On the generalized absolute Cesàro summability of double orthogonal series

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Dedicated to Professor Károly Tandori on his 60th birthday

Introduction. As usual we denote by σ_n^α the n-th Cesàro means of order α of a single numerical series $\sum a_n$. The following definition is due to FLETT [1]: A series $\sum a_n$ is said to be summable $|C, \alpha, \gamma|_\varkappa$ $(\alpha > -1, \gamma \geqq 0, \varkappa \geqq 1)$ if the series $\sum n^{\varkappa\gamma+\varkappa-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^\varkappa$ converges.

Very recently MÓRICZ [3] introduced a definition of $|C, (\alpha, \beta)|_\varkappa$ summability for a double series

(1) $$\sum_{i,k} a_{ik},$$

namely, series (1) is summable $|C, (\alpha, \beta)|_\varkappa$ $(\alpha > -1, \beta > -1, \varkappa \geqq 1)$ if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{\varkappa-1} |\Delta_{mn}^{\alpha\beta}|^\varkappa < \infty,$$

where

(2) $$\Delta_{mn}^{\alpha\beta} = \sigma_{mn}^{\alpha\beta} - \sigma_{m-1,n}^{\alpha\beta} - \sigma_{m,n-1}^{\alpha\beta} + \sigma_{m-1,n-1}^{\alpha\beta} \quad (m, n = 1, 2, \ldots)$$

and

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^\alpha} \frac{1}{A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^\alpha A_{n-k}^\beta a_{ik} \quad (m, n = 0, 1, \ldots)$$

is the rectangular $(C, (\alpha, \beta))$ mean of double series (1) with the binomial coefficients

$$A_0^\alpha = 1, \quad A_n^\alpha = \frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{n!}, \quad (n = 1, 2, \ldots).$$

Considering the rectangular partial sum of series (1)

$$s_{mn} = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik} \quad (m, n = 0, 1, \ldots)$$

Received October 12, 1984.

we have the identity

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{i=0}^{m}\sum_{k=0}^{m} A_{m-i}^{\alpha-1}A_{n-k}^{\beta-1}s_{ik} \quad (m, n = 0, 1 \ldots)$$

which is the rectangular $(c, (\alpha, \beta))$ mean of the sequence $\{s_{ik}\}$. Denote by $\tau_{mn}^{\alpha\beta}$ the rectangular $(c, (\alpha, \beta))$ mean of the sequence $\{ika_{ik}\}$, that is,

$$\tau_{mn}^{\alpha\beta} = \frac{1}{A_m^\alpha}\frac{1}{A_n^\beta}\sum_{i=1}^{m}\sum_{k=1}^{n} A_{m-1}^{\alpha-1}A_{n-k}^{\beta-1}ika_{ik} \quad (m, n = 1, 2, \ldots). \tag{3}$$

Now we introduce the definition of the $|C, (\alpha, \beta), (\mu, \nu)|_\varkappa$ summability as follows: Double series (1) is said to summable $|c, (\alpha, \beta), (\mu, \nu)|_\varkappa$ $(\alpha > -1, \beta > -1, 0 \leqq \mu < 1, 0 \leqq \nu < 1, \varkappa \geqq 1)$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}|\tau_{mn}^{\alpha\beta}|^\varkappa < \infty.$$

The identity

$$\tau_{mn}^{\alpha\beta} = mn(\sigma_{mn}^{\alpha\beta} - \sigma_{m-1,n}^{\alpha\beta} - \sigma_{m,n-1}^{\alpha\beta} + \sigma_{m-1,n-1}^{\alpha\beta}) \quad (m, n = 1, 2, \ldots) \tag{4}$$

shows that our definition is an extension of the Flett's definition, and — by (2) and (4) — it is a generalisation of the Móricz's definition. Here we mention a very useful identity

$$\tau_{mn}^{\alpha\beta} = \alpha\beta(\sigma_{mn}^{\alpha\beta} - \sigma_{mn}^{\alpha,\beta-1} - \sigma_{mn}^{\alpha-1,\beta} + \sigma_{mn}^{\alpha-1,\beta-1}) \quad (m, n = 1, 2, \ldots),$$

too.

Our first result extends a theorem of Flett ([1], Theorem 1) for summability $|C, (\alpha, \beta), (\mu, \nu)|_\varkappa$.

Theorem 1. *If* $\varrho \geqq \varkappa > 1$, $\mu \geqq 0$, $\nu \geqq 0$, $\alpha > \mu - 1$, $\beta > \nu - 1$ *and* $\min(\delta, \bar{\delta}) > \varkappa^{-1} - \varrho^{-1}$ *then the inequality*

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varrho\mu-1}n^{\varrho\nu-1}|\tau_{mn}^{\alpha+\delta,\beta+\bar{\delta}}|^\varrho\Big\}^{1/\varrho} \leqq K\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}|T_{mn}^{\alpha\beta}|^\varkappa\Big\}^{1/\varkappa} \tag{5}$$

*holds,**) *so the summability* $|C, (\alpha, \beta), (\mu, \nu)|_\varkappa$ *implies the summability*

$$|C, (\alpha+\delta, \beta+\bar{\delta}), (\mu, \nu)|_\varrho.$$

In 1960 Tandori [5] published the very interesting

Theorem A. *The condition*

$$\sum_{m=0}^{\infty}\Big(\sum_{n=2^m+1}^{2^{m+1}} a_n^2\Big)^{1/2} < \infty$$

*) K will denote a positive constants not necessarily the same at each occurrence.

is necessary and sufficient that for any orthonormal system $\{\varphi_n(x)\}$ *on the interval* $(0, 1)$, *the series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

be summable $|C, 1, 0|_1$ *almost everywhere in* $(0, 1)$.

This result has a lot of generalisations and extensions. For example, in the first step, it was extended for $|C, \alpha, 0|_1$ $(\alpha > -1)$ summability by LEINDLER [2], and using the Leindler's method, for $|C, \alpha, \gamma|_\varkappa$ $(\alpha > -1, 0 \leqq \gamma < 1, \varkappa \geqq 1)$ summability by the author [4].

Denote by I the two-dimensional unit interval $(0, 1)\times(0, 1)$ and let $\{\varphi_{ik}(x, y)\}$ be an orthonormal system on I. Very recently MÓRICZ [3] proved the following theorems.

Theorem B. *If* $\alpha > 1/2$, $\beta > 1/2$, $1 \leqq \varkappa \leqq 2$ *and*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\sum_{i=2^m}^{2^{m+1}-1} \sum_{k=2^n}^{2^{n+1}-1} a_{ik}^2 \Big)^{\varkappa/2} < \infty \tag{6}$$

then the double orthogonal series

$$\sum_{i,k} a_{ik} \varphi_{ik}(x, y) \tag{7}$$

is $|C, (\alpha, \beta), (0, 0)|_\varkappa$ *summable almost everywhere on* I.

Theorem C. *If* $\alpha > 1/2, \beta > 1/2$ *and in the case* $\varkappa = 1$ *condition* (6) *is not satisfied, then the two-dimensional Rademacher series*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} r_i(x) r_k(y) \quad (r_i(x) = \operatorname{sgn} \sin 2^i \pi x) \tag{8}$$

is not $|C, (\alpha, \beta), (0, 0)|_1$ *summable almost everywhere on* I.

Generalizing these results we have two theorems.

Theorem 2. *Let* $\alpha > 1/2$, $\beta > 1/2$, $0 \leqq \mu < 1$, $0 \leqq \nu < 1$, $1 \leqq \varkappa \leqq 2$. *If*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\sum_{i=2^m}^{2^{m+1}-1} \sum_{k=2^n}^{2^{n+1}-1} i^{2\mu} k^{2\nu} a_{ik}^2 \Big)^{\varkappa/2} < \infty, \tag{9}$$

then the inequality

(10)

$$\iint_I \Big(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu-1} n^{\varkappa\nu-1} |\tau_{mn}^{\alpha\beta}(x, y)|^\varkappa \Big) dx\, dy \leqq K \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Big(\sum_{i=2^m}^{2^{m+1}-1} \sum_{k=2^n}^{2^{n+1}-1} i^{2\mu} k^{2\nu} a_{ik}^2 \Big)^{\varkappa/2}$$

*holds**) and double orthogonal series* (7) *is* $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ *summable almost everywhere on* I.

Theorem 3. *If series* (8) *is summable* $|C,(\alpha,\beta),(\mu,\nu)|_\varkappa$ $(\alpha>1/2, \beta>1/2, 0\leqq\mu<1, 0\leqq\nu<1, 1\leqq\varkappa\leqq2)$ *on a subset* $E\subset I$ *with positive measure then* (9) *holds.*

Proof of Theorem 1. The proof is based on the identity

$$\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}=\frac{1}{A_m^{\alpha+\delta}}\frac{1}{A_n^{\beta+\bar\delta}}\sum_{i=1}^m\sum_{k=1}^n A_{m-i}^{\delta-1}A_{n-k}^{\bar\delta-1}A_i^\alpha A_k^\beta\tau_{ik}^{\alpha\beta} \tag{11}$$

which can be proved by definition (3) and the elementary identity of binomial coefficients

$$\sum_{n=0}^m A_{m-n}^{\delta-1}A_n^{\alpha-1}=A_m^{\alpha+\delta-1}.$$

Let S denote the expression on the right of (5), λ and ω be numbers to be choosen later, but certainly such that $\lambda>(\varkappa')^{-1}$ $(\varkappa^{-1}+(\varkappa')^{-1}=1)$, $0<\omega<1$, $\min(\delta,\bar\delta)>1>1-(\varkappa'(1-\omega))^{-1}$. Using (11) and applying Hölder's inequality with indices $\varrho,\varkappa'$ and $\varkappa\varrho/\varrho-\varkappa$ (if $\varrho=\varkappa$ then with indices ϱ and $\varkappa'$ only) we have

$$|\tau_{mn}^{\alpha+\delta,\beta+\bar\delta}|\leqq Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\sum_{i=1}^m\sum_{k=1}^n(m-i+1)^{\delta-1}(n-k+1)^{\bar\delta-1}i^\alpha k^\beta|\tau_{ik}^{\alpha\beta}|=$$

$$=Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\sum_{i=1}^m\sum_{k=1}^n\{(m-i+1)^{(\delta-1)\omega}(n-k+1)^{(\bar\delta-1)\omega}\times$$

$$\times i^{\alpha+\lambda-(\varkappa\mu-1)(\varrho-\varkappa)/\varkappa\varrho}k^{\beta+\lambda-(\varkappa\nu-1)(\varrho-\varkappa)/\varkappa\varrho}|\tau_{ik}^{\alpha\beta}|^{\varkappa/\varrho}\}\times$$

$$\times\{(m-i+1)^{(\delta-1)(1-\omega)}(n-k+1)^{(\bar\delta-1)(1-\omega)}i^{-\lambda}k^{-\lambda}\}\{i^{\varkappa\mu-1}k^{\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{(\varrho-\varkappa)/\varkappa\varrho}\leqq$$

$$\leqq Km^{-\alpha-\delta}n^{-\beta-\bar\delta}\{\sum_{i=1}^m\sum_{k=1}^n(m-i+1)^{\varrho(\delta-1)\omega}(n-k+1)^{\varrho(\bar\delta-1)\omega}\times \tag{12}$$

$$\times i^{\varrho\alpha+\varrho\lambda-(\varkappa\mu-1)(\varrho-\varkappa)/\varkappa}k^{\varrho\beta+\varrho\lambda-(\varkappa\nu-1)(\varrho-\varkappa)/\varkappa}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{1/\varrho}\times$$

$$\times\{\sum_{i=1}^m\sum_{k=1}^n(m-i+1)^{(\delta-1)(1-\omega)\varkappa'}(n-k+1)^{(\bar\delta-1)(1-\omega)\varkappa'}i^{-\lambda\varkappa'}k^{-\lambda\varkappa'}\}^{1/\varkappa'}\times$$

$$\times\{\sum_{i=1}^m\sum_{k=1}^n i^{\varkappa\mu-1}k^{\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^\varkappa\}^{(\varrho-\varkappa)/\varkappa\varrho}.$$

Having

$$\sum_{i=1}^m\sum_{k=1}^n(m-i+1)^{(\delta-1)(1-\omega)\varkappa'}(n-k+1)^{(\bar\delta-1)(1-\omega)\varkappa'}i^{-\lambda\varkappa'}k^{-\lambda\varkappa'}\leqq$$

$$\leqq Km^{(\delta-1)(1-\omega)\varkappa'+1-\lambda\varkappa'}n^{(\bar\delta-1)(1-\omega)\varkappa'+1-\lambda\varkappa'}$$

**) $\tau_{mn}^{\alpha\beta}(x,y)$ is the rectangular $(C,(\alpha,\beta))$ mean of the sequence $\{ika_{ik}\varphi_{ik}(x,y)\}$.

and consideiing that the last factor in (12) is less than $S^{1-\varkappa/\varrho}$ we get

$$|\tau_{mn}^{\alpha+\delta,\beta+\bar{\delta}}| \leqq Km^{-\alpha-1-\delta\omega+\omega-\lambda+(\varkappa')^{-1}} n^{-\beta-1-\bar{\delta}\omega+\omega-\lambda+(\varkappa')^{-1}} S^{1-\varkappa/\varrho} \times$$

$$\times \Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{c}(n-k+1)^{\bar{c}} i^{a+\varkappa\mu-1} k^{b+\varkappa\nu-1} |\tau_{ik}^{\alpha\beta}|^{\varkappa}\Big\}^{1/\varrho},$$

where $a=\varrho(\alpha+\lambda+\varkappa^{-1}-\mu)$, $b=\varrho(\beta+\lambda+\varkappa^{-1}-\nu)$, $c=\varrho(\delta-1)\omega$ and $\bar{c}=\varrho(\bar{\delta}-1)\omega$. Since $\varrho\mu-1-\varrho(-\alpha-1-\delta\omega+\omega-\lambda+(\varkappa')^{-1})=-a-c-1$, and similarly $\varrho\nu-1-\varrho(-\beta-1-\bar{\delta}\omega+\omega-\lambda+(\varkappa')^{-1})=-b-\bar{c}-1$ for any $M\geqq 1$ and $N\leqq 1$ we obtain

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\varrho\mu-1} n^{\varrho\nu-1} |\tau_{mn}^{\alpha+\delta,\beta+\bar{\delta}}|^{\varrho} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{m=1}^{M}\sum_{n=1}^{N} m^{-a-c-1} n^{-b-\bar{c}-1} \sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{c}(n-k+1)^{\bar{c}} i^{a+\varkappa\mu-1} k^{b+\varkappa\nu-1} |\tau_{ik}^{\alpha\beta}|^{\varkappa} =$$

$$= KS^{\varrho-\varkappa}\sum_{m=1}^{M} m^{-a-c-1}\sum_{i=1}^{m}(m-i+1)^{c} i^{a+\varkappa\mu-1}\sum_{k=1}^{N} k^{b+\varkappa\nu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa}\sum_{n=k}^{N} n^{-b-\bar{c}-1}(n-k+1)^{\bar{c}} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{k=1}^{N} k^{\varkappa\nu-1}\sum_{m=1}^{M} m^{-a-c-1}\sum_{i=1}^{m}(m-i+1)^{c} i^{a+\varkappa\mu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa}$$

provided that $\bar{c}>-1$ and $b>0$, which is possible by choosing $\omega=\varkappa'/\varkappa'+\varrho$ and $(\varkappa')^{-1}-(1+\beta-\nu)<\lambda<(\varkappa')^{-1}$. With $\omega=\varkappa'/\varkappa'+\varrho$ and $(\varkappa')^{-1}-(1+\alpha-\mu)<\lambda<(\varkappa')^{-1}$ the inequalities $c>-1$ and $a>0$ are also fulfilled, so using the above method, we get

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\varrho\mu-1} n^{\varrho\nu-1} |\tau_{mn}^{\alpha+\delta,\beta+\bar{\delta}}|^{\varrho} \leqq$$

$$\leqq KS^{\varrho-\varkappa}\sum_{k=1}^{N} k^{\varkappa\nu-1}\sum_{i=1}^{M} i^{\varkappa\mu-1}|\tau_{ik}^{\alpha\beta}|^{\varkappa} \leqq KS^{\varrho}$$

and this, assuming that M and N tend to infinity, proves inequality (5), moreover means the $|C,(\alpha+\delta,\beta+\bar{\delta}),(\mu,\nu)|_{\varrho}$ summability of double series (1).

Proof of Theorem 2. Applying Hölder's inequality and considering (3) we have the estimations:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^{\varkappa}\,dx\,dy \leqq$$

$$\leqq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\Big(\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^2\,dx\,dy\Big)^{\varkappa/2} =$$

$$= K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1} m^{\varkappa\mu-1}n^{\varkappa\nu-1}\Big(\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^2\,dx\,dy\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=2^p}^{2^{p+1}-1} m^{\varkappa\mu-1}2^{q\varkappa(\nu-1/2)}\Big(\sum_{n=2^q}^{2^{q+1}-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^2\,dx\,dy\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\Big(\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\iint_I |\tau_{mn}^{\alpha\beta}(x,y)|^2\,dx\,dy\Big)^{\varkappa/2} \equiv \sum\nolimits_1.$$

A routine calculation gives

$$\sum\nolimits_1 = K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\left(\sum_{m=2^p}^{2^{p+1}-1}\sum_{n=2^q}^{2^{q+1}-1}\sum_{i=1}^{m}\sum_{k=1}^{n}\left(\frac{A_{m-i}^{\alpha-1}A_{n-k}^{\beta-1}}{A_m^{\alpha}A_n^{\beta}}\right)^2 i^2k^2a_{ik}^2\right)^{\varkappa/2} =$$

$$= K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1/2)}2^{q\varkappa(\nu-1/2)}\left(\sum_{m=2^p}^{2^{p+1}-1}\sum_{i=1}^{m}\left(\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right)^2 i^2\sum_{k=1}^{2^{q+1}-1}k^2a_{ik}^2\sum_{n=\max(2^q,k)}^{2^{q+1}-1}\left(\frac{A_{n-k}^{\beta-1}}{A_n^{\beta}}\right)^2\right)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{q=0}^{\infty} 2^{q\varkappa(\nu-1)}\sum_{p=0}^{\infty} 2^{p\varkappa(\mu-1/2)}\left(\sum_{k=1}^{2^{q+1}-1}k^2\sum_{i=1}^{2^{p+1}-1}i^2a_{ik}^2\sum_{m=\max(2^p,i)}^{2^{p+1}-1}\left(\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right)^2\right)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{q=0}^{\infty} 2^{q\varkappa(\nu-1)}\sum_{p=0}^{\infty} 2^{p\varkappa(\mu-1)}\Big(\sum_{k=1}^{2^{q+1}-1}\sum_{i=1}^{2^{p+1}-1} i^2k^2a_{ik}^2\Big)^{\varkappa/2} \equiv \sum\nolimits_2.$$

To approach the form of our condition we go on as follows:

$$\sum\nolimits_2 = K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{p\varkappa(\mu-1)}2^{q\varkappa(\nu-1)}\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{q}\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^2k^2a_{ik}^2\Big)^{\varkappa/2} \leqq$$

$$\leqq K\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{\varrho\varkappa(\mu-1)}2^{q\varkappa(\nu-1)}\sum_{m=0}^{p}\sum_{n=0}^{q}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^2k^2a_{ik}^2\Big)^{\varkappa/2} =$$

$$= K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^2k^2a_{ik}^2\Big)^{\varkappa/2}\sum_{p=m}^{\infty}\sum_{q=n}^{\infty} 2^{q\varkappa(\nu-1)}2^{\varrho\varkappa(\nu-1)} \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2^{m\varkappa(\mu-1)}2^{n\varkappa(\nu-1)}\Big(\sum_{i=2^m}^{2^{m+1}-1}\sum_{k=2^n}^{2^{n+1}-1} i^2k^2a_{ik}^2\Big)^{\varkappa/2}).$$

Regarding condition (9) and inequality (10) we can apply the Levi's theorem, so our proof is completed.

Proof of Theorem 3. This proof is very similar to that of Theorem C, but certain modifications are required to obtain a wider range of parameters μ, ν and $\varkappa$. Since these are not quite obvious we give the proof here. We need a

Lemma (MÓRICZ [3], Lemma 2). *For every finite sum*

$$P(x, y) = \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik} r_i(x) r_k(y) \quad (M \geqq m \geqq 0, \quad N \geqq n \geqq 0)$$

and for any set $E \subset I$ of positive measure, there exist an integer n_0 and a constant K_0 such that if $\max(m, n) \geqq n_0$ *then*

$$\iint_E |P(x, y)|\, dx\, dy \geqq K_0 \Big(\sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2 \Big)^{1/2}.$$

To begin the proof of Theorem 3, without loss of generality, we may assume that $a_{ik}=0$ for $i, k = 0, 1, \ldots, n_0-1$. By Egorov's theorem there exist a constant K^* and a set $E^* \subset E$ of positive measure such that for every $(x, y) \in E^*$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} |\Delta_{mn}^{\alpha\beta}(x, y)|^{\varkappa} \leqq K^* \tag{13}$$

where $\Delta_{mn}^{\alpha\beta}(x, y)$ is the suitable difference (see (2)) in the case of series (8). Using Hölder's inequality we obtain

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|^{\varkappa}\, dx\, dy \geqq$$

$$\geqq K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{\varkappa\mu+\varkappa-1} n^{\varkappa\nu+\varkappa-1} \Big(\iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy \Big)^{\varkappa} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{m=2^{p-1}}^{2^{p+1}-1} m^{\varkappa\mu+\varkappa-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} n^{\varkappa\nu+\varkappa-1} \Big(\iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy \Big)^{\varkappa} \geqq \tag{14}$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{q\varkappa\nu} \sum_{m=2^{p-1}}^{2^{p+1}-1} m^{\varkappa\mu+\varkappa-1} \Big(\sum_{n=2^{q-1}}^{2^{q+1}-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy \Big)^{\varkappa} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} \iint_{E^*} |\Delta_{mn}^{\alpha\beta}(x, y)|\, dx\, dy \Big)^{\varkappa} \equiv \sum.$$

Setting

$$\delta_{pq}^{\alpha\beta}(x, y) = \sigma_{2^{p+1}-1,\, 2^{q+1}-1}^{\alpha\beta}(x, y) - \sigma_{2^{p-1}-1,\, 2^{q+1}-1}^{\alpha\beta}(x, y) -$$

$$- \sigma_{2^{p+1}-1,\, 2^{q-1}-1}^{\alpha\beta}(x, y) + \sigma_{2^{p-1}-1,\, 2^{q-1}-1}^{\alpha\beta}(x, y)$$

it is easy to see that

$$\delta_{pq}^{\alpha\beta}(x, y) = \sum_{m=2^{p-1}}^{2^{p+1}-1} \sum_{n=2^{q-1}}^{2^{q+1}-1} \Delta_{mn}^{\alpha\beta}(x, y). \tag{15}$$

Applying now Lemma and using the monotonicity of the binomial coefficients we may write

$$\iint_{E^*} |\delta_{pq}^{\alpha\beta}(x, y)|\, dx\, dy \geqq K_0 \left(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} \left(\frac{A_{2^{p+1}-1-i}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \right)^2 \left(\frac{A_{2^{q+1}-1-k}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \right)^2 a_{ik}^2 \right)^{1/2}.$$

Using this fact and (15) we continue estimation (14) as follows

$$\sum \geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \left(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} \left(\frac{A_{2^{p+1}-1-i}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \right)^2 \left(\frac{A_{2^{q+1}-1-k}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \right)^2 a_{ik}^2 \right)^{\varkappa/2}.$$

Considering that there exists a constant $K_{p,q,\alpha,\beta}$ such that

$$\frac{A_{2^{p+1}-1-i}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \frac{A_{2^{q+1}-1-k}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \geqq \frac{A_{2^p}^{\alpha}}{A_{2^{p+1}-1}^{\alpha}} \frac{A_{2^q}^{\beta}}{A_{2^{q+1}-1}^{\beta}} \geqq K_{p,q,\alpha,\beta} > 0$$

we have

$$\sum \geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{i=2^{p-1}}^{2^{p+1}-1} \sum_{k=2^{q-1}}^{2^{q+1}-1} a_{ik}^2 \Big)^{\varkappa/2} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} 2^{p\varkappa\mu} 2^{q\varkappa\nu} \Big(\sum_{i=2^p}^{2^{p+1}-1} \sum_{k=2^q}^{2^{q+1}-1} a_{ik}^2 \Big)^{\varkappa/2},$$

and by (13) and (14) inequality (9) holds.

References

[1] T. M. Flett, Some more theorems concerning the absolute summability of Fourier series and power series, *Proc. London Math. Soc.*, **8** (1958), 357—387.

[2] L. Leindler, Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **22** (1961), 243—268.

[3] F. Móricz, On the $|C, \alpha, \beta|$-summability of double orthogonal series, *Acta Sci. Math.*, **48** (1985), 315—328.

[4] I. Szalay, On generalized absolute Cesàro summability of orthogonal series, *Acta Sci. Math.*, **32** (1971), 51—57.

[5] K. Tandori, Über die orthogonalen Funktionen. IX. Absolute Summation, *Acta Sci. Math.*, **21** (1960), 292—299.

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