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Representation of functionals via summability methods. I

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Dedicated to Professor K. Tandori on his 60th birthday

§ 1. Introduction

Summability theory has benefited from functional analysis: several of its fundamental results have source at the main principles of the latter. In this paper and in the continuation of it we show that conversely, some problems concerning functionals and measures can be solved by the aid of summability methods.

Let $C(K)$ be the sup-normed Banach space of real valued continuous functions defined on the compact Hausdorff space K. The representation problem of the bounded linear functionals on $C(K)$ has a long history. It was shown by HADAMARD [3] in 1903 that every $L \in C^*(K)$, where $K=[0, 1]$, has the form

$$Lf = \lim_{n\to\infty} \int_0^1 f(x) p_n(x)\, dx$$

where $\{p_n(x)\}$ is a suitable sequence of continuous functions. The so called Riesz representation theorem, which asserts that every $L \in C^*(K)$ has the form

$$Lf = \int_K f\, d\mu \tag{1.1}$$

with a suitable signed Borel measure μ, was proved for $K=[0, 1]$ by F. RIESZ [5] in 1909, for metrizable K by BANACH and SAKS [1, 6] in 1937—38 and for every K by KAKUTANI [4] in 1941.

Here we present another way for representing every bounded linear functional which, as it seems, have been overlooked so far. This is the form

$$Lf = \lim_{n\to\infty} \frac{c_1 f(x_1) + \ldots + c_n f(x_n)}{n} \tag{1.2}$$

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with appropriate c_k's and x_k's. Naturally, (1.1) is a more convenient form than (1.2), nevertheless, (1.2) has some advantages: (1.2) may be exact up to the domain of L, (cf. Theorem 1 below), the c_k's and x_k's can be obtained, at least for positive L, in a constructive way, the representation (1.2) can be extended to larger spaces, finally a quite similar representation can be given for subadditive and homogeneous functionals: all we have to do is to replace lim by limsup.

The paper is organized as follows. In § 2 we give the representation (1.2) for $K=[0, 1]$ and treat the analogous problem with $c_k\equiv 1$. In § 3 we investigate the subadditive functionals and quasinorms and, finally, in § 4 the generalization to metrizable K's is given.

There will be a forthcoming paper with the following content: (1.2) can be extended to the space $Q[0, 1]$ of functions having discontinuities only of the first kind and $Q[0, 1]$ is maximal among certain "natural" spaces with this property; we shall determine those functionals of $R[0, 1]$, the space of Riemann-integrable functions, which have the form (1.2) and give an application to density measures and, finally, we also characterize those summability methods by which the $(C, 1)$-method in (1.2) can be replaced.

§ 2. Functionals in $C[0, 1]$

Let $c=\{c_k\}_{k=1}^{\infty}$ be a bounded sequence of real numbers and $X=\{x_k\}_{k=1}^{\infty}\subseteq[0, 1]$ a sequence from $[0, 1]$. For an $f\in C[0, 1]$ we define

$$L_{c,X}f = \lim_{n\to\infty}\frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n} \tag{2.1}$$

if the limit on the right exists and let $D_{c,X}$ be the domain of $L_{c,X}$. Clearly, $D_{c,X}$ is a closed subspace of $C[0, 1]$ and $L_{c,X}$ is a bounded linear functional on $D_{c,X}$, $\|L_{c,X}\|\leqq \sup_i |c_i|$.

Our first result states that every bounded linear functional has this form.

Theorem 1. *If $D\subseteq C[0, 1]$ is a closed subspace and $L:D\to\mathbf{R}$ is a bounded linear functional on D then there are sequences c and X such that $L=L_{c,X}$, $D=D_{c,X}$.*

Corollary 1. *If $L\in C^*[0, 1]$ then there are sequences $\{c_k\}$, $|c_k|\leqq\|L\|$ and $\{x_k\}\subseteq[0, 1]$ such that*

$$Lf = \lim_{n\to\infty}\frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n} \tag{2.2}$$

holds for every $f\in C[0, 1]$.

Corollary 2. *If $D\subseteq C[0, 1]$ is a closed subspace and L is a bounded linear* *functional on D then there is a sequence of polynomials* $\{p_n\}$ *such that*

$$\int_0^1 |p_n| = O(1) \quad (n = 1, 2, \ldots),$$

$$Pf \stackrel{\text{def}}{=} \lim_{n\to\infty} \int_0^1 fp_n$$

exists if and only if $f \in D$, *furthermore,* $Pf = Lf$ *for all* $f \in D$.

Let us call a functional of the form (2.1) partial weighted $(C, 1)$-functional, and a one with domain $C[0, 1]$ a weighted $(C, 1)$-functional. If for all k we have $c_k = 1$ then we call $L_{c,X} = L_X$ a partial $(C, 1)$-functional or $(C, 1)$-functional according as $\operatorname{Dom} L_X \subseteqq C[0, 1]$ or $\operatorname{Dom} L_X = C[0, 1]$, respectively. Thus, the (partial) $(C, 1)$-functionals have the form

$$L_X f = \lim_{n\to\infty} \frac{f(x_1) + \ldots + f(x_n)}{n} \quad (f \in \operatorname{Dom} L_X) \tag{2.3}$$

with a sequence $X = \{x_k\} \subseteqq [0, 1]$. It is clear that every such L_X is a positive linear (partial) functional of norm 1 ($L_X 1 = 1$) which shall be abbreviated in the following as: L_X is a PL1 (partial) functional.

By Theorem 1 every bounded partial linear functional (i.e. a functional with domain $\subseteqq C[0, 1]$) is a partial weighted $(C, 1)$-functional. Now what about PL1 functionals? Does every partial PL1 functional have the form (2.3)? The answer is given in

Theorem 2. *Let* $D \subseteqq C[0, 1]$ *be a closed subspace and* L *a* PL1 *functional on* D. *The following assertions are equivalent to each other:*

(i) *L has the form* (2.3), *i.e. there exists a sequence X with* $L = L_X$, $D = \operatorname{Dom} L_X$,

(ii) *to every* $f \in C[0, 1] \setminus D$ *there are two* PL1 *extensions, say* $L_f^{(1)}$ *and* $L_f^{(2)}$, *of* L *to* $C[0, 1]$ *for which* $L_f^{(1)} f \neq L_f^{(2)} f$,

(iii) *D contains the constants, and if for an* $f \in C[0, 1]$ *we have*

$$\inf_{g \in D, g \geqq f} Lg = \sup_{g \in D, g \leqq f} Lg \tag{2.4}$$

then $f \in D$.

E.g. if $D = \{f | f(0) = f(1)\}$ and $Lf = f(1/2)$ for f in D, then there is no X with $(L, D) = (L_X, \operatorname{Dom} L_X)$. Indeed, for $f(x) = x$ (2.4) is satisfied, but $f \notin D$. In other words, the partial PL1 functionals of the form (2.3) are the ones which have no unique extension to any larger subspace of $C[0, 1]$.

Corollary 3. *If L is an arbitrary* PL1 *functional on* $C[0, 1]$ *then there exists* *a sequence* $\{x_k\}\subseteq[0, 1]$ *with*

$$Lf = \lim_{n\to\infty} \frac{f(x_1)+\ldots+f(x_n)}{n} \tag{2.5}$$

for every $f\in C[0, 1]$.

Corollary 4. *Let* $T=(t_{nk})_{n,k=1}^{\infty}$ *be a non-negative summation matrix,* $D\subseteq C[0, 1]$ *a closed subspace containing the constants and* $L\colon D\to\mathbf{R}$ *a partial* PL1 *functional. If there is a sequence* $\{x_k\}\subseteq[0, 1]$ *such that*

$$Lf = T-\lim_k f(x_k) \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} f(x_k) \quad (f\in D)$$

and the limit on the right does not exist for any $f\notin D$, *then* L *is a partial* $(C, 1)$*-functional.*

This corollary tells us that the $(C, 1)$-method is the strongest one from the point of view of the representation of PL1 functionals.

In connection with the representation (2.3) the following very natural questions arise: when do we have $\operatorname{Dom} L_X=C[0, 1]$, and in this case for which other sequences $Y=\{y_k\}\subseteq[0, 1]$ do we have $L_X=L_Y$? The answers are given by

Proposition. (i) *The limit*

$$\lim_{n\to\infty} \frac{1}{n}(f(x_1)+\ldots+f(x_n)) \tag{2.6}$$

exists for every $f\in C[0, 1]$ *if and only if there is a sequence* $\{z_m\}\subseteq[0, 1]$ *dense in* $[0, 1]$ *such that* $\{x_k\}$ *has density in every interval* $[0, z_m]$.

(ii) *Two sequences* X *and* Y *determine the same* PL1 *functional (via* (2.3)*) if and only if there is a dense sequence* $\{z_m\}$ *in* $[0, 1]$ *such that* X *and* Y *have the same density in every interval* $[0, z_m]$.

Remark. If we allow the sequence $\{c_k\}$ in (2.1) to be unbounded then (2.1) still defines a (possibly unbounded) linear functional L on some linear subspace of $C[0, 1]$. However, if the domain of L is $C[0, 1]$ (or any closed subspace of it) then, by the uniform boundedness principle, the obtained L is bounded, so we have lost very little in assuming $\{c_k\}$ to be bounded.

Proofs. In the proofs of the above statements the following lemma will be useful.

Lemma 1. *Let* $g_1, g_2, \ldots$ *be arbitrary functions from* $C[0, 1]$ *and* L *a partial linear functional with* $g_j\in\operatorname{Dom} L$ *for* $j=1, 2, \ldots$. *If there are partial* $(C, 1)$*-functionals* $L_1, L_2, \ldots$ *with* $g_j\in\operatorname{Dom} L_n$ *for* $n, j=1, 2, \ldots$ *such that*

$$\lim_{n\to\infty} L_n g_j = L g_j \quad (j = 1, 2, \ldots)$$

then there exists a partial $(C, 1)$-functional L' with

$$L'g_j = Lg_j \quad (j = 1, 2, \ldots).$$

Proof. For the sake of brevity we introduce the notation

$$\sigma_n(\{x_k\}, f) = \frac{1}{n}(f(x_1)+\ldots+f(x_n)).$$

Let L_n be represented in the form (2.3) by the sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$ i.e. let

$$\lim_{m\to\infty} \sigma_m(\{x_k^{(n)}\}, f) = L_n f \quad (f\in \mathrm{Dom}\, L_n).$$

We define the increasing sequence $\{m_i\}$, $\{M_i\}$ and $\{N_i\}$ in succession so that the following conditions be satisfied:

$|L_n g_j - Lg_j| < 1/i$ for $1 \leqq j \leqq i$ and $n \geqq m_i$, $|\sigma_n(\{x_k^{(m_i)}\}, g_j) - Lg_j| < 1/i$ for $1 \leqq j \leqq i$ and $n \geqq M_i$, $M_{i+1}/N_i < 1/i$, $(\sum_{j=1}^{i-1} N_j)/N_i < 1/i$ for $i = 1, 2, \ldots$, and finally we put $K_0 = 0$ and $K_i = \sum_{j=1}^{i} N_j$ for $i = 1, 2, \ldots$. Let $x_n = x_{n-K_{i-1}}^{(m_i)}$ for $K_{i-1} < n \leqq K_i$. We claim that the partial $(C, 1)$-functional L' represented by the sequence $\{x_k\}_{k=1}^{\infty}$ is suitable for us.

Indeed, let j be an arbitrary but fixed natural number. For $i > j$, $K_i < n \leqq K_{i+1}$ we distinguish two cases according as $n - K_i$ is less than M_{i+1} or not.

1) $n - K_i < M_{i+1}$. By the definitions

$$|\sigma_n(\{x_k\}, g_j) - Lg_j| \leqq |\sigma_n(\{x_k\}, g_j) - \sigma_n(\{x_k^{(m_i)}\}, g_j)| +$$

$$+ |\sigma_n(\{x_k^{(m_i)}\}, g_j) - Lg_j| \leqq \frac{1}{n}\Big(\sum_{r=1}^{K_{i-1}} \max |g_j| + \sum_{r=K_i+1}^{n} \max |g_j| +$$

$$+ \sum_{n=N_i+1}^{n} \max |g_j|\Big) + \frac{1}{i} = \frac{1}{n} \max |g_j| \big(K_{i-1} + (n - K_i) + n - N_i\big) + \frac{1}{i} \leqq$$

$$\leqq \frac{1}{i}(1 + 4 \max |g_j|).$$

2) $n - K_i \geqq M_{i+1}$. We obtain similarly

$$|\sigma_n(\{x_k\}, g_j) - Lg_j| = \Bigg|\frac{1}{n} \sum_{r=1}^{K_{i-1}} (g_j(x_r) - Lg_j) +$$

$$+ \frac{N_i}{n}(\sigma_{N_i}(\{x_k^{(m_i)}\}, g_j) - Lg_j) + \frac{n-K_i}{n}(\sigma_{n-K_i}(\{x_k^{(m_{i+1})}\}, g_j) - Lg_j)\Bigg| \leqq$$

$$\leqq \frac{K_{i-1}}{n}(|Lg_j| + \max |g_j|) + \frac{N_i}{in} + \frac{n-K_i}{n(i+1)} \leqq \frac{1}{i}(|Lg_j| + \max |g_j| + 1)$$

and the proof is over.

We shall prove our theorems and their corollaries in the following order: Corollary 3, Theorem 2, Corollary 1, Theorem 1, Corollaries 2, 4, Proposition.

Proof of Corollary 3. For a natural number n let $g_0^{(n)}\equiv 1$ and for $i=1,2,\ldots,2^n$

$$g_i^{(n)}(x)=\begin{cases}0 & \text{if } 0\leqq x\leqq (i-1)/2^n\\ 1 & \text{if } i/2^n\leqq x\leqq 1\\ \text{linear on} & [(i-1)/2^n, i/2^n].\end{cases}\tag{2.7}$$

Since L is positive with unit norm we have $L1=1$ and

$$0\leqq Lg_{2^n}^{(n)}\leqq\ldots\leqq Lg_1^{(n)}\leqq Lg_0^{(n)}=1.$$

To every $\varepsilon>0$ there are integers $0<m_{2^n}<\ldots<m_1<m_0$ such that

$$\left|\frac{m_i}{m_0}-Lg_i^{(n)}\right|<\varepsilon\quad(0\leqq i\leqq 2^n)\tag{2.8}$$

be satisfied. Let $x_1=x_2=\ldots=x_{m_{2^n}}=1$, $x_{m_{2^n}+1}=\ldots=x_{m_{2^n-1}}=1-(1/2^n)$, $\ldots$, $x_{m_2+1}=\ldots$ $\ldots=x_{m_1}=1/2^n$, $x_{m_1+1}=\ldots=x_{m_0}=0$. Clearly for every $0\leqq i\leqq 2^n$ we have

$$\sum_{j=1}^{m_0} g_i^{(n)}(x_j)=m_i$$

i.e., by (2.8),

$$\left|\frac{1}{m_0}\sum_{j=1}^{m_0} g_i^{(n)}(x_j)-Lg_i^{(n)}\right|<\varepsilon\quad(0\leqq i\leqq 2^n).$$

The sequence $x_1,x_2,\ldots,x_{m_0},x_1,\ldots,x_{m_0},x_1,\ldots$ represents a $(C,1)$-functional $L_\varepsilon^{(n)}$ with

$$|L_\varepsilon^{(n)}g_i^{(n)}-Lg_i^{(n)}|<\varepsilon\quad(0\leqq i\leqq 2^n).$$

Putting here $\varepsilon=1,1/2,\ldots$, Lemma 1 yields a partial $(C,1)$-functional L_n with

$$L_n g_i^{(n)}=Lg_i^{(n)}\quad(0\leqq i\leqq 2^n).$$

But then the same equality holds for the linear combinations of the $g_i^{(n)}$'s and among them there is any $g_j^{(m)}$ with $m\leqq n$. Thus,

$$\lim_{n\to\infty} L_n g_i^{(m)}=Lg_i^{(m)}$$

for all m and $0\leqq i\leqq 2^m$ and another application of Lemma 1 yields a partial $(C,1)$-functional L' with

$$L'g_i^{(m)}=Lg_i^{(m)}\quad(m=1,2,\ldots,\ 0\leqq i\leqq 2^m).\tag{2.9}$$

Since the linear combinations of the $g_i^{(m)}$'s are dense in $C[0,1]$ and both L and L' have norm one, the equality $L=L'$ readily follows from (2.9).

Proof of Theorem 2. (i) $\Rightarrow$ (ii). If $f\notin D$ then, by assumption, there are two subsequences $\{n_k^{(1)}\}$ and $\{n_k^{(2)}\}$ of the natural numbers such that

$$\lim_{k\to\infty}\sigma_{n_k^{(1)}}(X,f)\neq\lim_{k\to\infty}\sigma_{n_k^{(2)}}(X,f) \tag{2.10}$$

and both of these limits exist. Let us define the partial functional L' and L'' by

$$L'g=\lim_{k\to\infty}\sigma_{n_k^{(1)}}(X,g),\quad L''g=\lim_{k\to\infty}\sigma_{n_k^{(2)}}(X,g).$$

Since

$$L'g\leqq\sup g\,(g\in\operatorname{Dom}L'),\quad L''g\leqq\sup g\ (g\in\operatorname{Dom}L''),$$

L' and L'' can be extended by the Hahn—Banach theorem to $C(0,1)$ so that the previous inequalities remain valid for all $g\in C[0,1]$. The obtained functionals $L_f^{(1)}$ and $L_f^{(2)}$ are clearly PL1 functionals and, by (2.10), $L_f^{(1)}f=L'f\neq L''f=L_f^{(2)}f$.

(ii)$\Rightarrow$(i). By assumption to every $f\notin D$ there are two PL1 extensions $L_f^{(1)}$ and $L_f^{(2)}$ of L with $L_f^{(1)}f\neq L_f^{(2)}f$, say $L_f^{(1)}f<L_f^{(2)}f$. Then there is a neighbourhood U_f of f and an $\varepsilon_f>0$ such that

$$L_f^{(1)}g\leqq L_f^{(2)}g-\varepsilon_f\quad\text{for all}\quad g\in U_f.$$

Since $C[0,1]\setminus D$ is a separable metric space it satisfies the Lindelöf property, so that

$$C[0,1]\setminus D=\bigcup_{m=1}^{\infty}U_{f_m}$$

for some sequence $\{f_m\}_1^\infty\subseteqq C[0,1]\setminus D$. Let $\{L_n\}$ be a sequence of the functionals $\{L_{f_m}^{(1)},L_{f_m}^{(2)}\}_{m=1}^\infty$ which contains every $L_{f_m}^{(1)}$ and $L_{f_m}^{(2)}$.

By the above proved Corollary 3 there are sequences $\{x_k^{(n)}\}_{k=1}^\infty$ representing L_n in the sense (2.5). Now let $\{x_n\}$ be any sequence guaranteed by the following lemma:

Lemma 2. *If $\{x_k^{(n)}\}_{k=1}^\infty$, $n=1,2,\dots$ are the just introduced sequences then there is a union $\{x_n\}_{n=1}^\infty=\bigcup_{n=1}^{\infty}\{x_k^{(n)}\}_{k=1}^\infty$ of these sequences such that*

(i) *every $\{x_k^{(n)}\}_{k=1}^\infty$ is a subsequence, say $\{x_{j_k^{(n)}}\}_k$, of $\{x_n\}$ and it has upper density 1 in $\{x_n\}$, i.e.*

$$\limsup_{k\to\infty}k/j_k^{(n)}=1,$$

(ii) *for every m there are four indices $n_1(m)$, $n_2(m)$, $k_1(m)$ and $k_2(m)$ such that*

(α) $m-(k_1(m)+k_2(m))=o(m)\quad(m\to\infty)$

(β) *the finite sequences $\{x_k^{(n_1(m))}\}_{k=1}^{k_1(m)}$ and $\{x_k^{(n_2(m))}\}_{k=1}^{k_2(m)}$ form two disjoint subsequences of $\{x_k\}_{k=1}^m$ and*

(γ) *for a dense countable subset $D'\subset D$ and for every $f\in D'$ we have*

$$\sigma_{k_1(m)}(\{x_k^{(n_1(m))}\},f)=L_{n_1(m)}f+o(1)\quad(k_1(m)\to\infty)$$

$$\sigma_{k_2(m)}(\{x_k^{(n_2(m))}\},f)=L_{n_2(m)}f+o(1)\quad(k_2(m)\to\infty).$$

(ii) (α) and (β) say that for every m the sequence $\{x_k\}_1^m$ is essentially formed from two initial segments $\{x_k^{(n_1(m))}\}_{k=1}^{k_1(m)}$ and $\{x_k^{(n_2(m))}\}_{k=1}^{k_2(m)}$. The proof of this lemma is straightforward, we omit it.

Returning to the proof of (ii)$\Rightarrow$(i) we claim that the partial $(C, 1)$-functional L' respresented by the sequence $\{x_n\}$ satisfies $L'=L$, $\mathrm{Dom}\, L'=D$. Indeed, for $f\in D'$ (cf. (ii) γ, in the preceding lemma) we have $L_n f=Lf$ for every n, hence, by Lemma 2, (ii)

$$\sigma_m(\{x_n\}, f) = o_m(1)+\frac{k_1(m)}{m}\sigma_{k_1(m)}(\{x_k^{(n_1(m))}\}, f)+$$
$$+\frac{k_2(m)}{m}\sigma_{k_2(m)}(\{x_k^{(n_2(m))}\}, f) = o_m(1)+\frac{k_1(m)}{m}(Lf+o_{k_1(m)}(1))+$$
$$+\frac{k_2(m)}{m}(Lf+o_{k_2(m)}(1)) = Lf+o_m(1)$$

where $o_m(1)$ denotes a quantity that tends to zero together with m. The relation above shows $f\in\mathrm{Dom}\, L'$ and $L'f=Lf$. Since this holds for every $f\in D'$ and D' is dense in D we can conclude that $D\subset\mathrm{Dom}\, L'$ and L' agrees with L on D. On the other hand, if $f\notin D$ then $f\in U_{f_n}$ for some n and thus, by our construction and Lemma 2, (i)

$$\liminf_{m\to\infty}\sigma_m(\{x_k\}, f) \leqq \limsup_{m\to\infty}\sigma_m(\{x_k\}, f)-\varepsilon_n$$

i.e. $f\notin\mathrm{Dom}\, L'$ and so $L=L'$ has been verified.

The equivalence of (ii) and (iii) is clear from any standard proof of the Hahn—Banach extension theorem.

We have completed the proof of Theorem 2.

Proof of Corollary 1. By Riesz' decomposition theorem $L=\alpha L_1-\beta L_2$ where $\alpha\geqq 0$, $\beta\geqq 0$, $\alpha+\beta=\|L\|$ and L_1 and L_2 are PL1 functionals. If L_1 and L_2 are represented by the sequences $\{x_k^{(1)}\}$ and $\{x_k^{(2)}\}$ via (2.5) and $\{n_k^{(1)}\}$ and $\{n_k^{(2)}\}$ are disjoint subsequences of the natural numbers N with density $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$, respectively, furthermore $N=\{n_k^{(1)}\}\cup\{n_k^{(2)}\}$ then the sequences

$$x_n=\begin{cases} x_k^{(1)} & \text{if} \quad n=n_k^{(1)} \\ x_k^{(2)} & \text{if} \quad n=n_k^{(2)}, \end{cases} \qquad c_n=\begin{cases} \|L\| & \text{if} \quad n=n_k^{(1)} \\ -\|L\| & \text{if} \quad n=n_k^{(2)} \end{cases}$$

clearly satisfy the requirements of Corollary 1.

The Proof of Theorem 1 is similar to that of Theorem 2 if we notice that to every $f\notin D$ there are two extensions of L, say $L_f^{(1)}$ and $L_f^{(2)}$, for which $L_f^{(1)}f\neq L_f^{(2)}f$, $\|L_f^{(1)}\|$, $\|L_f^{(2)}\|\leqq\|L\|+1$ and if we apply Corollary 1 instead of Corollary 3.

Corollary 2 follows easily from Theorem 1 since the Dirac measures δ_{x_i} can be approximated by polynomials p_i^* satisfying $p_i^* \geqq 0$, $\int_0^1 p_i^* = 1$ and $\max_{|x-x_i| \geqq 1/i} p_i^*(x) < 1/i$ in the sense $\int_0^1 f p_i^* = f(x_i) + o(1)$ $(f \in C[0, 1])$.

In the Proof of Corollary 4 the fact that to every $f \notin D$ there are two PL1 extensions $L_f^{(1)}$ and $L_f^{(2)}$ of L with $L_f^{(1)} f \neq L_f^{(2)} f$ can be proved exactly as in the case of the $(C, 1)$-method in Theorem 2 (use that by $1 \in D$, $L1 = 1$), and we have to apply only Theorem 2.

Proof of the Proposition. (i). Let

$$\tau(z) = \liminf_{n\to\infty} \sigma_n(\{x_k\}, \chi_{[0,z]})$$

and

$$\mu(z) = \limsup_{n\to\infty} \sigma_n(\{x_k\}, \chi_{[0,z]}),$$

$$\chi_{[0,z]}(x) = \begin{cases} 1 & \text{if } 0 \leqq x \leqq z \\ 0 & \text{if } z < x \leqq 1 \end{cases}$$

be the lower and upper density of $\{x_k\}$ in $[0, z]$. τ and μ are increasing functions, so they are continuous everywhere but a denumerable set. If $\varepsilon > 0$ and

$$g_{z,\varepsilon}(x) = \begin{cases} 1 & \text{if } x \leqq z \\ 0 & \text{if } x \geqq z+\varepsilon \\ \text{linear on } [z, z+\varepsilon] \end{cases}$$

then

$$\tau(z) \leqq \mu(z) \leqq \lim_{n\to\infty} \sigma_n(\{x_k\}, g_{z,\varepsilon}) \leqq \tau(z+\varepsilon)$$

and so $\tau(z) = \mu(z)$ at every point z where τ is continuous, and this proves the necessity of the condition.

Conversely, if $\tau(z_n) = \mu(z_n)$ for every z_n in a dense set then the limit (2.3) exists for every f which is the linear combination of the characteristic funtions of the intervals $[0, z_n]$ and every continuous function can be approximated uniformly by such f's.

(ii) can be proved similarly.

We have completed our proofs.

§ 3. Subadditive functions and quasinorms

In this paragraph we present some representation theorems for subadditive functionals which are very close in spirit to the results of the previous chapter.

Recall that a functional τ: $C[0, 1] \to \mathbf{R}$ is called subadditive if

$$\tau(f+g) \leqq \tau(f) + \tau(g) \tag{3.1}$$

is satisfied for all $f, g \in C(0, 1)$. It is positive homogeneous if $\tau(\lambda f)=\lambda\tau(f)$ for all $f \in C[0, 1]$ and $\lambda \geqq 0$. If τ is both subadditive and positive homogenenous then we call it convex functional. Quasinorms are the non-negative convex even functionals, i.e. besides (3.1) they satisfy $\tau(f)\geqq 0$, $\tau(\lambda f)=|\lambda|\tau(f)$ for all f and λ.

If $\{c_k\}\subseteqq \mathbf{R}$, $|c_k|=O(1)$ and $\{x_k\}\subseteqq[0, 1]$ are two sequences then each of the following defines a bounded convex functional on $C[0, 1]$

$$\tau(f) = \limsup_{n\to\infty} \frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n}, \tag{3.2}$$

$$\tau(f) = \limsup_{n\to\infty} \left|\frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n}\right|, \tag{3.3}$$

$$\tau(f) = \limsup_{n\to\infty} \frac{|c_1|\,|f(x_1)|+\ldots+|c_n|\,|f(x_n)|}{n}. \tag{3.4}$$

Obviously the τ in (3.3) and (3.4) is a quasinorm, furthermore $|f|\leqq|g|$ implies $\tau(f)\leqq\tau(g)$ in (3.4). Now all of these statements have converses:

Theorem 3. *Every bounded convex functional τ on $C[0, 1]$ has the form* (3.2) *with suitable sequences* $\{c_k\}\subseteqq R$, $|c_k|=O(1)$ *and* $\{x_k\}\subseteqq[0, 1]$.

Theorem 4. *Every bounded quasinorm on $C[0, 1]$ has the form* (3.3).

Theorem 5. *Every bounded quasinorm τ on $C[0, 1]$ with the property*

$$\tau(f) \leqq \tau(g) \quad \text{whenever} \quad |f| \leqq |g|$$

has the form (3.4).

E.g. every L^p-norm $(1\leqq p<\infty)$

$$\tau_p(f) = \left\{\int_0^1 |f|^p\right\}^{1/p}$$

has the form (3.4) with suitable $\{c_k\}$ and $\{x_k\}$.

From our results one can deduce other representation theorems, e.g. Theorem 4 implies that every bounded quasinorm τ on $C[0, 1]$ has the form

$$\tau(f) = \limsup_{n\to\infty} \frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n} - \liminf_{n\to\infty} \frac{c_1 f(x_1)+\ldots+c_n f(x_n)}{n}.$$

We mention also that, as can be seen easily from the proofs, the sequences $\{c_i\}$ in Theorems 3—5 can be chosen so that they also satisfy $|c_i|\leqq\|\tau\|$.

We also give the characterization of those convex functionals which can be obtained from (3.2)—(3.4) with $c_k \equiv 1$ i.e. which have the forms

$$\tau(f) = \limsup_{n\to\infty} \frac{f(x_1)+\ldots+f(x_n)}{n}, \tag{3.5}$$

$$\tau(f) = \limsup_{n\to\infty} \left|\frac{f(x_1)+\ldots+f(x_n)}{n}\right|, \tag{3.6}$$

$$\tau(f) = \limsup_{n\to\infty} \frac{|f(x_1)|+\ldots+|f(x_n)|}{n}, \tag{3.7}$$

respectively.

Clearly, we have $\|\tau\|=1$ in these cases.

Theorem 6. *For a convex functional τ with norm* 1 *the following assertions are equivalent:*

(i) *τ has the form* (3.5),
(ii) $\tau(1)=-\tau(-1)=1$,
(iii) $\tau(f+c)=\tau(f)+c$ *for* $f\in C[0, 1]$, $c\in \mathbf{R}$,
(iv) $\tau(f)\leqq \max f$ for $f\in C[0, 1]$,
(v) *if $L\leqq\tau$ is a linear functional then L is positive and has norm* 1 *(i.e. L is* PL1 *functional).*

Theorem 7. *For a quasinorm τ with norm* 1 *the following assertions are equivalent:*

(i) *τ has the form* (3.6),
(ii) (α) $\max\big(\tau(f+c), \tau(f-c)\big)=\tau(f)+c$ *for all f and* $c\geqq 0$ *and*
(β) $|f|\leqq g$ *implies* $\tau(f)\leqq\tau(g)$ *for all f and g,*
(iii) $\tau(f)=\max\big(\mu(f), \mu(-f)\big)$ $(f\in C[0, 1])$ *with a μ satisfying any of the conditions of Theorem 6.*

Theorem 8. *For a quasinorm τ with norm* 1 *the following assertions are equivalent:*

(i) *τ has the form* (3.7),
(ii) (α) $\tau(f)\leqq\tau(g)$ *whenever* $|f|\leqq|g|$ *and*
(β) $\tau(f+c)=\tau(f)+c$ *for all* $f\geqq 0$ *and* $c\geqq 0$,
(iii) $\tau(f)=\mu(|f|)$ *with a μ satisfying any of the conditions of Theorem 6.*

Remarks. (1) Most of our results have analogues for superadditive functionals i.e. for functionals satisfying

$$\tau(f+g) \geqq \tau(f)+\tau(g),$$

naturally we have to use lim inf instead of lim sup. We do not go into the details.

(2) Here, again, we might restrict ourselves to bounded sequences $\{c_k\}$ because of the uniform boundedness principle.

Proofs. First we verify Theorem 6.

(i)⇒(ii) is obvious.

(ii)⇒(iii). By the subadditivity we have

$$\tau(f)+c=\tau(f)-\tau(-c)\leqq\tau(f+c)\leqq\tau(f)+\tau(c)=\tau(f)+c.$$

(iii)⇒(iv). Since τ has norm 1 we obtain

$$\tau(f)=\tau(f-\min f)+\min f\leqq\|\tau\|\,\|f-\min f\|+\min f=$$
$$=\max(f-\min f)+\min f=\max f.$$

(iv)⇒(v). If $f\geqq 0$ then we have

$$Lf=-L(-f)\geqq-\tau(-f)\geqq-\max(-f)\geqq 0,$$

i.e. L is positive, furthermore

$$1=\|\tau\|\geqq\tau(1)\geqq L1=-L(-1)\geqq-\tau(-1)\geqq-\max(-1)=1$$

i.e. $L1=1$ which, together with the positivity of L prove that $\|L\|=1$.

(v)⇒(i). By the Hahn—Banach theorem and (v)

$$\tau(f)=\sup_{\substack{L\leqq\tau\\ \|L\|=1,\ L\text{ positive}}} Lf \qquad (f\in C[0,1]).$$

Since $C[0,1]$ is separable and any convex functional τ with norm 1 satisfies

$$\tau(f)-\varepsilon\leqq\tau(f)-\tau(f-g)\leqq\tau(g)\leqq\tau(f)+\tau(g-f)\leqq\tau(f)+\varepsilon$$

provided $\|g-f\|\leqq\varepsilon$, we obtain at once that there is a sequence L_n of PL1 functionals for which $L_n\leqq\tau$ and

$$\tau(f)=\sup_n L_nf \quad (f\in C[0,1]).$$

If L_n is represented by the sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$ in the sense of Corollary 3 and if $\{x_n\}$ the sequence associated with $\{x_k^{(n)}\}_{k=1}^{\infty}$, $n=1,2,\dots$ by Lemma 2 (i) then an easy calculation gives (3.5).

We have completed our proof.

The Proof of Theorem 3 is much the same as that of (v)⇒(i) above if we use Corollary 1, the formula

$$\tau(f)=\sup_{L\leqq\tau} Lf \quad (f\in C[0,1])$$

and the fact that $L\leqq\tau$ implies $-\|\tau\|\,\|f\|\leqq-\tau(-f)\leqq-L(-f)=Lf\leqq\tau(f)\leqq\|\tau\|\,\|f\|$, i.e. $\|L\|\leqq\|\tau\|$.

Proof of Theorem 7. (i)⇒(ii) is obvious because $\tau(f+c)=\tau(f)+c$ or $\tau(f-c)=\tau(f)+c$ according as

$$\tau(f)=\limsup_{n\to\infty}\sigma_n(\{x_k\},f)$$

or

$$\tau(f)=\limsup_{n\to\infty}\sigma_n(\{x_k\},-f)$$

respectively.

(ii)⇒(iii). First of all we notice that for $f\geqq 0$ and $c\geqq 0$ we have

$$\tau(f+c)=\max\big(\tau(f+c),\ \tau(f-c)\big)=\tau(f)+c \tag{3.8}$$

because $|f-c|\leqq f+c$ implies $\tau(f-c)\leqq\tau(f+c)$.

Now let us define μ by $\mu(f)=\tau(f+c)-c$ where $f\in C[0,1]$ and c is a constant with $f+c\geqq 0$. By (3.8) μ is uniquely defined and an easy consideration yields that μ is a convex functional with $\mu(1)=-\mu(-1)=1$. Since for large $c>0$

$$-\tau(f)=-\tau(-f)\leqq-\tau(-f)+\tau(f+c)+\tau(-f)-c=$$
$$=\mu(f)\leqq\tau(f)+\tau(c)-c=\tau(f),$$

μ also has norm 1. Thus, μ satisfies the condition of Theorem 6. Applying the previous inequality also to $-f$ we obtain

$$\max\big(\mu(f),\ \mu(-f)\big)\leqq\tau(f)$$

and here the equality sign holds for all f because of (ii), α, which proves (ii)⇒(iii).

(iii)⇒(i). If μ is represented by $\{x_k\}$ in the sense of (3.5) (see Theorem 6), then we have (3.6) for this $\{x_k\}$ because

$$\limsup_{n\to\infty}|s_n|=\max\big(\limsup_{n\to\infty}s_n,\ \limsup_{n\to\infty}(-s_n)\big)$$

for every sequence $\{s_n\}$.

The proof is complete.

The proof of Theorem 4 is easy on the ground of Theorem 3. By Theorem 3 there are sequences $\{c_k\}$, $\{x_k\}$ for which

$$\tau(f)=\tau(\pm f)=\limsup_{n\to\infty}\pm\frac{c_1f(x_1)+\ldots+c_nf(x_n)}{n}$$

and this immediately gives (3.3).

Proof of Theorem 8. Again, (i)⇒(ii) is obvious.

(ii)⇒(iii). Arguing as in the proof of (ii)⇒(iii) in Theorem 7 we obtain that $\tau(f)=\mu(f)$ for all non-negative f with a μ satisfying the conditions of Theorem 6. This also proves our assertion because for every $f\in C[0,1]$

$$\tau(f)=\tau(|f|)=\mu(|f|).$$

(iii)$\Rightarrow$(i). It is clear, that if μ has the form (3.5) then $\tau(f)=\mu(|f|)$ has the form (3.7) with the same sequence $\{x_k\}$.

Proof of Theorem 5. Let us consider the positive cone $C_+=\{f\in C[0,1] \mid f\geqq 0\}$. For every $f\in C_+$ there is, by the Hahn—Banach theorem, a linear functional L_f with $L_f f=\tau(f)$, $|L_f g|\leqq\tau(g)$ $(g\in C[0,1])$. Let

$$L_f^+ g = \sup_{0\leqq h\leqq g} L_f h, \quad g\in C_+$$

and

$$L_f^+ g \overset{\text{def}}{=} L_f^+ g^+ - L_f^+ g^-, \quad g\in C[0,1]$$

where $g=g^+-g^-$ is the decomposition of g into its positive and negative parts. Then L_f^+ is a positive linear functional on $C[0,1]$ (the positive part of L_f) with the properties

$$L_f^+ g = L_f^+(g^+-g^-) \leqq L_f^+ g^+ = \sup_{0\leqq h\leqq g^+} L_f h \leqq$$

$$\leqq \sup_{0\leqq h\leqq g^+} \tau(h) \leqq \tau(g^+) \leqq \tau(|g|) = \tau(g) \quad (g\in C[0,1]),$$

$$\tau(f) \geqq L_f^+ f \geqq L_f f = \tau(f), \quad \|L_f^+\| \leqq \|\tau\|.$$

Thus, for all $f\in C_+$

$$\tau(f) = \sup_{\substack{\|L\|\leqq\|\tau\|,\, L \text{ positive} \\ L\leqq\tau}} Lf$$

and this yields again a sequence $\{L_n\}$ of positive linear functionals such that $\|L_n\|\leqq \leqq\|\tau\|$, $L_n\leqq\tau$ and

$$\tau(f) = \sup_n L_n f \quad (f\in C_+).$$

By Corollary 3 every L_n has the form

$$L_n f = \lim_{m\to\infty} \frac{\|L_n\| f(x_1^{(n)}) + \ldots + \|L_n\| f(x_m^{(n)})}{m}$$

with a suitable sequence $\{x_k^{(n)}\}_{k=1}^{\infty}$. Now Lemma 2 (i) gives a sequence $\{x_n\}$ and also a corresponding sequence $\{c_n\}$ (every c_n is some $\|L_k\|$) with

$$\tau(f) = \tau(|f|) = \sup_n L_n(|f|) = \limsup_{m\to\infty} \frac{c_1|f(x_1)| + \ldots + c_m|f(x_m)|}{m}$$

and we are done.

We have completed our proofs.

§ 4. Extension to compact metric spaces

Theorem 9. *All of the results of* §§2 *and* 3 *hold if we replace in them* $C[0, 1]$ *by* $C(K)$ *where* K *is an arbitrary compact metric space.*

Naturally, in Corollary 2 the term "polynomial" must be replaced by "generalized polynomial" corresponding to a system satisfying the assumptions of the Stone—Weierstrass approximation theorem.

If K is a compact Hausdorff space then the metrizability of K is equivalent to the separability of $C(K)$. Now what about nonseparable spaces? Does Theorem 9 hold without the metrizability assumption? The answer is no: if K is a non-separable compact topological group with Hausdorff topology and μ $(\mu(K)=1)$ is the left invariant Haar-measure on K then

$$Lf = \int_K f\,d\mu$$

is not a $(C, 1)$-functional. Indeed, if $\{x_k\}_{k=1}^{\infty}$ is any sequence from K then there is a function $f\in C(K), f\geqq 0, f\not\equiv 0$ such that f is zero on the closure of $\{x_k\}$, but, by the properties of μ, $Lf>0$.

Now at this point one might suspect that the metrizability of K is necessary in Theorem 9. However, this again turns out to be false: if K is the one point (so called Alexandroff) compactifications of a non-countable discrete space, i.e.

$$K = \{x_\alpha\}_{\alpha\in A}\cup\{w\}, \quad |A| > \aleph_0$$

then every continuous function is constant on $K\setminus$\{a countable set\} and hence for every complex Borel measure μ

$$\int f\,d\mu = \sum_{\alpha\in A} f(x_\alpha)\mu(\{x_\alpha\}) + f(w)\Big(\mu(K) - \sum_{\alpha\in A}\mu(\{x_\alpha\})\Big)$$

(take into account that in the sums we have $\mu(\{x_\alpha\})\neq 0$ for at most a countable set of the α's), and it is obvious that the functional on the right hand side is a $(C, 1)$-functional.

We were not able to give necessary and sufficient conditions for a compact Hausdorff space K that every $L\in C^*(K)$ be a weighted $(C, 1)$-functional.

Proof of Theorem 9. Since $C(K)$ is separable, all of our considerations remain valid for $C(K)$ if we can prove the analogue of Corollary 3. Examining the proof of Corollary 3 we can see that it is enough to show that every PL1 functional L is the weak*-limit of a sequence of functionals of the form

$$L^n f = \frac{1}{n}\big(f(x_1) + \dots + f(x_n)\big).$$

Since the extremal points of the weakly compact and convex set of all PL1 functionals are exactly the point evaluations (≡ functionals corresponding to point masses), the required statement follows from the Krein—Milman theorem [2, p. 440]: if M is a compact closed subset of a locally convex linear topological space then M is the closure of the convex hull of its extremal points. We have completed our proof.

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