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On the rate of approximation by orthogonal series

H. SCHWINN

Dedicated to Professor K. Tandori on his sixtieth birthday

1. Introduction

Let $\{\varphi_n(x)\}$ be a normalized system of orthogonal functions (ONS) with respect to the space $L^2[0, 1]$. We ask for additional conditions on coefficients $\{c_n\}$ with $\sum_{n=0}^{\infty} c_n^2 < \infty$ such that the partial sums $\{s_n(x)\}$ of the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x)$ are convergent to a limit function $f(x)$, uniquely a.e. determined by the Riesz—Fischer theorem, with a certain speed. K. TANDORI [10] proved the following basic result:

Theorem A. *Assume that $\{\lambda(n)\}$ is an increasing sequence tending to ∞. If $\sum_{n=2}^{\infty} c_n^2 \lambda^2(n) (\ln n)^2 < \infty$, then the estimate*

$$f(x) - s_n(x) = o_x\left(\frac{1}{\lambda(n)}\right) \quad a.e. \tag{1}$$

holds.

Asking for the finality of Theorem A as a consequence of a result of L. Leindler ([7], Hilfssatz 2) it follows that in case $\lambda(n+1) > C^*\lambda(n)$ $(C^* > 1)$ the factor $(\ln n)^2$ may be omitted. On the other side, for certain sequences increasing slowly enough, V. A. ANDRIENKO [2] proved the finality of Theorem A. Later on V. I. KOLYADA [6] proved the following result:

Theorem B. *Assume that the positive increasing sequence $\{\lambda(n)\}$ is such that*

$$\ln n = o(\lambda(n))$$

and that there exists a sequence $\{\nu_n\}$ with the properties:

$$\mu(n) = \nu_{n+1} - \nu_n \geqq 2, \quad 1 < \varrho \leqq \frac{\lambda(\nu_{n+1})}{\lambda(\nu_n)} \leqq r.$$

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If $\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty$, *then we have the estimate*

$$f(x) - s_n(x) = o_x\left(\frac{\ln \mu(q_{n+1})}{\lambda(n+1)}\right) \quad a.e.;$$

where q_n *is defined with the aid of the strictly increasing function* $v(t)$ *with* $v(n) = v_n$ *and its inverse* $v^{-1}(t)$ *by* $q_n = [v^{-1}(n)]$.

V. I. Kolyada also proved in [6] the finality of Theorem B in the following way: the speed $\ln \mu(q_{n+1})/\lambda(n+1)$ may not be replaced by a speed $(A(n))^{-1}$ tending faster to zero, i.e. if $A(n) \cdot \ln \mu(q_{n+1})/\lambda(n+1) \to \infty$.

In this paper we want to establish a general condition for estimations of type (1), which is also necessary for a special class of coefficients $\{c_n\}$. In the following let $\{\lambda(n)\}$ be a nondecreasing sequence tending to infinity. We consider in dependence of a fixed chosen constant $q > 1$ the uniquely determined sequence of increasing natural numbers $\{\mu_k\}$ with

(2) $$\lambda(\mu_{k+1}) \geqq q \cdot \lambda(\mu_k) \quad \text{and} \quad \lambda(\mu_{k+1} - 1) < q \cdot \lambda(\mu_k) \quad (k = 0, 1, \ldots).$$

Theorem 1. *Let*

$$\sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) (\ln (n - \mu_k + 2))^2 < \infty$$

be fulfilled. Then the estimation

(3) $$f(x) - s_n(x) = o_x\left(\frac{1}{\lambda(n+1)}\right) \quad a.e.$$

holds.

We can extend this statement to partial sums $\{s_{n_i}(x)\}$, where $\{n_i\}$ is an increasing sequence of natural numbers. With respect to the above considered sequence $\{\mu_n\}$, let $I(k)$ be defined by

(4) $$n_{I(k)-1} < \mu_k - 1 \leqq n_{I(k)} \quad (k = 1, 2, \ldots).$$

Then $I(k+1) - I(k)$ indicates the number out of $\{n_i\}$ between $\mu_k - 1$ and $\mu_{k+1} - 1$. The above definition also admits the case $I(k) = I(k+1)$; therefore let $\{k_j\}$ denote the sequence of those numbers when $I(k_j+1) - I(k_j) > 0$. Putting

(5) $$C_i = \Big\{ \sum_{n=n_{i-1}+1}^{n_i} c_n^2 \Big\}^{1/2} \quad (i = 0, 1, \ldots; \quad n_{-1} = -1)$$

we prove

Theorem 2. *Let*

$$\sum_{j=1}^{\infty} \sum_{i=I(k_j)+1}^{I(k_j+1)} C_i^2 \lambda^2(n_{i-1}+1) (\ln (i - I(k_j) + 2))^2 < \infty$$

be fulfilled. Then for $\{s_{n_i}(x)\}$ *the estimation*

$$f(x)-s_{n_i}(x) = o_x\left(\frac{1}{\lambda(n_i+1)}\right) \quad a.e.$$

holds.

Theorem 3. *Let*

$$\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) < \infty$$

be fulfilled and let $\alpha(n)$ *be defined by* $\alpha(n)=\ln(n-\mu_k+2)$ *if* $\mu_k \leqq n < \mu_{k+1}$. *Then the estimation*

$$f(x)-s_n(x) = o_x\left(\frac{\alpha(n)}{\lambda(n+1)}\right) \quad a.e.$$

holds.

It is possible to show that the conditions of these theorems are also necessary if the coefficients $\{c_n\}$ resp. $\{C_i\}$ are nonincreasing in a restricted sense. The following theorem is close to K. Tandori's theorem [9] on the necessity of the condition of coefficients in the Rademacher—Menchoff-theorem (cf. G. ALEXITS [1, p. 83]).

Theorem 4. *If* $c_n \geqq c_{n+1}$ *for* $\mu_k \leqq n \leqq \mu_{k+1}-2$; $k=1,2,\ldots,$ *and*

$$\sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) (\ln(n-\mu_k+2))^2 = \infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\overline{\lim_{n\to\infty}}\, \lambda(n+1)|f(x)-s_n(x)| = \infty \quad (x\in[0,1]). \tag{6}$$

Remark. V. A. ANDRIENKO and L. V. G'RNEVSKA [3] have proved that $\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty$ implies estimation (3) if $\{\varphi_n(x)\}$ defines a convergence system (i.e. $\sum_{n=0}^{\infty} c_n^2 < \infty$ implies the convergence a.e. of $\sum_{n=0}^{\infty} c_n \varphi_n(x)$); they further proved that in (3) $\{\lambda(n)\}$ must not be replaced by a sequence $\{\Lambda(n)\}$ tending faster to infinity. By Lemma 3 we can conclude that $\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty$ is also necessary, for in the case $\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) = \infty$ it always exists a convergence system such that estimation (3) fails.

In a way similiar to that of L. CSERNYÁK and L. LEINDLER [4] used to extend K. TANDORI's theorem [9] to subsequences $\{s_{n_i}(x)\}$, we can prove with terms (5)

Theorem 5. *If* $C_i \geqq C_{i+1}$ *for* $\mathrm{l}(k_j) \leqq i \leqq \mathrm{l}(k_j+1)-2,\ j=1,2,\ldots,$ *and*

$$\sum_{j=1}^{\infty} \sum_{i=\mathrm{l}(k_j)}^{\mathrm{l}(k_{j+1})-1} C_i^2 \lambda^2(n_i+1)\big(\ln(i-l(k_j)+2)\big)^2 = \infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\overline{\lim_{i\to\infty}}\, \lambda(n_i+1)|f(x)-s_{n_i}(x)| = \infty \quad (x\in[0,1]).$$

Obviously Theorems 2 and 4 are generalizations of Theorems 1 and 3. But the result of Theorem 3 is a necessary step in the proof of Theorem 4 and the proof of Theorem 2 is based on Theorem 1. The finality of Theorem 3 follows finally with

Theorem 6. *If* $c_n \geqq c_{n+1}$ *for* $\mu_k \leqq n \leqq \mu_{k+1}-2;\ k=1,2,\ldots,$ *and*

$$\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) = \infty,$$

then there exists an ONS $\{\varphi_n(x)\}$ *with*

$$\overline{\lim_{n\to\infty}}\, \frac{\lambda(n+1)}{\alpha(n)} |f(x)-s_n(x)| = \infty \quad (x\in[0,1]).$$

2. Proof of Theorems 1, 2 and 3

The following result will be essential.

Lemma 1. *For any ONS* $\{\varphi_n(x)\}$ *the following estimation holds:*

$$\int_0^1 \Big\{\max_{1\leqq i\leqq j\leqq N} |c_i\varphi_i(x)+\ldots+c_j\varphi_j(x)|^2\Big\} dx \leqq K_1\Big(c_1^2+\sum_{n=2}^{N} c_n^2(\ln n)^2\Big).^{1)}$$

Proof: cf. K. Tandori [11, Satz VII]; see also A. Zygmund [13, p. 193].

Proof of Theorem 1. In the first step we prove the assertion for the partial sums $s_{\mu_k-1}(x)$, $k=1,2,\ldots$; namely by (2) we get

$$\sum_{k=1}^{\infty} \int_0^1 \lambda^2(\mu_k)\cdot(f(x)-s_{\mu_k-1}(x))^2 dx = \sum_{k=1}^{\infty} \lambda^2(\mu_k) \sum_{j=k}^{\infty} \sum_{n=\mu_j}^{\mu_{j+1}-1} c_n^2 =$$

$$= \sum_{j=1}^{\infty} \sum_{n=\mu_j}^{\mu_{j+1}-1} c_n^2 \sum_{k=1}^{j} \lambda^2(\mu_k) = O(1) \sum_{n=\mu_1}^{\infty} c_n^2\lambda^2(n) < \infty.$$

[1)] $K_1, K_2, \ldots$ denote absolute constants.

With the aid of *B*. Levi's theorem we conclude

$$(7) \qquad f(x)-s_{\mu_k-1}(x)=o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..}$$

For the remaining partial sums Lemma 1 leads us with

$$\delta_k(x)=\max_{\mu_k\leqq n<\mu_{k+1}-1}|s_n(x)-s_{\mu_{k+1}-1}(x)|$$

to

$$\sum_{k=1}^{\infty}\int_0^1\lambda^2(\mu_k)\,\delta_k^2(x)\,dx\leqq K_2\sum_{k=1}^{\infty}\lambda^2(\mu_k)\sum_{n=\mu_k}^{\mu_{k+1}-1}c_n^2(\ln(n-\mu_k+2))^2=$$

$$=O(1)\sum_{k=1}^{\infty}\sum_{n=\mu_k}^{\mu_{k+1}-1}c_n^2\lambda^2(n)\,(\ln(n-\mu_k+2))^2<\infty.$$

This shows

$$s_n(x)-s_{\mu_{k+1}-1}(x)=o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e.} \quad (\mu_k\leqq n\leqq\mu_{k+1}-1);$$

by $\lambda(n+1)\leqq q\cdot\lambda(\mu_k)$ for $\mu_k\leqq n\leqq\mu_{k+1}-2$ (cf. (2)) together with (7) it follows

$$f(x)-s_n(x)=O(1)\{|f(x)-s_{\mu_{k+1}-1}(x)|+|s_{\mu_{k+1}-1}(x)-s_n(x)|\}=o_x\left(\frac{1}{\lambda(n+1)}\right) \quad \text{a.e.,}$$

thus Theorem 1 is proved.

Proof of Theorem 2. We represent $\{s_{n_i}(x)\}$ as (direct) partial sums $\{S_i(x)\}$ of an appropriate orthogonal series with coefficients (5); instead of $\{\lambda(n)\}$ the sequence $\{\Lambda(i)\}$ with $\Lambda(i)=\lambda(\mu_k)$ if $\mu_k\leqq n_i+1<\mu_{k+1}$ is taken. Here with respect to (2) $\{l(k_j)\}$ assumes the role of $\{\mu_k\}$ (cf. (4)). Theorem 1 gives $f(x)-S_i(x)=f(x)-s_{n_i}(x)=$ $=o_x\left(\frac{1}{\Lambda(i+1)}\right)$; noting that $\lambda(n_i+1)=O(\lambda(\mu_k))$ for $n_i+1<\mu_{k+1}$ the assertion follows immediately.

Proof of Theorem 3. By the proof of Theorem 1 it yields

$$(8) \qquad f(x)-s_{\mu_k-1}(x)=o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..}$$

Now, for the partial sums $s_n^*(x)$ of the series $\sum_{n=0}^{\infty}c_n^*\varphi_n(x)$ with $c_n^*=c_n\cdot(\ln(n-\mu_k+2))^{-1}$ if $\mu_k\leqq n<\mu_{k+1}$; $k=0,1,\ldots,$ the proof of Theorem 1 has shown that

$$\hat{\delta}_k(x)=\max_{\mu_k\leqq n<\mu_{k+1}}|s_n^*(x)-s_{\mu_{k+1}-1}^*(x)|=o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e.}$$

and

$$\delta_k^*(x)=\max_{\mu_k\leqq n<\mu_{k+1}}|s_n^*(x)-s_{\mu_k-1}^*(x)|=o_x\left(\frac{1}{\lambda(\mu_k)}\right) \quad \text{a.e..}$$

By Abel's transformation (cf. G. ALEXITS [1; p. 68]) we get with $\mu_k \leqq n < \mu_{k+1}$

$$|s_n(x)-s_{\mu_k-1}(x)| = \Big|\sum_{\nu=\mu_k}^{n} c_\nu \varphi_\nu(x)\Big| = \Big|\sum_{\nu=\mu_k}^{n} \ln(\nu-\mu_k+2) c_\nu^* \varphi_\nu(x)\Big| =$$

$$= \Big|\sum_{\nu=\mu_k}^{n-1} \big(\ln(\nu-\mu_k+2)-\ln(\nu+1-\mu_k+2)\big)\big(s_\nu^*(x)-s_{\mu_k-1}^*(x)\big)+$$

$$+\ln(n-\mu_k+2)\big(s_n^*(x)-s_{\mu_k-1}^*(x)\big)\Big| \leqq 3\ln(n-\mu_k+2)\delta_k^*(x).$$

This proves $\lambda(\mu_k)(\ln(n-\mu_k+2))^{-1}(s_n(x)-s_{\mu_k-1}(x)) \to 0$ a.e. $(n\to\infty)$, the assertion follows by (8) and (2).

3. Proof of Theorems 4, 5 and 6

To prove the necessity of the conditions stated in these theorems we need some auxiliary results. We use the following lemma of K. TANDORI [9] (cf. G. ALEXITS [1, p. 87]) which plays an important role in the proof of divergence phenomena of orthogonal series in general.

Lemma 2. *Let $\{a_n\}$ be a nonincreasing sequence of positive real numbers, and let $N_r=2^{r+2}-4$, $r=0, 1, \ldots$. Then, for every r, there exists a measurable set F_r with measure*

$$\mu(F_r) \geqq K_1^* \min\{1, N_{r+1} a_{N_{r+1}}^2 (\ln N_{r+1})^2\} \quad (K_1^*>0),$$

and an ONS $\{\Phi_n(x)\}$ consisting of piecewise functions, such that

(a) *the sets $F_0, F_1, \ldots$ are stochastically independent*[1)]

(b) *for all $x\in F_r$ it exists a number $n_{r(x)}<2^{r+2}$ such that $\Phi_{N_r}(x), \ldots, \Phi_{N_r+n_{r(x)}}(x)$ are of the same sign and*

$$|\Phi_{N_r}(x)+\ldots+\Phi_{N_r+n_{r(x)}}(x)| \geqq \frac{K_2^*}{a_{N_{r+1}}} \quad (K_2^*>0).$$

Remark. The proof of the lemma shows that F_r may be chosen as a simple set (i.e. consisting of a finite number of segments) and with the additional property: if $\Phi_0(x), \ldots, \Phi_{N_{r+1}-1}(x)$ are constant in a segment I^*, then either $F_r\cap I^*=\emptyset$ or $I^*\subset F_r$.

To prove the necessity of the condition in Theorem 4 we first state

1) $F_0, F_1, \ldots$ are stochastically independent with respect to $[0, 1]$, if $k_1<k_2\ldots<k_l$ then $\mu(F_{k_1}\cap F_{k_2}\cap \ldots \cap F_{k_l})=\mu(F_{k_1})\mu(F_{k_2})\ldots\mu(F_{k_l})$.

Lemma 3. *Let $\{c_n\}$ be an arbitrary sequence of real numbers. If condition*

$$\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) = \infty$$

is fulfilled, then there exists an ONS $\{\varphi_n(x)\}$ consisting of piecewise constant functions which forms a convergence system with

$$\overline{\lim_{n\to\infty}} \lambda(n+1)|f(x)-s_n(x)| = \infty.$$

Proof. We can find a nonincreasing positive sequence $\{\varepsilon_n\}\to 0$ with

$$\sum_{n=1}^{\infty} c_n^2 \lambda^2(n) \varepsilon_n^2 = \infty. \tag{9}$$

We define the system of functions by induction. In the basic step we put with $r_1(x)=\mathrm{sign}\,(\sin 2\pi x)$ $(0\leqq x\leqq 1)$

$$\varphi_0(x) = r_1(x).$$

Now let $\varphi_0(x), \ldots, \varphi_{m-1}(x)$ be defined. The segments where each of these functions are constant are denoted by $I_0^{(m)}, \ldots, I_{q_m}^{(m)}$ (with $\sum_{l=0}^{q_m} \mu(I_l^{(m)})=1$). Putting

$$\gamma(m) = \begin{cases} 1 & \text{if } c_m = 0 \text{ or } c_m^2\lambda^2(m)\varepsilon_m^2 > 1, \\ c_m^2\lambda^2(m)\varepsilon_m^2 & \text{elsewhere,} \end{cases}$$

we choose in each $I_l^{(m)}=(s_l^{(m)}, t_l^{(m)})$ a partial segment $J_l^{(m)}=(u_l^{(m)}, v_l^{(m)})$ with $\mu(J_l^{(m)})=$ $=\gamma(m)\mu(I_l^{(m)})$ and with $u_l^{(m)}=s_l^{(m)}$. In general, for a segment $J=(u, v)$ and a function $f(x)$ defined on $[0, 1]$, let the denotation

$$f(J; x) = \begin{cases} f\left(\dfrac{x-u}{v-u}\right) & \text{if } x\in J \\ 0 & \text{if } x\notin J \end{cases} \tag{10}$$

be valid. Then we put

$$\varphi_m(x) = \frac{1}{\sqrt{\gamma(m)}} \sum_{l=0}^{q_m} r_1(J_l^{(m)}; x).$$

It is easy to verify that $\varphi_0(x), \ldots, \varphi_m(x)$ constitute a set of orthogonal and normalized functions.

The sets

$$G_m = \bigcup_{l=0}^{q_m} J_l^{(m)}, \quad m = 1, 2, \ldots,$$

are stochastically independent. Thus by the second Borel—Cantelli lemma (cf. W. FELLER [5, p. 155]) we deduce that with $\mu(G_m)=\gamma(m)$ and $\sum_{n=1}^{\infty} \mu(G_m)=\infty$ (cf. (9)), for $\bar{G}=\overline{\lim}\, G_m$ $\mu(\bar{G})=1$ holds. Taking $x_0\in\bar{G}$ we can find an infinite set of numbers m with

$$\lambda(m)|c_m\varphi_m(x)| = \frac{|c_m|\lambda(m)}{\sqrt{\gamma(m)}} \geqq \frac{1}{\varepsilon_m}.$$

Because of $\varepsilon_m\to 0$ and because of the estimate

$$|f(x)-s_{m-1}(x)| \geqq |c_m\varphi_m(x)|-|f(x)-s_m(x)|$$

the above stated equality contradicts the estimation $f(x)-s_{m-1}(x)=O_x\left(\frac{1}{\lambda(m)}\right)$ a.e.. Changing the values of $\{\varphi_n(x)\}$ in $[0, 1]-\bar{G}$ in an appropriate way, we get the assertion of Lemma 3.

To prove that $\{\varphi_n(x)\}$ is a convergence system, we mention a lemma of D. E. Menchoff ([8], Lemma 2) proving that the following conditions are sufficient for $\{\psi_n(x)\}$ to be a convergence system: Let the segments $I_l^{(n)}$, $l=1, \ldots, p_n$, with $I_l^{(n)}\cap I_{l_0}^{(n)}=\emptyset$ if $l\neq l_0$ and $\sum_{l=1}^{p_n}\mu(I_l^{(n)})=1$, satisfy

(i) $\psi_n(x)$ has constant value in $I_l^{(n)}$, $l=1, \ldots, p_n$;

(ii) $\int_{I_l^{(n)}}\psi_m(x)\;dx=0$ for $l=1, \ldots, p_n$; $m=n+1, n+2, \ldots$;

(iii) $\lim_{n\to\infty}\max_l \mu(I_l^{(n)})=0$;

(iv) if $m>n$ then for every $I_l^{(n)}=(s, t)$ $(1\leqq l\leqq p_n)$ it exists an index $\sigma=\sigma(l, n, m)$ such that for $I_\sigma^{(m)}=(u, v)$ yields $u=s$.

It is obvious, that the above mentioned functions $\{\varphi_n(x)\}$ and the sets $J_l^{(m)}$ $(l=0, 1, \ldots, q_m;\ m=1, 2, \ldots)$ satisfy conditions (i)—(iv), which completes the proof.

Proof of Theorem 4. We first mention that the case $\sum_{n=1}^{\infty}c_n^2\lambda^2(n)=\infty$ is treated in Lemma 3. Therefore, in the following, we may assume that $\sum_{n=1}^{\infty}c_n^2\lambda^2(n)<\infty$. For the terms

$$\sigma_k = \begin{cases} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2\lambda^2(n)\big(\ln(n-\mu_k+2)\big)^2 & \text{if } \mu_{k+1}-\mu_k>4 \\ 0 & \text{if } \mu_{k+1}-\mu_k\leqq 4 \end{cases}$$

where $\{\mu_k\}$ is given in (2), the series $\sum_{k=1}^{\infty}\sigma_k$ is divergent. Again we can find a nonincreasing sequence $\{\varepsilon_k\}\to 0$ with

$$\sum_{k=1}^{\infty}\sigma_k\varepsilon_k^2=\infty. \tag{11}$$

On the basis of this series we define an ONS which satisfies (6). By the aid of the Rademacher functions $r_n(x)=\operatorname{sign}(\sin 2^n\pi x)$, $n=0, 1, \ldots$, we take

$$\varphi_n(x)=r_n(x),\quad n=0, 1, \ldots, \mu_1-1.$$

Now let $\varphi_0(x), \ldots, \varphi_{\mu_k-1}(x)$ be defined and let us denote by $I_l^{(k)}$, $l=0, \ldots; q_k$; the segments in which each of these functions have constant value; we may assume that either $I_l^{(k)} \subset I_{l_0}^{(k-1)}$ or $I_l^{(k)} \cap I_{l_0}^{(k-1)} = \emptyset$. In the case of $\mu_{k+1}-\mu_k \leqq 4$ we put (by transformation (10))

$$\varphi_n(x) = \sum_{l=0}^{q_k} r_{n-\mu_k+1}(I_l^{(k)};\, x), \quad n = \mu_k, \ldots, \mu_{k+1}-1.$$

Otherwise if $\mu_{k+1}-\mu_k > 4$ we select $r_0 = r_0(k) \geqq 0$ such that with the numbers $\{N_r\}$ defined in Lemma 2 $N_{r_0+1} < \mu_{k+1}-\mu_k \leqq N_{r_0+2}$ is satisfied. Then we refer to Lemma 2 and set

$$a_n = c_{\mu_k+n}\varepsilon_k\lambda(\mu_k), \quad n = 0, 1, \ldots, N_{r_0+1}. \tag{12}$$

Let I_l' and I_l'' denote the two halves of I_l, $l=0, 1, \ldots, q_k$; then we put with the functions $\{\Phi_n(x)\}$ out of Lemma 2

$$\varphi_{\mu_k+n}(x) = \sum_{l=0}^{q_k} \{\Phi_n(I_l';\, x) - \Phi_n(I_l'';\, x)\}, \quad n = 1, \ldots, N_{r_0+1}.$$

In the case $N_{r_0+1}+1 < \mu_{k+1}-\mu_k$, we again select all segments I_l^*, $l=0, 1, \ldots; q_k^*$; in which the already stated functions have constant value. We then put

$$\varphi_{\mu_k+N_{r_0+1}+n}(x) = \sum_{l=0}^{q_k^*} r_n(I_l^*;\, x), \quad n = 1, \ldots, \mu_{k+1}-N_{r_0+1}-1.$$

With the transformation $(0, 1) \to I_l'$ resp. $(0, 1) \to I_l''$ if $\mu_{k+1}-\mu_k > 4$ the sets F_r, $r=0, 1, \ldots, r_0(k)$, considered in Lemma 2 will be transformed into the sets $F_r(I_l')$ resp. $F_r(I_l'')$. Then we set

$$G_r^{(k)} = \bigcup_{l=0}^{q_k} \{F_r(I_l') \cup F_r(I_l'')\}.$$

Let $l_1, l_2, \ldots$ denote those numbers with $\mu_{l_j+1}-\mu_{l_j} > 4$. Then the following is true:

(i) the sets $G_r^{(l_j)}$, $r=0, 1, \ldots, r_0(l_j)$, are stochastically independent (which follows by the definition and by the Remark to Lemma 2); furthermore

$$\mu(G_r^{(l_j)}) = \sum_{l=0}^{q_{l_j}} \{\mu(F_r(I_l')) + \mu(F_r(I_l''))\} = \mu(F_r) \sum_{l=0}^{q_{l_j}} \{\mu(I_l') + \mu(I_l'')\} = \mu(F_r) \geqq$$
$$\geqq K^* \min\{1,\, N_{r+1} c^2_{\mu_{l_j}+N_{r+1}} \varepsilon^2_{l_j} \lambda^2(\mu_{l_j})(\ln N_{r+1})^2\};$$

(ii) if $x \in G_r^{(l_j)}$, then there exists a number $n_r(x) = n_r(l_j;\, x) < 2^{r+2}$ with

$$|\varphi_{\mu_{l_j}+N_r}(x) + \ldots + \varphi_{\mu_{l_j}+N_r+n_r(x)}(x)| \geqq \frac{B^*}{c_{\mu_{l_j}+N_{r+1}}\varepsilon_{l_j}\lambda(\mu_{l_j})}$$

and the considered functions have the same sign at x. By the second Borel—Cantelli lemma, using $c_n \geqq c_{n+1}$ and $\lambda(n) \leqq q\lambda(\mu_{l_j})$ if $\mu_{l_j} \leqq n \leqq \mu_{l_{j+1}}-1$, furthermore that

$$\sum_{j=1}^{\infty} \sum_{r=0}^{r_0(l_j)} \mu(G_r^{(l_j)}) = \infty, \tag{13}$$

(cf. (i), (11)) for $\bar{G} = \overline{\lim}\, G_r^{(l_j)}$ we have $\mu(\bar{G}) = 1$. On the other side we get by (ii) with $x \in G_r^{(l_j)}$, $0 \leqq r \leqq r_0(l_j)$; $j = 1, 2, \ldots$, that for a suitable $n_r(x)$ it holds

$$\lambda(\mu_{l_j} + N_r) \Big| \sum_{n=\mu_{l_j}+N_r}^{\mu_{l_j}+N_r+n_r(x)} c_n \varphi_n(x) \Big| \geqq \frac{B^*}{\varepsilon_{l_j}} \to \infty,$$

whence (6) obviously follows (changing the values of $\{\varphi_n(x)\}$ in a suitable set of measure 0, if necessary).

The proof of Theorem 5 is based on Theorem 4, the method being close to that of L. CSERNYÁK and L. LEINDLER [4] where the following extension of K. Tandori's theorem [9] is proved: Let $\{C_i\}$ (cf. (5)) be a nonincreasing sequence. If $\sum_{i=2}^{\infty} C_i^2 (\ln i)^2 = \infty$, then it exists an ONS $\{\varphi_n(x)\}$ with $\overline{\lim}_{i\to\infty} |f(x) - s_{n_i}(x)| = \infty$. But at first we want to mention a result concerning the Rademacher functions $\{r_n(x)\}$. For any sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n^2 < \infty$, and for $f(x) \sim \sum_{n=0}^{\infty} c_n r_n(x)$ given by the Riesz—Fischer theorem it holds

$$A\Big\{\sum_{n=0}^{\infty} c_n^2\Big\}^{1/2} \leqq \int_0^1 |f(x)|\,dx \leqq B\Big\{\sum_{n=0}^{\infty} c_n^2\Big\}^{1/2}. \tag{14}$$

(A, B absolutely constant; cf. A. ZYGMUND [12, p. 213]). On the basis of estimation (14) L. Csernyák and L. Leindler proved

Lemma 4. *For an arbitrary sequence* $\{c_n\}$ *let the sets* $E_{n,m}$ *be defined by*

$$E_{n,m} = \Big\{x: \Big|\sum_{\nu=n}^{n+m} c_\nu r_\nu(x)\Big| > \frac{1}{2}\Big\{\sum_{\nu=n}^{n+m} c_\nu^2\Big\}^{1/2}\Big\}.$$

Then the sets $E_{n,m}$ *are simple sets with*

$$\mu(E_{n,m}) \geqq \frac{A^2}{4},$$

A given by (14).

Proof of Theorem 5. (a) By Theorem 4 we can find for the sequence $\{C_i\}$ an ONS $\{\Phi_i(x)\}$ such that for the partial sums $\{S_i(x)\}$ of the series $\sum_{i=0}^{\infty} C_i \Phi_i(x)$ we have

$$\overline{\lim_{i\to\infty}}\, \lambda(n_i+1)|f(x) - S_i(x)| = \infty \quad (x \in [0, 1]).$$

As in [4] it will be proved that with the aid of $\{\Phi_i(x)\}$ we can set up an ONS $\{\varphi_n(x)\}$ for which the assertion of the theorem is true. We outline the proof and refer the reader to [4] for complete argumentation.

We distinguish two cases: $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)=O(1)$ and $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)\neq O(1)$ (for definition of $\mathrm{I}(k_j)$ cf. the preliminary remark regarding Theorem 2) and treat only the (more complicate) latter one.

When $\mathrm{I}(k_j+1)-\mathrm{I}(k_j)\neq O(1)$ let $l_1^*, l_2^*, \dots$ denote those indices with

$$\mathrm{I}(l_j^*+1)-\mathrm{I}(l_j^*) > 4.$$

By the proof of Theorem 4 we can find some simple sets $G_r^{(l_j^*)}$, $r=0, 1, \dots, r_0(l_j^*)$, with

$$\sum_{j=1}^{\infty}\sum_{r=0}^{r_0(l_j^*)} \mu(G_r^{(l_j^*)}) = \infty$$

(cf. (13)), and for $x\in G_r^{(l_j^*)}$ numbers $i_0=\mathrm{I}(l_j^*)+N_r$ and $n_r(x)$ such that

$$\lambda(n_{i_0}+1)\Big|\sum_{i=i_0}^{i_0+n_r(x)} C_i\Phi_i(x)\Big| > \frac{B^*}{\varepsilon_{l_j^*}} \quad (\{\varepsilon_n\}\to 0), \tag{15}$$

and these functions are of the same sign on $G_r^{(l_j^*)}$.

(b) Next, corresponding to l_j^* and $r\leqq r_0(l_j^*)$ the number $\varkappa(j, r)$ is defined by

$$\varkappa(j, r) = \max\{(n_{i+1}-n_i)\colon \mathrm{I}(l_j^*)+N_r \leqq i < \mathrm{I}(l_j^*)+N_{r+1}\},$$
$$r = 0, 1, \dots, r_0(l_j^*);\quad j = 1, 2, \dots.$$

Now let us choose a fixed l_j^* and r. With the above value $\varkappa=\varkappa(j, r)$ we divide [0, 1] $Q^*=2^{\varkappa+1}$ partial segments with equal length $I_q=(u_q, v_q)$, $1\leqq q\leqq Q^*$. With respect to some n_{i_0}, where $n_{\mathrm{I}(l_j^*)+N_r}\leqq n_{i_0}<n_{\mathrm{I}(l_j^*)+N_{r+1}}$, the number of segments I_q, where

$$\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}r_n(x) > \frac{A}{2}\Big\{\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}^2\Big\}^{1/2}, \tag{16}$$

is at least $p^*=2^{-3}A^2Q^*$ bearing in mind that $r_n((1/2)+x)=r_n((1/2)-x)$. Let us then change the segments I_q and simultaneously the corresponding values of the functions $r_n(x)$, $n=1, \dots, n_{i_0+1}-n_{i_0}$, in such a way that (16) holds for the first p^* segments. The new functions are denoted by $r_{i_0,n}(x)$, $n=1, \dots, n_{i_0+1}-n_{i_0}$, i.e. it yields

$$\sum_{n=1}^{n_{i_0+1}-n_{i_0}} c_{n_{i_0}+n}r_{i_0,n}(x) > \frac{A}{2}\Big\{\sum_{n=n_{i_0}+1}^{n_{i_0+1}} c_n^2\Big\}^{1/2},$$
$$x\in I_q;\quad q = 1, \dots, p^*.$$

With the ONS $\{\Phi_i(x)\}$ mentioned in (a), we consider the functions (cf. (10))

$$g_{i_0 q}(x) = \Phi_{i_0}(I_q;\ x), \quad q = 1, \dots, Q^*,$$

and we put

$$\gamma_{n_{i_0}+n}(x) = \sum_{q=1}^{Q^*} r_{i_0, n}(x)\, g_{i_0 q}(x), \quad n = 1, \dots, n_{i_0+1} - n_{i_0}. \tag{17}$$

As in [4] we can prove that $\{\gamma_n(x)\}$ are orthogonal and normalized functions.

(c) For those n_i which are not covered by (b), namely when $i^* = \mathrm{I}(l_j^*) + N_{r_0(l_j^*)+1}$ and $n_{i^*} \leqq n_i < n_{\mathrm{I}(l_{j+1}^*)}$ we put $Q^* = 2^{n_{i+1} - n_i + 1}$ and define $\gamma_{n_i+1}(x), \dots, \gamma_{n_{i+1}}(x)$ analogously to (17) in (b).

(d) For fixed l_j^* and r with respect to the transformation $(0, 1) \to I_q$ in (b) the set $G_r^{(l_j^*)}$ will be transformed into the set $G_r^{(l_j^*)}(I_q)$. Taking

$$\bar{G}_r^{(l_j^*)} = \bigcup_{q=1}^{Q^*} G_r^{(l_j^*)}(I_q)$$

we get

$$\mu(\bar{G}_r^{(l_j^*)}) \geqq 2^{-4} A^3 \mu(G_r^{(l_j^*)}). \tag{18}$$

On the other side, with $i_0 = \mathrm{I}(l_j^*) + N_r$ we can find for $x \in \bar{G}_r^{(l_j^*)}$, e.g. $x \in G_r^{(l_j^*)}(I_{q_0})$, a number $n_r(x)$ with (cf. (15), (16))

$$\Big|\sum_{i=i_0}^{i_0+n_r(x)} \sum_{k=n_i+1}^{n_{i+1}} c_k \gamma_k(x)\Big| = \Big|\sum_{i=i_0}^{i_0+n_r(x)} g_{iq_0}(x) \sum_{k=n_i+1}^{n_{i+1}} c_k r_{i, k-n_i}(x)\Big| > \frac{A \cdot B^*}{2\varepsilon_{l_j} \lambda(\mu_{l_j^*})}. \tag{19}$$

(e) In the last step we have to give the ONS $\{\varphi_n(x)\}$ asked for. At first we put

$$\varphi_n(x) = r_n(x), \quad n = 0, 1, \dots, n_{\mathrm{I}(l_1^*)}.$$

Now let with $i_0 = \mathrm{I}(l_j^*) + N_\nu$ the functions $\varphi_0(x), \dots, \varphi_{n_{i_0}}(x)$ be defined. Denoting with $J_\sigma = J_\sigma^{(i_0)}$, $\sigma = 1, \dots, R(i_0)$ all those partial segments where these functions have constant value, we may assume that $J_\sigma^{(i_0)} \cap J_{\sigma'}^{(i_0-1)} = J_\sigma^{(i_0)}$ or that $J_\sigma^{(i_0)} \cap J_{\sigma'}^{(i_0-1)} = \emptyset$. The two halves of J_σ may be marked by J'_σ and J''_σ.

In the case $\nu = r_0(l_j^*) + 1$ and $\mathrm{I}(l_{j+1}^*) - \mathrm{I}(l_j^*) > N_{\nu+1}$, we put with $R = R(i_0)$ (cf. (10))

$$\varphi_k(x) = \sum_{\sigma=1}^{R} \big(\gamma_k(J'_\sigma;\ x) - \gamma_k(J''_\sigma;\ x)\big), \quad k = n_{i_0}+1, \dots, n_{\mathrm{I}(l_{j+1}^*)};$$

otherwise if $\nu \leqq r_0(l_j^*)$, then we take

$$\varphi_k(x) = \sum_{\sigma=1}^{R} \big(\gamma_k(J'_\sigma;\ x) - \gamma_k(J''_\sigma;\ x)\big), \quad k = n_{i_0}+1, \dots, n_{\mathrm{I}(l_j^*)+N_{\nu+1}}.$$

In the last case we also take up the set

$$H_\nu^{(j)} = \bigcup_{\sigma=1}^{R} \left(G_\nu^{(l_j^*)}(J'_\sigma) \cup G_\nu^{(l_j^*)}(J''_\sigma)\right),$$

whereby $G_\nu^{(l_j^*)}$ is transformed into $G_\nu^{(l_j^*)}(J'_\sigma)$ when (0, 1) is transformed into J'_σ.

The sets $H_\nu^{(j)}$, $\nu=0, 1, \ldots, r_0(l_j)$; $j=1, \ldots$ are stochastically independent and

$$\sum_{j=1}^{\infty} \sum_{\nu=0}^{r_0(l_j^*)} \mu(H_\nu^{(j)}) = \infty$$

(cf. (18)), i.e.

$$\mu(\overline{\lim}\, H_\nu^{(j)}) = 1.$$

But with $x \in H_\nu^{(j)}$ we can find $n(x)$ such that for $i^* = \mathrm{I}(l_j^*) + N_\nu$ then e.g. for $x \in G_\nu^{(l_j^*)}(J'_\sigma)$

$$\Big|\sum_{i=i^*}^{i^*+n(x)} \sum_{n=n_i+1}^{n_{i+1}} c_n \varphi_n(x)\Big| = \Big|\sum_{i=i^*}^{i^*+n(x)} \sum_{n=n_i+1}^{n_{i+1}} c_n \gamma_n(J'_\sigma;\, x)\Big| \geqq \frac{AB^*}{2\varepsilon_{l_j^*} \lambda(\mu_{l_j^*})}$$

(cf. (19)) which is not bounded. But this contradicts the estimation $f(x) - s_{n_i}(x) = = O_x\left(\frac{1}{\lambda(n_i+1)}\right)$. Changing the values of $\{\varphi_n(x)\}$ on a set with measure 0 we get the assertion of Theorem 5.

(f) In the case $\mathrm{I}(k_j+1) - \mathrm{I}(k_j) = O(1)$ we proceed as in (c).

The proof of Theorem 6 is similiar to that of Theorem 4, taking

$$\sigma_k^2 = \begin{cases} \sum_{n=\mu_k}^{\mu_{k+1}-1} c_n^2 \lambda^2(n) & \text{if } \mu_{k+1} - \mu_k > 4, \\ 0 & \text{if } \mu_{k+1} - \mu_k \leqq 4, \end{cases}$$

in case $\mu_{k+1} - \mu_k \neq O(1)$ with

$$\sum_{k=1}^{\infty} \sigma_k^2 \varepsilon_k^2 = \infty \quad (\{\varepsilon_k\} \to 0)$$

and putting in (12)

$$a_n = c_{\mu_k+n} \varepsilon_k, \quad n = 0, 1, \ldots, N_{r_0+1};$$

otherwise, if $\mu_{k+1} - \mu_k = O(1)$ the assertion follows by Lemma 3.

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MATHEMATISCHES INSTITUT
ARNDSTRASSE 2
6300 GIESSEN, GERMANY