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On G-finite W^*-algebras*

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Dedicated to Professor Károly Tandori on his 60th birthday

Let M be a W^*-algebra and G a group of *-automorphisms of M. In [3] we have proved that if there exists a faithful G-invariant *normal* state φ on M, then for every $t \in M$, the w^*-closure of the convex hull of the orbit of t under G contains a unique G-invariant element t^G. (In fact, we have proved this result under the more general assumption that the family of G-invariant normal states on M is faithful, i.e., M is G-finite. If M is σ-finite, for example, if M is an operator algebra in a separable Hilbert space, then this assumption obviously implies the existence of a faithful G-invariant normal state on M). In the present paper we shall prove that the assumption of normalcy of φ is superfluous in this theorem in case G contains all inner automorphisms of M. In fact, we shall prove the stronger result that the mapping $t \to t^G$ is normal, i.e., M is G-finite [3]. Under additional hypotheses, we shall also prove that φ is itself a normal state.

At the end of the paper we shall make two comments on our paper [4].

Proposition. *Let M be an Abelian W^*-algebra and G a group of *-automorphisms of M. If there exists a faithful G-invariant (not necessarily normal) positive linear form φ on M, then M is G-finite. (For the notion of G-finiteness, cf.* [3].*)*

Proof. Let e be the least upper bound of the supports of all G-invariant normal positive linear forms on M. According to [3], we have to prove that $e=1$. Assume on the contrary that $e \neq 1$. Since e is a G-invariant projection in M, the restrictions of the elements of G to the W^*-algebra $M(1-e)=(1-e)M(1-e)$ form a group G_{1-e} of *-automorphisms of $M(1-e)$. By the definition of e, the only G_{1-e}-invariant normal positive linear form on $M(1-e)$ is the zero functional. Consequently, to prove the theorem, i.e., to obtain a contradiction to the assumption $e \neq 1$, it is suf-

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ficient to show that there is a nonzero G_{1-e}-invariant normal positive linear form on $M(1-e)$. Since the restriction φ_{1-e} of φ to $M(1-e)$ is a faithful G_{1-e}-invariant normal positive linear form on $M(1-e)$, we may assume that $e=0$ and thus $M=M(1-e)$, $G_{1-e}=G$ and $\varphi_{1-e}=\varphi$. In other words, we have to prove that under the hypotheses of the theorem, there exists a nonzero G-invariant normal positive linear form on M.

Let S denote the family of those projections p in M for which $t\to\varphi(tp)=\varphi(ptp)$ is a *normal* positive linear form on M. We are going to show that

(1) if $p, q\in S$, then $p\vee q\in S$

(2) $\sup S=1$.

To prove (1), let $p, q\in S$. Then by the commutativity of M, we have $p\vee q= =p+q-pq$ and thus the functional $t\to\varphi(t(p\vee q))=\varphi(tp)+\varphi(t(q-pq))$ is a normal positive linear form on M. (Because $p, q\in S$ and $q-pq\leqq q$.) This proves (1). To prove (2), we have to show that every nonzero projection p in M majorizes a nonzero projection belonging to S. This can be shown by using arguments of J. Dixmier [1], which originate from Lebesgue's work. Let p be a nonzero projection in M. Consider a *normal* positive linear form μ on M, such that $\mu(p)\geqq\varphi(p)$. Me are going to prove that there exists a nonzero projection q in M, such that $\varphi(r)\leqq\mu(r)$ for every projection r in M, such that $r\leqq q$. Then the spectral decomposition theorem will imply that $\varphi(t)\leqq\mu(t)$ for every $t\in Mq$, $t\geqq 0$. Since every positive linear form majorized by a *normal* positive linear form is normal [1], this will prove (2). Now assume on the contrary that every nonzero projection $q\leqq p$ in M majorizes a nonzero projection $r\in M$ such that $\varphi(r)>\mu(r)$. By Zorn's lemma, there exists a maximal family C of mutually orthogonal nonzero projections s in M such that $\varphi(s)>\mu(s)$ for $s\in C$. By the indirect hypothesis, $\sum_{s\in C}s=p$. Then $\varphi(p)=\varphi(\sum_{s\in C}s)\geqq\sum_{s\in C}\varphi(s)> >\sum_{s\in C}\mu(s)=\mu(p)$, which contradicts the choice of μ (i.e., that $\mu(p)\geqq\varphi(p)$). Consequently, there is a nonzero projection $q\leqq p$ in M, such that $\varphi(r)\leqq\mu(r)$ for every projection $r\in M$ majorized by q. Hence (2) is proved.

Since S is an upward directed set, it may serve as an index set for generalized sequences. We shall prove that

$$(*)\qquad \psi(t)=\lim_{p\in S}\varphi(tp),\quad t\in M$$

exists (and is finite). First let $t\geqq 0$. If $p\leqq q$ and $p, q\in S$, then the equality $tq=tp+t(q-p)$ shows that $tq\geqq tp$. By the positivity of φ, the function $p\to\varphi(tp)$ is a nondecreasing nonnegative numerical-valued function on S and $\varphi(tp)\leqq\varphi(t)$ for $p\in S$. Consequently, the finite limit $\lim_{p\in S}\varphi(tp)$ exists and is equal to $\sup\{\varphi(tp)\colon p\in S\}$:

$$(**)\qquad \lim_{p\in S}\varphi(tp)=\sup\{\varphi(tp)\colon p\in S\},\quad t\geqq 0.$$

The existence of $\lim_{p\in S}\varphi(tp)$ for all $t\in M$ follows by linearity.

It is clear that ψ is a positive linear form on M. Moreover, ψ is normal. Indeed, it is an elementary observation that for $p \in S$, the functional $t \to \varphi(tp)$ is normal on M. The normalcy of ψ follows from $(**)$ by using the elementary result that the supremum of an upward directed family of normal positive linear forms is normal [1].

Now we are going to prove that ψ is G-invariant. First let us consider any element p_0 of S, g_0 of G and t_0 of M. Since φ is G-invariant and $p_0 \in S$, the linear form $t \to \varphi(tg_0(p_0))$ is normal. Consequently, $g_0(p_0) \in S$. We have

$$\psi(g_0(t_0)g_0(p_0)) = \lim_{p \in S} \varphi(g_0(t_0)g_0(p_0)p) = \lim_{p \geqq g_0(p_0), p \in S} \varphi(g_0(t_0)g_0(p_0)p) =$$
$$= \lim_{p \geqq g_0(p_0), p \in S} \varphi(g_0(t_0)g_0(p_0)) = \varphi(g_0(t_0)g_0(p_0)) = \varphi(g_0(t_0 p_0)) = \varphi(t_0 p_0) =$$
$$= \lim_{p \geqq p_0, p \in S} \varphi(t_0 p_0 p) = \lim_{p \in S} \varphi(t_0 p_0 p) = \psi(t_0 p_0).$$

So we have shown that $\psi(g_0(t_0)g_0(p_0)) = \psi(t_0 p_0)$. By using property (2) of S, proved above, we can let p_0 w^*-converge to 1 in this equality. Then relying on the normalcy of ψ and on the continuity properties of g_0, we obtain that $\psi(g_0(t_0)) = \psi(t_0)$. Since $g_0 \in G$ and $t_0 \in M$ have been chosen arbitrary, we have proved that ψ is G-invariant.

Finally, ψ is not identically zero. It is in fact faithful. Indeed, if $t \in M$, $t \geqq 0$ and $t \neq 0$, then $\psi(T) = \sup \{\varphi(tp) \colon p \in S\}$. By property (2) of S, we have $\varphi(tp) \neq 0$ for some $p \in S$. Therefore, $\psi(t) > 0$ and the proof of our proposition is complete.

Theorem. *Let M be a W^*-algebra and G a group of $*$-automorphisms of M containing the inner automorphism group. If there exists a faithful G-invariant (not necessarily normal) positive linear form on M, then M is G-finite.*

Proof. Since G contains the inner automorphism group, φ is central [1]. Let $t \in M$ be such that $t^*t = 1$. Then $\varphi(1 - tt^*) = \varphi(1) - \varphi(tt^*) = \varphi(t^*t) - \varphi(tt^*) = 0$. Since $1 - tt^*$ is a projection, the faithfulness of φ implies that $tt^* = 1$, i.e., M is a finite W^*-algebra. Let Z denote the center of M. It is easy to see that Z is invariant under the elements of G. Consequently, we can apply Proposition to Z, the restriction G_Z of G to Z and the restriction of φ to Z. We obtain that Z is G_Z-finite. The G-finiteness of M follows from (simple) results of [3].

Corollary. *Suppose that under the hypotheses of the theorem, for every element t of the center Z of M, the uniformly closed convex hull of the orbit of t under G contains at least one G-invariant element. If the restriction of φ to the algebra Z^G of G-invariant elements of Z is normal, then φ is normal on M.*

Proof. Let I be the inner automorphism group of M. We know [1] that for every $t \in M$, the norm closure of the convex hull of the orbit of t under I contains at least one element $t^\natural$ of Z. Moreover, by the hypotheses of the corollary, the norm closure of the convex hull of the orbit of $t^\natural$ under G contains a G-invariant element t_G. It is clear that t_G is a G-invariant element in the norm closure of the convex hull of the orbit of t under G. Now let $t \to t^G$ be the G-canonical mapping of M onto M^G [3]. Since t^G is the unique G-invariant element in the w^*-closure of the convex hull of the orbit of t under t^G, we have $t^G = t_G$ and t^G is in the *norm* closure of the convex hull of the orbit of t under G.

Since φ is norm-continuous and G-invariant, $\varphi(t) = \varphi(t^G)$, i.e., φ is the composition of the mappings $t \to t^G$ $(t \in M)$ and $s \to \varphi(s)$ $(s \in M^G)$. We know [3] that $t \to t^G$ is normal. If φ is normal on M^G, then this composite mapping, i.e., φ, is also normal on M.

Remark 1. We do not have to use generalized sequences in the proof of the Proposition if G, as a subset of the space of linear self-mappings of M is separable in the topology of pointwise w^*-convergence. In this case we can choose a dense countable subgroup $G_0 = \{g_1, g_2, \ldots\}$ of G, a non-zero projection q in M, such that φ is normal on Mq and take the ordinary limit $\psi(t) = \lim_{n\to\infty} \varphi(t[g_1(q) \vee \ldots \vee g_n(q)])$. It can be shown that ψ is a non-zero G-invariant normal positive linear form on M. It is easy to see that G is separable if the predual of M is separable. This will always be the case if we only consider W^*-algebras M of operators in a separable Hilbert space.

Remark 2. The assumption of Proposition that φ is faithful is essential. Indeed, let G be an abstract infinite Abelian group. Then G acts naturally on $M = l^\infty(G)$ as a group of $*$-automorphisms. A G-invariant state on $l^\infty(G)$ is nothing else but an invariant mean on G. We know that there are infinitely many invariant means on G, none of which are normal. In this situation the entire proof of the theorem is valid except that ψ will be identically zero. Indeed, S will consist of the characteristic functions of finite subsets of G. It is an elementary fact that every invariant mean is zero on such functions.

Finally, the author would like to make two comments on his paper [4]. The first comment is that in Proposition 2 and in its corollary the assumption that M is σ-finite should be replaced by the assumption that the predual of M is separable.

The second comment is that all the results of the above mentioned paper remain valid if G is only assumed to be an amenable group (instead of an Abelian one). Indeed, if $U_n \subset G$ is a summing sequence [2], then it is easy to prove that under the hypotheses of Lemma 1, the sequence $|U_n|^{-1} \sum_{g \in U_n} g(t)$ w^*-converges to t^G for every $t \in B^*$. The remaining results of the paper can be extended to amenable groups G without altering the proofs.

Added in proof: By adapting the method of proof of Proposition to the nonabelian case, Theorem can be broved withoit the assumption that G contains the inner automorphisms. As suggested by R. R. Smith, this can be done even in a simpler manner by appealing to the decomposition of a state into singular and normal parts.

References

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