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On the strong nilstufe of rank two torsion free groups

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1. Introduction

SZELE [7] defined the nilstufe of a group G to be n, n a positive integer, if there exists an associative ring R with additive group G such that $R^n \neq 0$, but for every associative ring R with additive group G the equality $R^{n+1}=0$ holds. If there exists no such positive integer n, we will say that G has nilstufe ∞. FEIGELSTOCK [2] defines the strong nilstufe in a similar manner but allows non-associative rings on G. The nilstufe and strong nilstufe of G will be denoted by $n(G)$ and $N(G)$, respectively.

Unless otherwise stated, all groups in this paper are abelian, rank two torsion-free with addition denoting the group operation. A multiplication on a group G is meant to be the multiplication of a ring R with additive group G.

In this note we study $N(G)$ by classifying G according to the cardinality of the type set, $T(G)$, of G. Here the type set of G means the set of types $t(g)$ of non-zero elements g in G. (See [3], p. 109, for a definition of type.)

By [5] if G is a rank two torsion-free non-nil group (i.e. $N(G)>1$), then the cardinality of $T(G)$ is at most three. In this work we will get the following results for non-nil rank two torsion-free groups:

(i) If the cardinality of $T(G)$ is equal to one then the type must be idempotent and $N(G)=\infty$.

(ii) If the cardinality of $T(G)$ is equal to two then

(a) if G is indecomposable then $N(G)=2$,

(b) if G is decomposable and $T(G)=\{t_1, t_2\}$ such that $t_1<t_2$, $t_1 t_2>t_2$ and $t_1^2 \neq t_1$, $t_2^2 \neq t_2$ then $N(G)=2$,

(c) in the remaining cases $N(G)=\infty$.

(iii) If the cardinality of $T(G)$ is equal to three then $N(G)=\infty$.

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Let x, y be independent elements of a group G of rank two. Each element w of G has a unique representation $w=ux+vy$, where u, v are rational numbers. Let

$$U = \{u \in Q \mid ux+vy \in G \text{ for some } v \in Q\}, \quad U_0 = \{u_0 \in Q \mid u_0 x \in G\},$$
$$V = \{v \in Q \mid ux+vy \in G \text{ for some } u \in G\}, \quad V_0 = \{v_0 \in Q \mid v_0 y \in G\}.$$

Clearly, U_0, V_0 are subgroups of U, V respectively, which are isomorphic to the pure subgroups $\langle x\rangle^*$ and $\langle y\rangle^*$ of G. ($\langle x\rangle^*$ denotes the pure subgroup of G generated by x.) We call U, U_0, V, V_0 the groups of rank one belonging to the independent set $\{x, y\}$ of G.

Proposition 1 ([1], p. 107). *Let G be a torsion-free abelian group of rank two. If U, U_0, V, V_0 are the groups of rank one belonging to $\{x, y\}$, then $U/U_0 \cong V/V_0$.*

Proposition 2 ([3], p. 114). *Let C be a pure subgroup of the torsion-free group A such that*

(a) *A/C is completely decomposable and homogeneous of type t,*

(b) *all the elements in A but not in C are of type t,*

then C is a direct summand of A.

Proposition 3. *Let A be a torsion-free group of rank two, $T(A)=\{t_1, t_2\}$ and $t_1<t_2$. Let $\{x, y\}$ be an independent set of A such that $t(x)=t_1$, $t(y)=t_2$. Assume U, U_0, V, V_0 are the rank one groups belonging to $\{x, y\}$. If $t(U_0)=t(U)$ then $\langle y\rangle^*$ is a direct summand of A. In particular, if $kU \leqq U_0$ or $kV \leqq V_0$ for some integer $k \neq 0$, then A is decomposable.*

Proof. We have $A/\langle y\rangle^* \cong U$, hence $t(A/\langle y\rangle^*)=t(U)$. Let a be in A but not in $\langle y\rangle^*$; then $t(a)=t_1$. By assumption we have $t(U)=t(U_0)=t_1$, therefore the type of all elements in A but not in $\langle y\rangle^*$ are equal to $t(U)=t(A/\langle y\rangle^*)$. By Proposition 2, $\langle y\rangle^*$ is a direct summand of A. In particular, if $kU \leqq U_0$ or $kV \leqq V_0$ for some integer $k \neq 0$, then because of $U/U_0 \cong V/V_0$ we have that $t(U)=t(U_0)$, and hence A is decomposable.

2. One-element type set

For this case we first assume that the group is indecomposable.

Proposition 4. *If G is an indecomposable and homogeneous group then any non-zero element of $E(G)$, the endomorphism monoid of G, is monic.*

Proof. Let $\varphi \in E(G)$, $0 \neq \operatorname{Ker} \varphi \neq G$. Then $r(G/\operatorname{Ker} \varphi)=1$ since $r(G)=2$ and $\operatorname{Ker} \varphi$ is a pure subgroup of G. We have $G/\operatorname{Ker} \varphi \cong \operatorname{Im} \varphi < G$. Assume $\bar{g}=$

$=g+\text{Ker}\,\varphi\in G/\text{Ker}\,\varphi$ and $g\notin\text{Ker}\,\varphi$. Then

$$t(\bar{g}) = t(G/\text{Ker}\,\varphi) = t(\text{Im}\,\varphi) \geqq t(G) = t(g).$$

On the other hand, $t(\bar{g})\leqq t(g)$, therefore $t(\bar{g})=t(g)$. Hence by Proposition 2 Ker φ is a summand of G. But G is indecomposable, so Ker $\varphi=0$, and φ is monic.

Lemma 1. *If G is an indecomposable and homogeneous group then any non-trivial ring over G is without zero divisors.*

Proof. Let $(G, *)$ be a ring over G and let $xy=0$ for some $x, y\in G$, $x\neq 0$, $y\neq 0$. By Proposition 4 any non-trivial element of $E(G)$ is monic. For the left multiplication L_x we have $L_x(y)=xy=0$, which implies that $L_x=0$, so

(1) $$x^2 = L_x(x) = 0.$$

Let $\{x, z\}$ be an independent set of G. Then we have

(2) $$xz = L_x(z) = 0.$$

Furthermore, since the right multiplication R_z is 0 or monic, and $R_z(x)=xz=0$, therefore $R_z=0$. Hence

(3) $$z^2 = R_z(z) = 0.$$

Taking now the left multiplication L_z, by (3) we get that L_z is 0, so

(4) $$zx = L_z(x) = 0.$$

By assumption $\{x, z\}$ is an independent set of G, consequently by (1), (2), (3) and (4) $(G, *)$ is a trivial ring. This shows that any non-trivial ring over G is without zero divisors.

We conclude from this lemma that, if G is an indecomposable and homogeneous group, then $N(G)=1$ or $N(G)=\infty$.

Now we assume G is decomposable.

Proposition 5 (Rédei—Szele [4]). *A ring R with rank one torsion-free additive group G is either an associative domain, or $R^2=0$. R is an integral domain if and only if $t(G)$ is idempotent.*

Proposition 6. *Let $G=H\oplus K$ and $r(H)=r(K)=1$. If $t(H)$ is idempotent then $N(G)=\infty$.*

Proof. If $t(H)$ is idempotent then by Proposition 5, H is an associative integral domain, whence $N(H)=\infty$. We define a ring $(G, *)$ by putting

$$(h, k)*(h', k') = (hh', 0).$$

This proves that $N(G)=\infty$.

Proposition 7. *Let A, B be torsion-free, homogeneous groups of finite ranks. If* $t(A)>t(B)$ *then* $\operatorname{Hom}(A, B)=0$.

Proof. The fact that homomorphisms are type increasing (i.e. type non-decreasing) yields the proposition.

Lemma 2. *Let $G=K\oplus H$ and let $r(K)=r(H)=1$, $t(H)=t(K)$. Then $N(G)>1$ implies that $t(G)$ is idempotent and $N(G)=\infty$.*

Proof. If $t(H)$ is idempotent then by Proposition 6, $N(G)=\infty$. If $t(H)$ is not idempotent then $t(G\otimes G)=t^2(G)>t(G)$, and by Proposition 7, $\operatorname{Hom}(G\otimes G, G)=0$. We have

$$\operatorname{mult}(G) \cong \operatorname{Hom}(G\otimes G, G),$$

therefore G is a nil group and so $N(G)=1$.

3. Two-element type set

Proposition 8. *If R is a finite rank, torsion-free ring without zero divisors, then R^+ is homogeneous.*

Proof. Let $\{x_1, \ldots, x_r\}$ be an independent subset of R^+. Let x be in R, $x\neq 0$. First we prove that $xx_1, \ldots, xx_r$ are independent. Suppose not. Then there exist integers $a_1, \ldots, a_r$ such that $a_1xx_1+\ldots+a_rxx_r=0$, i.e. $x(a_1x_1+\ldots+a_rx_r)=0$; but R has no zero divisors, therefore $a_1x_1+\ldots+a_rx_r=0$, which is a contradiction, since $\{x_1, \ldots, x_r\}$ is an independent set.

Hence if $x\neq 0\neq y$ belong to R, then

$$my = m_1xx_1+\ldots+m_rxx_r = x(m_1x_1+\ldots+m_rx_r)$$

implies that $t(y)\geqq t(x)$, and similarly $nx=n_1yx_1+\ldots+n_ryx_r=y(n_1x_1+\ldots+n_rx_r)$ implies that $t(x)\geqq t(y)$. Thus $t(x)=t(y)$, consequently R is homogeneous.

Lemma 3. *Let G be a torsion-free indecomposable abelian group of rank two. Let $T(G)=\{t_1, t_2\}$ be such that $t_1<t_2$. If $\{x, y\}$ is an independent set such that $t(x)=t_1$, $t(y)=t_2$, then all non-trivial rings on G satisfy the following multiplication table:*

$$x^2 = by, \quad xy = yx = y^2 = 0, \quad b \text{ is a rational number.}$$

Proof. Let $(G, *)$ be a non-trivial ring over G. Since $t_1<t_2$, in general we have

$$x^2 = ax+by, \quad xy = cy, \quad yx = dy, \quad y^2 = ey.$$

We are going to prove that $a=c=d=e=0$.

Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. We claim $xy=yx$. If not, then $c\neq d$, and for an arbitrary element $g=ux+vy$ of G,

$$gx = ux^2+vyx, \quad xg = ux^2+vxy, \quad gx-xg = v(d-c)y,$$

implying that $(d-c)v\in V_0$ for all $v\in V$. Hence there is an integer $k\neq 0$ such that $kV\leqq V_0$. Now by Proposition 3, $\langle y\rangle^*$ is a direct summand of G, which is a contradiction. Hence $c=d$ and $xy=yx=cy$.

We claim that $a=0$. If not, take two arbitrary elements $g_1=ux+vy$, $g_2=rx+sy$ of G. Then

$$g_1 g_2 = urx^2+(su+rv)xy+vsy^2 = aurx+(urb+suc+rvc+vse)y.$$

This implies that $aU^2\leqq U\leqq U^2$, whence $t(U)=t(U^2)$. Consequently,

(1) if $a\neq 0$ then $t(U)$ is idempotent.

G is not homogeneous, hence by Proposition 8, G should have two non-zero elements $X=rx+sy$, $Y=\alpha x+\beta y$ such that $XY=0$, i.e.

$$XY = (rx+sy)(\alpha x+\beta y) = a\alpha rx+(\alpha rb+s\alpha c+r\beta c+\beta se)y = 0.$$

Since x, y are independent elements, $a\alpha r=0$. By assumption $a\neq 0$, hence we should have one of the following cases:

(i) $\alpha=0, \quad r=0$, (ii) $\alpha=0, \quad r\neq 0$, (iii) $\alpha\neq 0, \quad r=0$.

In case (i), s and β must be non-zero, as $X\neq 0$, $Y\neq 0$. Hence $0=XY=s\beta y^2= =s\beta ey$, which implies that $e=0$.

In case (ii), $\{X=rx+sy, y\}$ is an independent set of G, and

$$0 = XY = (rx+sy)(\beta y) = \beta(rx+sy)y,$$

since $\alpha=0$. However, $Y\neq 0$, therefore $\beta\neq 0$, so that

(2) $$Xy = (rx+sy)y = 0.$$

Let H, H_0, F, F_0 be the rank one groups belonging to $\{X, y\}$, and let $g=hX+fy$ be an arbitrary element of G where $h\in H, f\in F$. By (2) and by the assumption $y^2=ey$ we have

$$gy = hXy+fy^2 = efy,$$

so we conclude that ef belongs to F_0 for all f in F. If $e\neq 0$ then there is an integer $k\neq 0$ such that $kF\leqq F_0$, so by Proposition 3, $\langle y\rangle^*$ is a direct summand of G, contradicting the indecomposability of G. Hence $e=0$.

Similarly, in case (iii) we also conclude that $e=0$. Therefore,

(3) if $a\neq 0$ then $e=0$.

Let $g=ux+vy$ be an arbitrary element of G with $u\in U$, $v\in V$. By (3) we have $gy=uxy+vy^2=cuy$, so if c is not zero then $cU\leqq V_0$, hence

(4) $$t(U)\leqq t(V_0).$$

Now, using (1) and (4) we prove that $t(U)=t(U_0)$. By (1), $t(U)$ is idempotent, therefore $h_p^U(1)=0$ or ∞ except for finitely many prime numbers. $U_0\leqq U$ implies that $t(U_0)\leqq t(U)$, so that $h_p^U(1)=0$ implies $h_p^{U_0}(1)=0$ and $h_p^U(1)<\infty$ implies $h_p^{U_0}(1)<\infty$. It remains to prove that $h_p^{U_0}(1)=\infty$ if $h_p^U(1)=\infty$. Let $1/p^n\in U$ and $h_p^U(1)=\infty$. Then by the definition of U there is $K/m\in V$ such that $g=(1/p^n)x+ +(K/m)y\in G$. Let $m=m'p^i$ where $(m',p)=1$. Then

$$g=(1/p^n)x+(K/m'p^i)y,\quad m'g=(m'/p^n)x+(K/p^i)y,\quad (m'g-K(y/p^i))=(m'/p^n)x.$$

By (4), $1/p^i\in V_0$, so that $1/p^n\in U_0$. This is correct for all $n<\infty$, hence $h_p^{U_0}(1)=\infty$, so we conclude that $t(U)\leqq t(U_0)$. But $t(U_0)\leqq t(U)$, therefore $t(U_0)=t(U)$. By Proposition 3, $\langle y\rangle^*$ will be a direct summand of G which is in contradiction with indecomposability. Consequently $c=0$.

By assuming $a\neq 0$ we got $c=0$ and $e=0$, that is $x^2=ax+by$, $xy=yx= =y^2=0$. Thus $\{z=ax+by, y\}$ is an independent set of G, and $z^2=a^2x^2+b^2y^2+ +2abxy=a^2x^2=a^2z$, $zy=yz=y^2=0$. Let W, W_0, T, T_0 be the rank one groups belonging to $\{z,y\}$. Let $g=wz+ty$ be an arbitrary element of G and $w\in W$, $t\in T$. Then $gz=wz^2=a^2wz$.

Since we supposed $a\neq 0$, we have $a^2W\leqq W_0\leqq W$, hence $t(W_0)=t(W)$. Again by Proposition 3, $\langle y\rangle^*$ is a direct summand of G which is a contradiction. All contradictions are due to the assumption $a\neq 0$. Consequently $a=0$.

So far we proved that

$$x^2=by,\quad xy=yx=cy,\quad y^2=ey.$$

Let $g=ux+vy$ be an arbitrary element of G. Then

$$gx=uby+vcy=(ub+cv)y,\quad gy=cuy+evy=(cu+ev)y,$$

hence

$$\begin{aligned} ub+cv&=v_0\\ cu+ev&=v_0'\end{aligned}\quad \text{for some } v_0,\ v_0' \text{ in } V_0.$$

This implies that $(c^2-be)v=v_0''$ for some v_0'' in V_0. If $c^2-be\neq 0$ then there is an integer $k\neq 0$ such that $kU\leqq U_0$, which implies by Proposition 3 that $\langle y\rangle^*$ is a direct summand of G. This is a contradiction. Therefore

(5) $$c^2-be=0.$$

If $b=0$ then $gy=uxy+vy^2=evy$. Again this is a contradiction, hence

(6) $$b\neq 0.$$

By (5) and (6)

$$e = 0 \quad \text{if and only if} \quad c = 0. \tag{7}$$

If $e \neq 0$ and $c \neq 0$ then $\{z_1 = -cx + by, y\}$ is an independent set of G. We get

$$z_1^2 = (-cx+by)^2 = c^2x^2 + b^2y^2 - 2cbxy = c^2by + eb^2y - 2c^2by =$$
$$= b(eb - c^2)y = 0 \quad \text{(by (5))},$$
$$z_1y = yz_1 = -cxy + by^2 = -c^2y + eby = (-c^2 + eb)y = 0 \quad \text{(by (5))},$$
$$y^2 = ey.$$

Let M, M_0, N, N_0 be the rank one groups belonging to $\{z_1, y\}$, and let $g = mz_1 + ny$ be an arbitrary element of G where $m \in M$ and $n \in N$. Then $gy = ny^2 = eny$, hence $eN \leqq N_0$. It follows now that there is an integer $k \neq 0$ such that $kN \leqq N_0$, so by Proposition 3, $\langle y \rangle^*$ is a direct summand of G, contradicting the indecomposability of G. Therefore $c = 0$ or $e = 0$, whence by (7) $c = 0$ and $e = 0$, completing the proof of Lemma 3.

Remark 1. In case no element of $T(G) = \{t_1, t_2\}$ is idempotent, let $\{x, y\}$ be an independent set of G such that $t(x) = t_1$, $t(y) = t_2$; $t_1 < t_2$ and $t_1 t_2 \neq t_2$. Then $x^2 = by$, $xy = yx = y^2 = 0$ for any ring over G.

Theorem 1. *Let $T(G) = \{t_1, t_2\}$, $t_1 < t_2$, and let $\{x, y\}$ be an independent set of G such that $t(x) = t_1$, $t(y) = t_2$. Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. If G is either indecomposable or neither t_1 nor t_2 is idempotent and $t_1 t_2 > t_2$, then G is a non-nil group if and only if $t(U^2) \leqq t(V_0)$.*

Proof. Suppose G is a non-nil group. By Lemma 3 and Remark 1 we have

$$x^2 = by, \quad xy = yx = y^2 = 0, \quad b \neq 0.$$

Let $g = ux + vy$, $h = rx + sy$ be arbitrary elements of G with $u, r \in U$ and $v, s \in V$. Then $gh = bury$, which implies that $bU^2 \leqq V_0$, that is $t(U^2) \leqq t(V_0)$.

Conversely, if $t(U^2) \leqq t(V_0)$ then there is an integer $b \neq 0$ such that $bU^2 \leqq V_0$. Let $g = ux + vy$, $h = rx + sy$ be arbitrary elements of G, and define a multiplication over G by $gh = bury$. This multiplication is a ring over G, hence G is a non-nil group.

Remark 2. Let G be decomposable and let $T(G) = \{t_1, t_2\}$ be such that $t_1^2 \neq t_1$, $t_2^2 \neq t_2$. In [6] it has been proved that $t_1 t_2 = t_2$ implies $N(G) = \infty$.

Remark 3. Under the hypothesis of Theorem 1, $N(G) = 1$ or 2. $N(G) = 2$ if and only if $t(U^2) \leqq t(V_0)$.

Remark 4. If $G = H \oplus K$ and at least one of $t(H)$ and $t(K)$ is idempotent then by Proposition 6, $N(G) = \infty$.

4. At least three-element type set

Proposition 9. *Let G be a torsion-free group of rank two and let $T(G)=\{t_0, t_1, t_2\}$. Let $x, y \in G$ be such that $t(x)=t_1$ and $t(y)=t_2$. Suppose that $t_0<t_1$, $t_0<t_2$. If t_1, t_2 are incomparable, then for any ring on G we have $x^2=ax$, $y^2=by$, $xy=yx=0$ for some $a, b \in Q$.*

Proof. The set $G(t_1)$ of elements g in G whose types are $\geqq t_1$ form a pure subgroup of G ([3], p. 109). Let $z \in G$ be such that $t(z)=t_0$. Then $z \notin G(t_1)$. Because of the purity of $G(t_1)$, $r[G(t_1)]=1$. Since $t(x^2) \geqq t(x)=t_1$, we have $x^2, x \in G(t_1)$, thus x^2 and x are dependent elements, that is $x^2=ax$ for some $a \in Q$. Similarly we conclude that $y^2=by$ for some $b \in Q$. By the same token $t(yx) \geqq t(x)$ implies that $yx, x \in G(t_1)$, hence $yx=ex$ for some $e \in Q$. Similarly we deduce that $yx=fy$ for some $f \in Q$. If $yx \neq 0$ then $t(yx)=t(x)=t(y)$. This contradicts our hypothesis, therefore $yx=0$. In the same way we conclude that $xy=0$.

If the cardinality of $T(G)$ is greater than three, then by [5] G is nil group, that is, $N(G)=1$.

If the cardinality of $T(G)$ is equal to three then by [5] $T(G)$ has one minimal and two maximal elements; let $\{x, y\}$ be an independent set of G such that $t(x)$ and $t(y)$ are maximal. By Proposition 9, for any ring over G we have $x^2=ax$, $xy=yx=0$, $y^2=by$, where a, b are rational numbers. If G is non-nil, then a or b is non-zero, say $a \neq 0$. Then $x^n=a^{n-1}x$, hence there is no integer n such that $x^n=0$. This implies that $N(G)=\infty$.

Theorem 2. *Let G be a rank two torsion-free group and let $T(G)=\{t_0, t_1, t_2\}$ be such that $t_0<t_1$, $t_0<t_2$. Let $\{x, y\}$ be an independent set such that $t(x)=t_1$, $t(y)=t_2$. Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. Then G is a non-nil group if and only if either $t(U_0)=t(U)$ and $t(U_0)$ is idempotent or $t(V_0)=t(V)$ and $t(V_0)$ is idempotent.*

Proof. Let G be a non-nil group. Then $x^2=ax$, $xy=yx=0$, $y^2=by$ and a or b is non-zero. We assume $a \neq 0$. Let $g=ux+vy$ be an arbitrary element of G where $u \in U$ and $v \in V$. Then $gx=ux^2+vyx=aux$. This implies that $au \in U_0$ for all $u \in U$, so it follows that $aU \subseteq U_0$. However, $U_0 \subseteq U$, therefore $t(U)=t(U_0)$. Furthermore, since $a \neq 0$, we have $x^2 \neq 0$, hence $t(U_0)$ is idempotent.

Conversely, if $t(U)=t(U_0)$ and $t(U_0)$ is idempotent then there is an integer m such that $mU \subseteq U_0$. Let $g=ux+vy$, $h=rx+sy$ be arbitrary elements of G, and define a multiplication over G by $gh=m^2urx$. This multiplication is a ring over G, therefore G is a non-nil group.

5. Concluding remarks

(a) Lemma 1 and Lemma 2 imply that if G is a homogeneous torsion-free group of rank two then $n(G)=N(G)$.

(b) Let G be a torsion-free group of rank two and let $T(G)=\{t_1, t_2\}$ be such that $t_1<t_2$. Then Lemma 3 shows that if G is indecomposable then $n(G)=N(G)$. However, in the decomposable case it has been shown in [6] by an example that, in general, $n(G)$ and $N(G)$ are not equal.

(c) If G is a torsion-free group of rank two with $|T(G)|\geqq 3$ then $n(G)=N(G)$.

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