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# On $E$-disjunctive inverse semigroups

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Throughout the paper, we have the following notations. Let $\Omega=\{\xi, \eta, \zeta, ...\}$ and $S$ be inverse semigroups. Let $E(\Omega)=\{\alpha, \beta, \gamma, \delta, ...\}$ be the set of all idempotents of $\Omega$. A congruence $\varrho$ on $S$ is said to be *idempotent-determined* or *I. D.* if $e \varrho b$ and $e\in E(S)$ imply $b\in E(S)$. Let $\mu$ $[\tau]$ be the greatest idempotent-separating [I. D.] congruence on $S$. Let $N=\bigcup\{G_\gamma: \gamma\in Y\}$ be a Clifford semigroup, that is, a semilattice of groups. $S$ is said to be *E-disjunctive* if $\tau=\iota$. All other definitions and notations follow the conventions of [1].

Green [2] proved that if $S$ is inverse, then $\mu\cap\tau=\iota$. It is easily shown that $S/\tau$ is $E$-disjunctive. Hence every inverse semigroup is isomorphic with a subdirect product of a fundamental inverse semigroup $S/\mu$ and an $E$-disjunctive inverse semigroup $S/\tau$. We shall discuss $E$-disjunctive inverse semigroups by using [2, 4]. We have immediately

Lemma 1. *Let $\Omega$ be a full inverse subsemigroup of an inverse semigroup $U$. If $\Omega$ is E-disjunctive, then $U$ is E-disjunctive.*

An inverse semigroup $S=S[N, \Omega]$ is called a *regular extension* of $N$ by $\Omega$ if $N\subseteqq S$ and there exists a homomorphism $\varkappa$ of $S$ onto $\Omega$ such that $\bigcup\{(\gamma\varkappa^{-1}): \gamma\in E(\Omega)\}$ is the decomposition of $N$ induced by the finest semilattice congruence on $N$ [4].

Let End $(N)$ be the set of all endomorphisms of $N$. Fix $c$ in $N$. Define $\bar{c}: N\to N$ by $u\mapsto cuc^{-1}$. Then $\bar{c}$ is the inner endomorphism induced by $c$. Let $\Omega$, $Y=E(\Omega)$ and $N$ be the same as above and let $1_\gamma$ be the identity element of the group $G_\gamma$. For each $\xi$ in $\Omega$, define $\bar{\xi}\in$End$(N)$ such that (i) $\bar{\xi}$ is the inner endomorphism $\bar{1}_\gamma$ if $\xi=\gamma\in Y$, and (ii) $\bar{\xi}$ maps $G_\gamma$ into $G_{\xi\gamma\xi^{-1}}$ $(\gamma\in Y)$, in particular, it maps $G_{\xi^{-1}\xi}$ onto $G_{\xi\xi^{-1}}$.

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For each pair $\xi, \eta$ in $\Omega$, define $C(\xi, \eta)\in G_{\xi\eta(\xi\eta)^{-1}}$ satisfying

(1) $C(\xi\xi^{-1}, \xi) = 1_{\xi\xi^{-1}} = C(\xi, \xi^{-1}\xi)$ for all $\xi$ in $\Omega$, and $C(\gamma, \delta) = 1_{\gamma\delta}$ for all $\gamma, \delta$ in $Y$,

(2) $C(\eta, \zeta)^{\xi} C(\xi, \eta\zeta) = C(\xi, \eta) C(\xi\eta, \zeta)$, where $u^{\xi} = u\bar{\xi}$,

(3) $\overline{\xi\eta} = \bar{\eta}\bar{\xi}\overline{C(\eta, \xi)}$.

Then $\{\bar{\xi}, C(\xi, \eta)\colon \xi, \eta\in\Omega\}$ is called a *factor set of N belonging to Ω*. Define a product in $N*\Omega=\{(a, \xi)\colon a\in G_{\xi\xi^{-1}}, \xi\in\Omega\}$ by $(a, \xi)(b, \eta)=(ab^{\xi}C(\xi, \eta), \xi\eta)$. Then $N*\Omega$ is a regular extension of $N$ by $\Omega$. And the converse is valid [4]. We identify $S[N, \Omega]$ with $N*\Omega$. We obtain the following theorem immediately.

Theorem 2. *Let $S=S[N, \Omega]$ be a regular extension. Then S is E-disjunctive if and only if there exists no pair $\gamma, \delta$ in $E(\Omega)$ with $\delta<\gamma$ satisfying*

$$\{(y, \eta)\colon \xi\gamma\eta\in E(\Omega),\quad y^{\overline{\xi\gamma}} = [C(\xi\gamma, \eta)xC(\xi, \gamma)]^{-1}\} =$$

$$= \{(y, \eta)\colon \xi\delta\eta\in E(\Omega),\ y^{\overline{\xi\delta}} = [C(\xi\delta, \eta)xC(\xi, \delta)]^{-1}\} \quad \textit{for all} \quad (x, \xi)\in S.$$

Let $S=S[N, \Omega]$. $S$ is called an *I-regular extension* if $C(\xi, \eta)=1_{\xi\eta(\xi\eta)^{-1}}$ for all $\xi, \eta\in\Omega$. Let $N$ be a Clifford semigroup with linking homomorphisms $\{\psi_{\gamma,\delta}\colon \delta<\gamma\}$. Then $N$ is called a *D-Clifford semigroup* if there is no pair $\delta, \gamma$, $\delta<\gamma$, in $Y$ such that $\psi_{\gamma\beta,\delta\beta}$ is $1-1$ for all $\beta$ in $Y$, and $N$ is called a *W-Clifford semigroup* if $\psi_{\gamma,\delta}$ is $1-1$ for all $\delta<\gamma$ in $Y$. KRGOVIĆ and ALIMPIĆ [3] state the following theorem.

Theorem 3. *Let N be a Clifford semigroup. Then N is E-disjunctive if and only if N is a D-Clifford semigroup.*

Every $E$-disjunctive inverse semigroup $S$ is found by constructing a regular extension of $N$ by $\Omega$. If $N$ is a $D$-Clifford semigroup, then $N$ is $E$-disjunctive by Theorem 3, and thus $S[N, \Omega]$ is $E$-disjunctive by Lemma 1. We need to describe the case $N$ is not $E$-disjunctive. Now we get Corollary 4 for the $E$-disjunctivity of $S[N, \Omega]$ and $\Omega$.

Corollary 4. *Let $S=S[N, \Omega]$ and let $Y=E(\Omega)$. Suppose that $(\bar{\xi}|G_{\beta})$ maps $G_{\beta}$ onto $G_{\xi\beta\xi^{-1}}$ for all $\xi\in\Omega$, $\beta\in Y$. If $\Omega$ is E-disjunctive, then S is E-disjunctive.*

*Let N be a W-Clifford semigroup and let S be an I-regular extension. If S is E-disjunctive, then Ω is E-disjunctive.*

In what follows let $Y$ be a semilattice and let $T_Y$ be the Munn semigroup. Finally we discuss the $E$-disjunctivity of $T_Y$ where $Y$ is discrete. $1_A$ denotes the identity mapping on the set $A$. $\Delta(\xi)$ $[\nabla(\xi)]$ means the domain [range] of $\xi$.

We give the preliminary lemmas without proof.

Lemma 5. *Let $\Omega$ be a full inverse subsemigroup of $T_Y$. Let $\xi, \eta \in \Omega$, $e, f \in Y$ and $\nabla(\xi)=Yl$, $\Delta(\eta)=Ym$. Then*

$\xi \mathbf{1}_{(Ye)} \eta$ *is idempotent if and only if* $(\eta|Ylem)=(\xi^{-1}|Ylem)$;

$\mathbf{1}_{(Ye)}\ \tau_\Omega\ \mathbf{1}_{(Yf)}$ *if and only if*

$$\{\eta\in\Omega\colon (\eta|Ylem) = (\xi^{-1}|Ylem)\} = \tag{4}$$
$$= \{\eta\in\Omega\colon (\eta|Ylfm) = (\xi^{-1}|Ylfm)\} \quad \textit{for all } \xi \textit{ in } \Omega^1.$$

Theorem 6. *Let $Y$ be a semilattice and $\Omega$ a full inverse subsemigroup of $T_Y$. Then $\Omega$ is $E$-disjunctive if and only if there is no pair $e, f$ in $Y$, $e<f$, such that*

$$(\eta|Yev) = (\zeta|Yev), \quad \Delta(\eta) = Yv = \Delta(\zeta), \quad \eta, \zeta\in\Omega, \quad v\in Y, \tag{5}$$
$$\textit{implies } (\eta|Yfv) = (\zeta|Yfv).$$

Proof. $\Omega$ is not $E$-disjunctive $\Leftrightarrow(\exists e, f$ in $Y$: $e<f$ and $\mathbf{1}_{(Ye)}\ \tau_\Omega\ \mathbf{1}_{(Yf)})$, $\Leftrightarrow(\exists e, f$ in $Y$: $e<f$ and

$$\left.\begin{matrix}(\eta_1|Yelm) = (\zeta_1|Yelm), \\ \Delta(\eta_1) = Ym, \quad \Delta(\zeta_1) = Yl, \quad \eta_1, \zeta_1\in\Omega\end{matrix}\right\} \Rightarrow (\eta_1|Yflm) = (\zeta_1|Yflm)). \tag{6}$$

Sufficiency. Suppose $\Omega$ is not $E$-disjunctive. Then there is a pair $e, f$, $e<f$, satisfying (6). We shall prove that (5) holds for such pair $e, f$. Assume that the set $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5). Putting $\eta_1=\eta$ and $\zeta_1=\zeta$ in (6), we have $m=v=l$. Since $(\eta_1|Yelm)=(\eta|Yev)=(\zeta|Yev)=(\zeta_1|Yelm)$, we obtain $(\eta|Yfv)=(\zeta|Yfv)$. Hence (5) holds.

Necessity. Assume that there exists a pair $e, f$, $e<f$, satisfying (5). If $\{\eta_1, \zeta_1\}$ satisfies the hypothesis in (6), we have $\Delta(\eta)=Yv=\Delta(\zeta)$ by setting $v=lm$, $\eta=(\eta_1|Yv)$, $\zeta=(\zeta_1|Yv)$. Since $(\eta|Yev)=(\eta_1|Yelm)=(\zeta_1|Yelm)=(\zeta|Yev)$, we find $(\eta_1|Yflm)=(\zeta_1|Yflm)$ by (5).

Let $e, f\in Y$, $e<f$. Define $[e,f]=\{g\in Y\colon e\leqq g\leqq f\}$. $Y$ is called *discrete* if $e<f$ implies $[e,f]$ to be finite. Let $Sc(e)=\{f\in Y\colon e<f,\ |[e,f]|=2\}$ for $e$ in $Y$. A *tree* $Y$ means a semilattice such that, for all $e, f, g\in Y$, if $e\leqq g$ and $f\leqq g$ then either $e\leqq f$ or $f\leqq e$.

Corollary 7. *Let $Y$ be a discrete tree. Then $T_Y$ is $E$-disjunctive if and only if there is no element $e$ in $Y$ satisfying*

$$|Sc(e)| = 1, \tag{7}$$

$$Ye \cong Yp \quad \textit{implies} \quad |Sc(p)| \leqq 1. \tag{8}$$

Proof. Sufficiency. Suppose $T_Y$ is not $E$-disjunctive. Then (5) holds for some $e, f$, $e<f$, in $Y$. We may assume that $f\in Sc(e)$. Now suppose that there exists $g\in Sc(e)$ such that $g\neq f$. Define $\eta$: $Yf\cong Yg$ by $x\eta=g$ if $x=f$, and $x\eta=x$ if

$x \leqq e$. Define $\zeta = 1_{(Yf)}$ and $v = f$. Though $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5), we obtain $(\eta|Yfv) \neq (\zeta|Yfv)$, a contradiction. Hence we have $|Sc(e)| = 1$.

Next assume that $Ye \cong Yp$ and $q, r \in Sc(p)$; $q \neq r$. Define $v = f$ and $\zeta$: $Yf \cong Yq$. Define $\eta$: $Yf \cong Yr$ by $x\eta = r$ if $x = f$, and $x\eta = x\zeta$ if $x \leqq e$. $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5), but we find $(\eta|Yfv) \neq (\zeta|Yfv)$. Hence $|Sc(p)| \leqq 1$.

Necessity. Let $Sc(e) = \{f\}$. We shall prove that $\{e, f\}$ satisfies (5). Suppose that $\{\eta, \zeta, v\}$, $\eta, \zeta \in T_Y$; satisfies the hypothesis in (5). From $Sc(e) = \{f\}$, we have $fv \leqq e$ or $fv = f$. In case $fv \leqq e$ we obtain $ev = fv$, hence $(\eta|Yfv) = (\zeta|Yfv)$. Assume $fv = f$. Then we find $e < f \leqq v$, and thus $ev = e$. Since $(\eta|Ye) = (\zeta|Ye)$, we can put $p = e\eta = e\zeta$. If $(\eta|Yfv) \neq (\zeta|Yfv)$, $f\eta = r$, $f\zeta = q$, then we obtain $r, q \in Sc(p)$, $r \neq q$, contrary to (8). Hence we conclude that $(\eta|Yfv) = (\zeta|Yfv)$.

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