1}[0,\infty]$ for $0<\delta\leqq\gamma$ (cf. [15, p. 54 ff]), it follows by (2.6) that in these cases one has indeed the direct estimate (cf. [5])

$$\|W_\gamma(t)f-f\|_p\leqq C\|W_\delta(t)f-f\|_p, \tag{4.5}$$

which, of course, is stronger than (4.4).

On the other hand, concerning the sharpness of (4.4) for $\delta>\gamma$, it is shown in [6] that for each $0<\mu<1/2$, $0<\nu<1$ there exists an element $f_{\mu,\nu}$ such that for e.g. $\gamma=1$, $\delta=2$, $p=1$

$$\|W_2(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1\begin{cases}=O(t^{2\mu})\\ \neq o(t^{2\mu})\end{cases}\quad(t\to 0+),$$

$$\limsup_{t\to 0+}\frac{\|W_1(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1}{\|W_2(t)f_{\mu,\nu}-f_{\mu,\nu}\|_1}t^\nu>0. \tag{4.6}$$

Thus an estimate of type (4.5) is impossible for $\delta>\gamma$, even for nonsmooth elements.

4.2. Marchaud-type inequalities. For $h\in\mathbf{R}^n$ let symmetric differences of order $2r$, $r\in\mathbf{N}$, be given by

$$\Delta_h^{2r}f:=(\Delta_h^*)^r f,\quad (\Delta_h^* f)(x):=f(x+h)-2f(x)+f(x-h), \tag{4.7}$$

corresponding to the multipliers $(2(\cos\langle h,x\rangle-1))^r$. Let $S_{n-1}:=\{\omega\in\mathbf{R}^n\colon |\omega|=1\}$.

Corollary 4.3. *Given $r,s\in\mathbf{N}$ and $1\leqq p<\infty$, there exists a constant C such that for every $f\in L^p(\mathbf{R}^n)$, $\omega\in S_{n-1}$, $t>0$*

$$\|\Delta_{t\omega}^{2r}f\|_p\leqq\frac{C}{t}\int_0^\infty\|\Delta_{u\omega}^{2s}f\|_p\min\{1,(t/u)^{2r+1}\}\,du. \tag{4.8}$$

Proof. To apply Theorem 3.8, consider

$$d(u):=2(1-\cos u),\quad \sigma(u):=\int_0^1 d^s(uv)(1-v)\,dv. \tag{4.9}$$

Obviously, $d^s(u)=\sum_{j=0}^s a_{sj}\cos ju$ with $a_{s0}>0$, and therefore

$$\sigma(u)=\frac{a_{s0}}{2}+\sum_{j=1}^s a_{sj}\frac{1}{2}\frac{d(ju)}{(ju)^2}.$$

Now, $d(u)/u^2$ and consequently (cf. (2.11)) $d(ju)/(ju)^2$ belong to $BV_{j+1}[0,\infty]$ so that in view of $\lim_{u\to\infty}\sigma(u)=a_{s0}/2\neq 0$ there exists $a>0$ such that

$$\sigma(u)\geqq a_{s0}/4\neq 0\quad\text{for}\quad u\geqq a,\quad \sigma\in BV_{j+1}[a,\infty].$$

Again by (2.11) this implies that $\sigma_t(u):=\sigma(u/t)\in BV_{j+1}[at,\infty]$ with

$$\|\sigma_t\|_{BV_{j+1}[at,\infty]}\leqq C\|\sigma\|_{BV_{j+1}[a,\infty]}\leqq C\|\sigma\|_{BV_{j+1}[0,\infty]},$$

uniformly for $t>0$. Since also $|\sigma_t(u)|\geqq a_{s0}/4$ for $u\geqq\alpha(t):=at$, the family $\{\sigma_t\}_{t>0}$ satisfies the Tauberian condition with $\alpha(t)=at$ and $\beta(t)=\infty$.

Thus, in order to apply (3.19), set

$$(t,\omega)\in\mathscr{J}=(0,\infty)\times S_{n-1},\quad \psi_{t,\omega}(x)=[t|\langle\omega,x\rangle|/a]^{2r},$$
$$\gamma_{t,\omega}(u)=au^{1/2r}/t,\quad \tau_{t,\omega}(x)=d^r(t|\langle\omega,x\rangle|).$$

Since $d(u)/(u/a)^2$ and hence $[d(u)/(u/a)^2]^r$ belong to $BV_{j+1}[0,\infty]$, it follows that $\tau_{t,\omega}$ is globally divisible of order $\psi_{t,\omega}$ (cf. (2.6, 11), (4.2)). Moreover,

$$(\gamma_{t,\omega}\circ\psi_{t,\omega})(x)=|\langle\omega,x\rangle|,$$

and therefore $(\sigma_u\circ\gamma_{t,\omega}\circ\psi_{t,\omega})(x)=\sigma_u(|\langle\omega,x\rangle|)\in M_p$. Thus $X=L^p$ is R^j_ψ- and $R^j_{\gamma\circ\psi}$- bounded and

$$\|T(\sigma_u\circ\gamma_{t,\omega}\circ\psi_{t,\omega})f\|_p:=\Big\|\int_0^1 \Delta^{2s}_{v\omega/u}f(x)(1-v)\,dv\Big\|_p\leqq u\int_0^{1/u}\|\Delta^{2s}_{z\omega}f\|_p\,dz.$$

Moreover, $\delta_{t,\omega}(u)=\gamma^{-1}_{t,\omega}(au)=(tu)^{2r}$, $\delta'_{t,\omega}(u)=2rt^{2r}u^{2r-1}$, and $\delta^{-1}_{t,\omega}(1)=\gamma_{t,\omega}(1)=1/t$. Hence with (3.19)

$$\|\Delta^{2r}_{t\omega}f\|_p=:\|T(\tau_{t,\omega})f\|_p\leqq C\int_0^{1/t}u\int_0^{1/u}\|\Delta^{2s}_{z\omega}f\|_p\,dz\,2rt^{2r}u^{2r-1}\,du\leqq$$

$$\leqq Ct^{2r}\int_t^\infty v^{-2r-2}\Big(\int_0^t+\int_t^v\Big)\|\Delta^{2s}_{z\omega}f\|_p\,dz\,dv=$$

$$=Ct^{2r}\Big[\int_0^t\|\Delta^{2s}_{z\omega}f\|_p\,dz\int_t^\infty v^{-2r-2}dv+\int_t^\infty\|\Delta^{2s}_{z\omega}f\|_p\,dz\int_z^\infty v^{-2r-2}dv\Big]=$$

$$=\frac{c}{t}\int_0^t\|\Delta^{2s}_{z\omega}f\|_p\,dz+Ct^{2r}\int_t^\infty z^{-2r-1}\|\Delta^{2s}_{z\omega}f\|_p\,dz=$$

$$=\frac{c}{t}\int_0^\infty\|\Delta^{2s}_{z\omega}f\|_p\min\Big\{1,\Big(\frac{t}{z}\Big)^{2r+1}\Big\}\,dz.$$

Let the $2r$th modulus of continuity of $f\in L^p(\mathbf{R}^n)$ be defined for $r\in\mathbf{N}$, $h>0$ by (cf. (4.7))

$$\omega_{2r}(h,f;\ L^p(\mathbf{R}^n)):=\sup\{\|\Delta^{2r}_{t\omega}f\|_p:\ \omega\in S_{n-1},\quad 0<t<h\}. \tag{4.10}$$

Then (4.8) implies the familiar Marchaud inequality (see also [1])

$$\omega_{2r}(h,f;\ L^p(\mathbf{R}^n))\leqq Ch^{2r}\int_h^\infty\omega_{2s}(u,f;\ FL^p(\mathbf{R}^n))\,u^{-2r-1}\,du. \tag{4.11}$$

Indeed, for $\omega\in S_{n-1}$, $0<t\leqq h$

$$\|\Delta_{t\omega}^{2r}f\|_p \leqq C\Big[\omega_{2s}(t,f;L^p(\mathbf{R}^n))+t^{2r}\int\limits_t^\infty \omega_{2s}(u,f;\ L^p(\mathbf{R}^n))u^{-2r-1}du\Big] \leqq$$

$$\leqq (2r+1)Ct^{2r}\int\limits_t^\infty \omega_{2s}(u,f;\ L^p(\mathbf{R}^n))u^{-2r-1}du$$

so that (4.11) follows since $\omega_{2r}(t,f;\ L^p(\mathbf{R}^n))$ as well as the right-hand side are increasing functions of t.

4.3. A semidiscrete difference scheme for the heat equation. Let $n=1$. In order to approximate the exact solution of the heat equation $(x\in\mathbf{R},\ t>0)$

$$d/dt\,u(x,t)=d^2/dx^2\,u(x,t),\ \ u(x,0)=f(x)\in L^p(\mathbf{R}),$$

given by the Gauss—Weierstrass means (cf. Section 4.1)

$$W_2(t^{1/2})f(x)=(4\pi t)^{-1/2}\int\limits_{-\infty}^{\infty} f(x-y)e^{-y^2/4t}dy, \tag{4.12}$$

consider the initial value problem for $h>0$

$$d/dt\,u_h(x,t)=h^{-2}[u_h(x+h,t)-2u_h(x,t)+u_h(x-h,t)],\quad u_h(x,0)=f(x).$$

This leads to the semidiscrete difference scheme (cf. [2, p. 69])

$$u_h(x,t)=D_h(t)f(x):=T(e^{-(t/h^2)d(hy)})f(x),$$

the function d being given by (4.9). Thus the multiplier $\varkappa_{h,t}(\in M_p(\mathbf{R})$, uniformly for $h,t>0)$ of the remainder $D_h(t)-W_2(t^{1/2})$ has the representation

$$\varkappa_{h,t}(x)=e^{-(t/h^2)d(hx)}-e^{-tx^2}\quad (h,\ t>0). \tag{4.13}$$

For example by the results obtained in [7] (see also [2, p. 72] for a concrete approach) it follows that for $t=h^2$

$$\|D_h(h^2)f-W_2(h)f\|_p\leqq C\omega_4(h,f;\ L^p(\mathbf{R})).$$

Theorem 3.8 now enables one to derive the following inverse estimate (cf. [2, p. 79]).

Corollary 4.4. *One has (cf.* (4.10) *with* $n=1$*)*

$$\omega_4(h,f;\ L^p(\mathbf{R}))\leqq C\int\limits_0^\infty \|D_u(u^2)f-W_2(u)f\|_p\min\{1,(h/u)^4\}\frac{du}{u}. \tag{4.14}$$

Proof. With $h\in\mathscr{J}=(0,\infty)$, $\tau_h(x)=d^2(h|x|)$ (cf. (4.9)) it follows as in Section 4.2 that $\{\tau_h\}_{h>0}$ is globally divisible of order $(h|x|)^4$. Consider

$$\sigma(s):=e^{-d(s)}-e^{-s^2},\quad \sigma_u(s):=\sigma((2\pi-1)s/u).$$

Since $\sigma \in BV_{j+1}[2\pi-1, 2\pi+1]$ and for $|s-2\pi| \leqq 1$

$$\sigma(s) = e^{-d(s-2\pi)} - e^{-s^2} \geqq e^{-(s-2\pi)^2} - e^{-(2\pi-1)^2} \geqq e^{-1} - e^{-(2\pi-1)^2} > 0,$$

$\{\sigma_u\}_{u>0}$ satisfies the Tauberian condition with $\alpha(u)=u$, $\beta(u)=[(2\pi+1)/(2\pi-1)]u$ (cf. (2.11)). Moreover, for $\gamma_h(s)=s^{1/4}/h(2\pi-1)$, thus $\delta_h(s)=(sh(2\pi-1))^4$, condition (3.18) holds true for $q=[(2\pi+1)/(2\pi-1)]^4$, and one has (cf. (4.13))

$$(\sigma_u \circ \gamma_h \circ \psi_h)(x) = \sigma(|x|/u) = \varkappa_{u^{-1}, u^{-2}}(x) \in M_p(\mathbf{R}).$$

Therefore by (3.20)

$$\|\Delta_h^4 f\|_p \leqq C \int_0^\infty \|D_{u^{-1}}(u^{-2})f - W_2(u^{-1})f\|_p \min\{1, (uh(2\pi-1))^4\} \frac{du}{u}.$$

Substituting $1/u=z$, the result follows.

Let us finally mention that one may employ the analysis outlined in [8] in order to discuss the sharpness of (4.14) in a sense similar to (4.6).

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