Acta Sci. Math., 48 (1985), 443—450

On the almost everywhere divergence of Lagrange interpolation on infinite interval

J. SZABADOS and P. VÉRTESI

To Professor K. Tandori on his 60th birthday

1. Introduction

1.1. Let $X=\{x_{kn}\}$, $n=1,2,\ldots$, $1\leqq k\leqq n$, be an interpolatory matrix in $\boldsymbol{R}:=(-\infty,\infty)$, i.e.

$$(1.1)\qquad -\infty<x_{nn}<x_{n-1,n}<\ldots<x_{2n}<x_{1n}<\infty,\quad n=1,2,\ldots.$$

Restricting ourselves to nodes which are uniformly bounded with n, say $-1\leqq x_{kn}\leqq 1$, $1\leqq k\leqq n$, $n=1,2,\ldots$, the behaviour of the corresponding Lagrange interpolatory polynomials

$$(1.2)\qquad L_n(f,X,x)=L_n(f,x)=\sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x)$$

for $f\in C$ (i.e. f is continuous on $[-1,1]$) were thoroughly investigated. The first result is due to FABER [1], proving that *for any interpolatory matrix* $X\subset[-1,1]$ *there exists an* $f\in C$ *such that*

$$(1.3)\qquad \varlimsup_{n\to\infty}\|L_n(f,X,x)\|=\infty.$$

Here we used the notations

$$(1.4)\qquad l_{kn}(X,x)=l_{kn}(x)=\frac{\omega_n(x)}{\omega_n'(x_{kn})(x-x_{kn})},\quad k=1,2,\ldots,n,\quad n=1,2,\ldots$$

(where $\omega_n(X,x)=\omega_n(x)=c_n\prod_{k=1}^{n}(x-x_{kn})$), $\|g(x)\|=\|g\|=\max_{-1\leqq x\leqq 1}|g(x)|$ for $g\in C$).

Almost seventy years later the second named author in a joint paper with P. Erdős proved the following conjecture of ERDŐS [2].

Received January 6, 1984.

Theorem 1.1 (Erdős and Vértesi [3]). *For any interpolatory matrix X in* $[-1, 1]$ *one can find a function* $F(x)\in C$ *such that*

$$\varlimsup_{n\to\infty} |L_n(F, X, x)| = \infty \quad \textit{for almost all} \quad x\in[-1, 1]. \tag{1.5}$$

1.2. Some years ago the second of us completed the corresponding result for the unit circle ([4]).

In the same paper results for the trigonometric case were proven.

2. Infinite interval

2.1. Contrary to the case of the finite interval we know very little on the infinite ones. If the nodes are unbounded, as far as we know, even the analogue of Faber's result has not been proved yet.

Generally, the corresponding theorems deal with

a) special nodes (Laguerre or Hermite nodes, see e.g. [7]);

b) nodes where the maximal distance of the consecutive ones tends to zero (see e.g. [8]);

c) a *finite* interval $[a, b]\subset R$ (see e.g. [9]).

2.2. To illustrate these we quote the following statement of divergence type. Let $X^{(\alpha)}=\{x_{kn}^{(\alpha)}\}$, $1\leq k\leq n$, $n=1, 2, \ldots$, where $x_{kn}^{(\alpha)}$ is the k-th root of the n-th Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha>-1$.

Theorem 2.1 (Povchun [7]). *Let* $X=X^{(\alpha)}$, $\alpha>-1$. *Then there exists a function* $f(x)$ *continuous on* $[0, \infty)$ *such that*

$$\varlimsup_{n\to\infty} |L_n(f, X^{(\alpha)}, x)| = \infty \quad \textit{a.e. on} \quad [0, \infty).$$

2.3. Dealing with infinite intervals the main problem may be formulated in the following way. The functions $F(x)$ serving as counter-example generally have the form $\sum_{k=1}^{\infty} c_k\varphi_k(x)$. Here $\sum|c_k|<\infty$ and $\varphi_k(x)$ are uniformly bounded polynomials, say, in $[-1, 1]$.

On the other hand, if $x\to\infty$, even the continuity of $F(x)$ is questionable. Of course, one can "cut" somehow $\varphi_k(x)$ but then we could not use the very useful property $L_N(\varphi_k, x)\equiv\varphi_k(x)$ if $N>\deg\varphi_k$.

In this paper we were able to overcome these problems and to generalize Theorem 1.1 to infinite intervals.

3. Results

3.1. First let $C(\boldsymbol{R}):=\{f(x); f(x)$ is continuous for any $x\in\boldsymbol{R}\}$. Our main statement is the following.

Theorem 3.1. *For any interpolatory matrix $X\subset\boldsymbol{R}$ one can find a function $F(x)\in C(\boldsymbol{R})$ such that*

$$\overline{\lim_{n\to\infty}}\,|L_n(F,x)|=\infty \quad \textit{for almost all} \quad x\in\boldsymbol{R}. \tag{3.1}$$

3.2. To prove this, first we use a special lemma which may deserve a separate formulation.

Lemma 3.2. *Let $\varepsilon>0$ be given. Then for any interpolatory matrix $X\subset\boldsymbol{R}$ and for any fixed $[a,b]\subset\boldsymbol{R}$ one can find a function $F_\varepsilon(x)=F(a,b,X,\varepsilon,x)\in C(\boldsymbol{R})$ such that*

$$\overline{\lim_{n\to\infty}}\,|L_n(F_\varepsilon,x)|=\infty \quad \textit{on} \quad S_\varepsilon \tag{3.2}$$

where $S_\varepsilon\subset[a,b]$ and $|S_\varepsilon|\geqq b-a-\varepsilon$.

3.3. Let us remark that the set of divergence in (3.1) *is dense and of second category on $\boldsymbol{R}$.* [1)]

4. Proof of Lemma 3.2

We shall use many ideas of [3], even our notations will be similar, too.

First a simple remark. If

$$\lambda_n(x):=\sum_{k=1}^{n}|l_k(x)|, \quad \lambda_n:=\max_{x\in[x_{nn},x_{1n}]}\lambda_n(x), \quad n=1,2,\ldots, \tag{4.1}$$

then

$$\lambda_n>\frac{\ln n}{8\sqrt{\pi}}, \quad n=1,2,\ldots. \tag{4.2}$$

---

1) Indeed, as Orlicz [6] proved:

If A is a topological space of second category, $D\subset A$ is a dense subset and $\{h_n\}$, $n=1,2,\ldots$, are continuous functions on A with

$$\overline{\lim_{n\to\infty}}\, h_n(x)=\infty \quad \textit{whenever} \quad x\in D$$

then the set S, $D\subset S\subset A$,

$$S:=\{x\in A;\ \overline{\lim_{n\to\infty}}\, h_n(x)=\infty\}$$

is dense and of second category in A.

Now let $P\subset\boldsymbol{R}$ be the set for which (3.1) holds. By $|\boldsymbol{R}\setminus P|=0$, P is dense in $\boldsymbol{R}$. If $h_n(x)=|L_n(f,x)|$ and $D=P$, we obtain statement 3.3.

This relation was proved by S. Bernstein and G. Faber (see e.g. NATANSON [5], Volume III, Ch. II, § 1). Actually, they supposed that the matrix X is contained in a *finite* interval but their proof does not use this fact.

4.1. First let us suppose that there exists a finite interval $[A, B]$ which contains X. It is enough to prove the lemma for $[a, b]\supseteq[A, B]$. Indeed, applying Theorem 1.1, we obtain an $F(x)$ which is continuous on $[a, b]$ and for which $\varlimsup_{n\to\infty} |L_n(F, x)|=\infty$ a.e. in $[a, b]$. If

$$F_\varepsilon(x):=\begin{cases} F(a) & \text{if} \quad x\leqq a \\ F(x) & \text{if} \quad a\leqq x\leqq b \\ F(b) & \text{if} \quad x\geqq b \end{cases}, \tag{4.3}$$

we obtain the lemma. (Remark that now $|S_\varepsilon|=b-a$, moreover F_ε and S_ε do not depend on ε.)

4.2. Now let us suppose that $\varlimsup_{n\to\infty} [\max(|x_{1n}|, |x_{nn}|)]=\infty$. For sake of simplicity we may assume that

$$\lim_{n\to\infty} x_{1n}=\infty \quad \text{and} \quad x_{11}<x_{12}<\dots \tag{4.4}$$

(otherwise we can select a subsequence having this property).

For an arbitrary sequence $A_n>0$, $A_n\nearrow\infty$, we can construct F_ε as follows. First, we cover all the points of X with an interval-system I_ε of total measure $\leqq\varepsilon$. We suppose that x_{kn}, say, is the middle point of its covering interval. Then obviously

$$\delta_n(\varepsilon):=\min_{x\in R\setminus I_\varepsilon} |l_{1n}(x)|>0, \quad n=1, 2, \dots. \tag{4.5}$$

I.e., if

$$F_\varepsilon(x):=\frac{A_1}{\delta_1(\varepsilon)} \quad \text{when} \quad x\in(-\infty, x_{11}], \tag{4.6}$$

then

$$|L_1(F_\varepsilon, x)|\geqq A_1 \quad \text{when} \quad x\in R\setminus I_\varepsilon. \tag{4.7}$$

4.3. Generally, if we defined F_ε on $(-\infty, x_{1,n-1}]$ $(n\geqq2)$, then we define F_ε on $(-\infty, x_{1n}]$ as follows. Let

$$F_\varepsilon(x):=F_\varepsilon(x_{1,n-1}) \quad \text{if} \quad x\in[x_{1,n-1}, \alpha_n] \tag{4.8}$$

where $\alpha_n:=\max(x_{2n}, x_{1,n-1})$, $n=2, 3, \dots$. Considering (4.4), $\alpha_n<x_{1n}$. So we can define the function $F_\varepsilon(x)$ at x_{1n}. First we remark that

$$\Delta_n(\varepsilon):=\max_{x\in R\setminus I_\varepsilon}\left|\frac{\sum_{k=2}^{n} F_\varepsilon(x_{kn})\, l_{kn}(x)}{l_{1n}(x)}\right|<\infty, \tag{4.9}$$

because $\deg l_{1n}\geqq\deg\left(\sum_{k=1}^{n}\dots\right)$, moreover, we excluded a certain neighbourhood of the poles of the rational function appearing in (4.9). Now let

$$F_\varepsilon(x_{1n}) := \frac{A_n}{\delta_n(\varepsilon)} + \Delta_n(\varepsilon). \tag{4.10}$$

Finally, let F_ε be linear if $x \in [\alpha_n, x_{1n}]$. By these we completed the definition of F_ε on $(-\infty, x_{1n}]$. Using again (4.4), we can say that F_ε is defined for any $x \in \boldsymbol{R}$, moreover $F_\varepsilon \in C(-\infty, \infty)$. Then for any $x \in \boldsymbol{R} \setminus I_\varepsilon$ we have

$$|L_n(F_\varepsilon, x)| = \left|\sum_{k=1}^n F_\varepsilon(x_{kn})\, l_{kn}(x)\right| = \left|l_{1n}(x)\, F_\varepsilon(x_{1n}) + \sum_{k=2}^n \dots\right| \geqq \tag{4.11}$$
$$\geqq |l_{1n}(x)|\,[|F_\varepsilon(x_{1n})| - \Delta_n(\varepsilon)] \geqq \delta_n(\varepsilon)\left[\frac{A_n}{\delta_n(\varepsilon)} + \Delta_n(\varepsilon) - \Delta_n(\varepsilon)\right] = A_n$$

(see (4.5), (4.9) and (4.10)), i.e. we proved

$$\overline{\lim_{n\to\infty}}\, |L_n(F_\varepsilon, x)| = \infty \quad \text{on} \quad \mathbf{R} \setminus I_\varepsilon$$

(see (4.11)) which is more than stated in Lemma 3.2.

5. A lemma

5.1. Later we need the next

Lemma 5.1. *If* $g_1, g_2, \ldots \in C(\boldsymbol{R})$ *and*

$$\overline{\lim_{n\to\infty}}\, g_n(x) = \infty \quad (x \in B,\ |B| < \infty), \tag{5.1}$$

then for arbitrary fixed A, ε *and* M $(A, \varepsilon > 0,\ M \geqq 1,$ *integer) there exist a set* $H \subseteqq B$ *and an index* $N \geqq M$ *such that* $|H| \leqq \varepsilon$ *and if* $x \in B \setminus H$ *then for a certain* $u(x)$ *we have*

$$g_{u(x)}(x) \geqq A \quad \textit{where} \quad M \leqq u(x) \leqq N. \tag{5.2}$$

(The proof of Lemma 5.1. is in [3, 4.4.13.].)

6. Proof of Theorem 3.1

6.1. For a fixed k, $k = 1, 2, \ldots$, applying Lemma 3.2 with the cast $\varepsilon := 2^{-k-1}$ and $[a, b] := I_k$ (I_k will be determined later), we get $F_k \in C(\boldsymbol{R})$, $\max_{x \in I_k} |F_k(x)| \leqq 1$, for which

$$\overline{\lim_{n\to\infty}}\, |L_n(F_k, x)| = \infty \quad \text{if} \quad x \in T_k,$$

where $T_k \subset I_k$ and $|T_k| \geqq |I_k| - 2^{-k-1}$ ($F_k(x)$ and T_k correspond to $F_\varepsilon(x)[\max_{a \leqq x \leqq b} |F_\varepsilon(x)|]^{-1}$ and S_ε, respectively).

Now let $k=1$ and $I_1:=[-1, 1]$. Using Lemma 5.1 with $g_n(x):=|L_n(F_1, x)|$, $A:=A_1:=2$, $\varepsilon:=\varepsilon_1:=2^{-2}$, $M:=m_1:=1$ and $B:=T_1$, we get the set $S_1\subset T_1$ with $|S_1|\geqq 2-2^{-1}$ and the index n_1 such that

(6.1) $$|L_{u(x)}(F_1, x)| \geqq A_1 > 1^3\mu_0 \quad \text{if} \quad x\in S_1,$$

moreover $m_1\leqq u(x)\leqq n_1$. Here $\mu_0:=1$ (S_1, n_1 and $u(x)$ correspond to $B\setminus H, N$ and $u(x)$, respectively).

Now let $a_1:=\min(-1, \min_{1\leqq n\leqq n_1} x_{nn})$, $b_1:=\max(1, \max_{1\leqq n\leqq n_1} x_{1n})$, and let $I_2:=$ $:=[a_1-1, b_1+1]$. Clearly $[-2, 2]\subseteqq I_2$. We can construct a polynomial $\varphi_1(x)$ for which $\varphi_1(x_{in}):=F_1(x_{in})$ $(1\leqq i\leqq n,\ 1\leqq n\leqq n_1)$ and $|\varphi_1(x)-F_1(x)|\leqq 1$ if $x\in I_2$ (see [10, Part 3, Chapter 2, § 3, Lemma 3]). Then, by $|F_1(x)|\leqq 1$ $(x\in I_1)$,

(6.2) $$|\varphi_1(x)| \leqq 2 \quad \text{if} \quad x\in I_1.$$

By $\varphi_1(x_{in})=F_1(x_{in})$, (6.1) holds for φ_1, too. Finally, let $\deg\varphi_1\leqq N_1$.

Now by induction, for any fixed $k\geqq 2$, using similar arguments and the notations

(6.3) $$\begin{cases} \mu_{k-1} := \max\limits_{1\leqq i\leqq n_{k-1}} \max\limits_{x\in I_k} \lambda_i(x), \\ A_k := k^3\mu_{k-1}+1, \quad \varepsilon_k := 2^{-k-1}, \quad m_k := N_{k-1}+1, \end{cases}$$

we get the set $S_k\subset I_k$, $|S_k|\geqq 2k-2^{-k}$, the index n_k and a polynomial $\varphi_k(x)$ of degree $\leqq N_k$ for which

(6.4) $$|\varphi_k(x)| \leqq 2 \quad \text{if} \quad x\in I_k,$$

moreover for a certain $u(x)$, $m_k\leqq u(x)\leqq n_k$,

(6.5) $$|L_{u(x)}(\varphi_k, x)| \geqq A_k > k^3\mu_{k-1} \quad \text{if} \quad x\in S_k.$$

Let $a_k:=\min(-k, \min_{1\leqq n\leqq n_k} x_{nn})$, $b_k:=\max(k, \max_{1\leqq n\leqq n_k} x_{1n})$ and let $I_{k+1}:=[a_k-1, b_k+1]$. Clearly $[-(k+1), k+1]\subseteqq I_{k+1}$ and $I_k\subseteqq I_{k+1}$ $(k=2, 3, \ldots)$.

By construction we may suppose that

(6.6) $$m_1 < n_1 < N_1 < m_2 < n_2 < N_2 < \ldots,$$

while by (6.3)

(6.7) $$\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \ldots.$$

6.2. Now let

(6.8) $$F(x) := \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{k^2\mu_{k-1}}$$

and

(6.9) $$S := \bigcap_{k=1}^{\infty} \Big(\bigcup_{i=k}^{\infty} S_i\Big).$$

First we state that $F\in C(\mathbf{R})$. To this end, let $x\in\mathbf{R}$ be arbitrary. By $[-s, s]\subseteqq I_s$, one can find a fixed s for which $x\in I_s$. If we prove that F is bounded on (the closed) I_s, then, by the continuity of φ_k, F will be continuous on I_s, especially at x.

Let $F(x)=(\sum_{k=1}^{s-1}+\sum_{k=s}^{\infty})\,c_k\varphi_k(x)$, where $c_k:=(k^2\mu_{k-1})^{-1}$. Here the first sum can be estimated by $\sum_{k=1}^{s-1} c_k \max_{x\in I_s}|\varphi_k(x)|:=\alpha_s<\infty$, the second one, using that $|\varphi_k(x)|\leqq 2$ because $x\in I_s\subseteqq I_k$ (see (6.4)), by $2\sum_{k=s}^{\infty}c_k<2\sum_{k=1}^{\infty}k^{-2}:=2E_2<\infty$ which was to be proven.

Now we prove that

$$\overline{\lim_{n\to\infty}}\,|L_n(F,x)|=\infty \quad \text{if} \quad x\in S. \tag{6.10}$$

Indeed, for any fixed $x\in S$ there exist infinitely many S_j, $j=r_1, r_2, \ldots$, for which $x\in S_j\subset I_j$. For each fixed j there exists an index $u_j=u(x)$ such that by (6.5)

$$|L_{u_j}(\varphi_j, x)|\geqq A_j,\quad x\in S_j,\ m_j\leqq u_j\leqq n_j. \tag{6.11}$$

For a fixed j $(j=r_1, r_2, \ldots)$ we write

$$L_{u_j}(F,x)=\Big(\sum_{k=1}^{j-1}+\sum_{k=j}+\sum_{k=j+1}^{\infty}\Big)c_kL_{u_j}(\varphi_k,x):=J_1+J_2+J_3.$$

In the first sum $L_{u_j}(\varphi_k,x)\equiv\varphi_k(x)$ because $\deg\varphi_k\leqq N_k\leqq N_{j-1}<m_j\leqq u_j$. I.e., $J_1=(\sum_{k=1}^{r_1-1}+\sum_{k=r_1}^{j-1})c_k\varphi_k:=J_4+J_5$. Here $|J_4|\leqq\sum_{k=1}^{r_1-1}c_k\max_{x\in I_{r_1}}|\varphi_k(x)|=\alpha_{r_1}$ further $|J_5|\leqq$ $\leqq 2\sum_{k=r_1}^{j-1}c_k<2E_2$ by $|\varphi_k(x)|\leqq 2$ because $x\in I_{r_1}\subseteqq I_k$ (see (6.4)). Thus $|J_1|\leqq\alpha_{r_1}+2E_2$.

For J_2, by (6.11) and (6.5) we get that $|J_2|\geqq A_jj^{-2}\mu_{j-1}^{-1}>j$.

Finally we estimate $J_3=\sum_{k=j+1}^{\infty}c_k\big(\sum_{i=1}^{u_j}\varphi_k(x_{iu_j})l_{iu_j}(x)\big)$. Here all the values $|\varphi_k(x_{iu_j})|\leqq 2$ by $x_{iu_j}\in[a_j,b_j]\subset I_{j+1}\subseteqq I_k$, $k\geqq j+1$. I.e., $|J_3|\leqq 2\sum_{k=j+1}^{\infty}c_k\lambda_{u_j}(x)\leqq$ $\leqq 2\sum_{k=j+1}^{\infty}c_k\mu_j<2E_2$ (see (6.3) and (6.7)).

Summarizing, we get that

$$|L_{u_j}(F,x)|\geqq|J_2|-|J_1|-|J_3|\geqq j-\alpha_{r_1}-4E_2,\quad j=r_1,r_2,\ldots,$$

which is $\geqq j/2$ if j is big enough.

6.3. To complete the proof of the theorem we show that

$$|\boldsymbol{R}\setminus S|=0. \tag{6.12}$$

Indeed, by (6.9), $\boldsymbol{R}\setminus S=\bigcup_{k=1}^{\infty}(\boldsymbol{R}\setminus Q_k)=\bigcup_{k=1}^{\infty}P_k$ if $Q_k:=\bigcup_{i=k}^{\infty}S_i$ and $P_k:=\boldsymbol{R}\setminus Q_k$. Obviously $Q_1\supset Q_2\supset\ldots$ which means $P_1\subset P_2\subset\ldots$. Here $|P_k|\leqq 2^{-k+1}$. Indeed, considering that S_i overlaps $[-i, i]$ except a set H_i of measure $\leqq 2^{-i}$, we get that $Q_k = \bigcup_{i=k}^{\infty} S_i$ overlaps R except the set $P_k \subseteqq \bigcup_{i=k}^{\infty} H_i$. Hence $|P_k| \leqq \sum_{i=k}^{\infty} |H_i| \leqq \leqq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}$. Using $P_1 \subset P_2 \subset \ldots$, we get that $|R \setminus S| = \left|\bigcup_{k=1}^{\infty} P_k\right| = \lim_{k\to\infty} |P_k| = 0$ which was to be proven.

References

[1] G. Faber, Über die interpolatorische Darstellung steiger Funktionen, *Jahresber. der Deutschen Mat. Ver.*, **23** (1914), 190—210.

[2] P. Erdős, Problems and results on the theory of interpolation. I, *Acta Math. Acad. Sci. Hungar.*, **9** (1958), 381—388.

[3] P. Erdős and P. Vértesi, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, *Acta Math. Acad. Sci. Hungar.*, **36** (1980), 71—89; **38** (1981), 263.

[4] P. Vértesi, On the almost everywhere divergence of Lagrange interpolation, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 367—377.

[5] I. P. Natanson, *Constructive Theory of Functions*, GITTL (Moscow—Leningrad, 1949). (Russian)

[6] W. Orlicz, Über Folgen linearer Operationer die von einem Parameter abhängigen, *Studia Math.*, **5** (1934), 160—170.

[7] L. P. Povchun, On the almost everywhere divergence of Lagrange interpolation on the Laguerre nodes, *Izv. Vysš. Učebn. Zaved. Matematika*, **7** (1976), 114—116. (Russian)

[8] V. A. Kutepov, Lagrange interpolation on unbounded sets, International Conference on the Theory of Approximation of Functions, USSR, KIEV, May 30—June 6, 1983. Abstracts, p. 109. (Russian)

[9] L. P. Povchun, Divergence of interpolation processes at a fixed point, *Izv. Vysš. Učebn. Zaved. Matematika*, **3** (1980), 56—60.

MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15.
1053 BUDAPEST, HUNGARY