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On a problem of Baldwin and Berman

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Introduction. In their paper [1] BALDWIN and BERMAN proved that there exists an algebra where the lengths of the Mal'cev chains are at most two, but in an appropriate factor algebra there exists no bound for the length at all. (For the definitions see the next part of this paper.) In the same paper they asked the following: Is it true for any algebra in which the length of any Mal'cev chain is at most one, that its factors have the same property. This question can be motivated by the following. If not only the length is one, but also the "direction" is given, then for each a and b in the algebra there is a unary algebraic function f, such that $f(a)=b$ and $f(b)=a$. This calls force, however, congruence permutability, which yields the positive answer. However, we shall give in this short paper a negative answer to this question. Instead of giving the construction only we shall show what is beyond it.

Rephrasing the problem. We have to deal with congruences of an algebra and some factors of it. The following is well known:

Proposition 1. *Let $\mathfrak{A}=(A; F)$ be any algebra, S the set of unary algebraic functions of $\mathfrak{A}$ and G the set of nonconstant unary algebraic functions (in the sense of Grätzer). Then S is a semigroup under composition, and the following hold:*

1. *The congruence lattices of $(A; F)$, $(A; S)$, $(A; G)$ coincide.*
2. *The same holds for $(A/\theta; F)$, $(A/\theta; S)$, $(A/\theta; G)$, where θ is any congruence of $\mathfrak{A}$.*

Thus, from now on, we have to deal with unary algebras only. We shall associate to each unary algebra a colored multigraph which will closely keep track of Mal'cev's lemma.

Definition 2. Let $(A; S)$ be a unary algebra. We associate to it a colored undirected multigraph $M(A; S)$ which will be called the Mal'cev graph of the

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algebra. The vertex set of this graph will be A. For distinct vertices a and b the edges of color (a, b) will be all pairs of the form $(f(a), f(b))$. (The edges may have more colors, at each vertex there is a loop of each color.)

It is clear, that the Mal'cev graph of a factor algebra is equal to the factor graph of the original Mal'cev graph, and the kernels are the same. The classes of the principal congruence $\theta(a, b)$ are the connected components of the (a, b)-colored graph. If one of the (a, b)-colored components has diameter at least n, this means that we need a Mal'cev chain of length n when considering $\theta(a, b)$. If n is the maximal length of the Mal'cev chains then the maximal diameter of the components of the (a, b)-colored graph is n. Otherwise, this coloring has infinite diameter.

In their example Baldwin and Berman construct a Mal'cev graph in which each color has diameter at most two, but a factor of it has infinite diameter in one color.

Now we reformulate their question:

Question. Suppose a Mal'cev graph has diameter one in each color. Does each factor of it possess the same property?

Theorem. *There exists a Mal'cev graph such that it has diameter one in each color and a factor of it has infinite diameter in some color.*

Next, we should like to mention, that a Mal'cev graph has diameter one in a color if and only if in this coloring all components are complete.

If the algebra $(A; G)$ is "connected", then it is very hard to describe the action of G. For this reason we investigate the case when it is not connected.

Proposition 3. *Let $(A; G)$ be a unary algebra. Assume G consists of non-constant operations and the diameter of $M(A; G)$ is equal to 1. If A is the disjoint union of B and C such that both $(B; G)$ and $(C; G)$ are subalgebras of $(A; G)$, then G acts semiregularly on A (i.e. for each $a \in A$ and $f, g \in G$, $f(a)=g(a)$ implies $f=g$).*

Proof. Suppose we have $f(a)=g(a)$ for $a \in A$ and $f, g \in G$. We shall show, if a belongs to either one of B and C then the action of f and g are the same on the other among B and C. Thus, a repetition of the argument will prove the statement. Indeed, suppose $a \in C$ and consider any $b \in B$. Since both $(f(a), f(b))$ and $(g(a), g(b))$ have color (a, b) and $f(a)=g(a)$, $(f(b), g(b))$ must have, also, the color (a, b), because of the condition on the diameter. However, $(C; G)$ is a subalgebra, therefore no element of G sends a to either of $f(b)$ and $g(b)$, for they both belong to B. Hence, we must have $f(b)=g(b)$, as it was stated.

Proposition 4. *Let $(A; S)$ be a unary algebra and let G be the subset of the semigroup S consisting of the non-constant operations and suppose G acts semi-regularly on A. Then, $M(A; G)$ as well as $M(A; S)$ have diameter 1 in each color if and only if G is a group the elements of which have order at most three.*

Proof. Suppose the Mal'cev graph has diameter 1. First of all we mention that all invertible elements of S belong, obviously, to G. Now, let $f \in G$ be such that $f^2 \neq 1$. Since f is not the identity, semiregularity gives us that for each $a \in A$ the element $b = f(a)$ differs from both a and $c = f(b)$. Again by semiregularity $f^2 \neq 1$ implies $c \neq a$. By the condition on the diameter there exists a $g \in S$ such that $\{g(a), g(b)\}$ equals $\{a, c\}$. Since $b \notin \{a, c\}$, $g(b) \neq b$, i.e. $g \neq 1$, yielding $g(a) \neq a$. Thus, we have $g(a) = c = f^2(a)$ and $gf(a) = g(b) = a$. Now, semiregularity implies $g = f^2$ and $1 = gf = f^2 f = f^3$, i.e. all elements of G have order at most three and they are invertible. Since the invertible elements of a monoid form a group, G satisfies the condition.

Let us suppose, conversely, that the condition on G is satisfied. We have to prove that if (a, b) and (b, c) have some color then so has (a, c). Let us notice that (u, v) always has color $(f(u), f(v))$ for $f \in G$, since G is a group. Hence, we may suppose that the color in question is (a, b). If a, b and c are not all different the statement is clear, so we may assume they are different. Then the condition on the color implies the existence of an $f \in G$ such that $\{b, c\} = \{f(a), f(b)\}$ and the semiregularity yields $f(b) \neq b$, since $f \neq 1$. Hence, we have $b = f(a)$ and $c = f(b) = f(f(a))$. Thus $f^2 \neq 1$ so, by the condition on G, we must have $f^3 = 1$, i.e. $f^2(a) = c$ and $f^2(b) = f^3(a) = 1(a) = a$ proving that the graph has diameter 1.

Remark. In Proposition 4 the conditions on the color and on the nonconstant operations are clearly not equivalent without semiregularity, and they do not imply, even together, the semiregularity. The latter is obvious in the case when A is a three-element set and S consists of all permutations and the constants.

The construction. By Proposition 4 the condition on the diameter will be satisfied if we start with two different copies B and C of a group G such that the order of its elements does not exceed three. It is easy to see, that the restriction of any congruence of such a structure to either one of B and C consists of the left cosets of some subgroups. (The action of the elements of G is defined by $f(g) = fg$.) We have to find two suitable subgroups such that in the factor algebra the diameter of the corresponding Mal'cev graph will be infinite. For the time being we are not interested in the condition on the order of the elements of G.

Proposition 5. *Let B and C be two copies of the group G on which the operations are defined by $f(g) = fg$ and let A be the disjoint union of B and C; further let S be the union of G and the constant maps on A. Consider two subgroups H and K of G.*

Then the equivalence relation on A the classes of which are the left cosets of H on B and the left cosets of K on C is a congruence relation on $(A; S)$ the factor algebra by which will be denoted by $(A'; S)$. Further, the graph of $M(A'; S)$ colored by (H, K) has the following properties:

(i) *Two different vertices of the graph are connected iff they are of the form fH and fK for suitable $f \in G$.*

(ii) *For any subset M of G the set of those elements which are connected with some element bH $(b \in M)$ consists of the elements cK, $c \in MHK$.*

(iii) *If H and K together generate G then the graph is connected.*

(iv) *If, in addition, no power of HK is equal to G then the graph has infinite diameter.*

Proof. The statement on the equivalence relation just as property (i) are trivial.

As far as property (ii) is concerned, it is, clearly, enough to prove it for singletons. By condition (i) bH is connected exactly with those fK for which f is contained in bH, i.e., which are contained in bHK.

Property (iii) is an obvious consequence of property (ii).

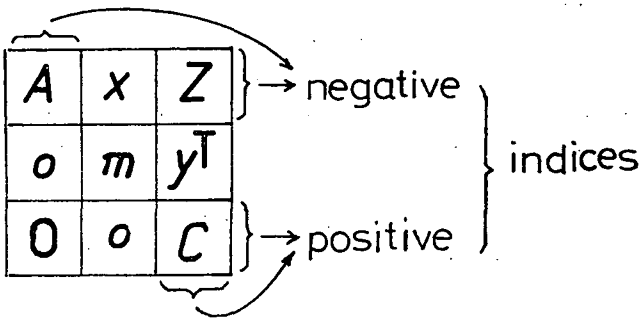
By property (ii) the distance between H and some other element is exactly k iff it is contained in a product $HKHKH\dots$ of $k+1$ factors but not in a product of $k-1$ factors. This proves, obviously, property (iv).

Remark. Just to answer the original question we should not need infinite diameter in the factor, diameter two would do as well. The above idea could give, trivially, such an example. Namely, let $\{1, 2, 3, 4\}$ be the underlying set of the algebra and let the permutation $(12)(34)$ be the only operation and factor out by the congruence generated by $(1, 2)$.

Now, in virtue of Proposition 5, we have to find, only, a group G generated by its subgroups H and K such that no power of HK equals G and the order of the elements does not exceed three. First, we shall construct a group satisfying all the conditions but the one about the order of the elements and then we shall choose it so that the group will have exponent three (i.e., all non-units have order three). In what follows, we shall give a general construction which will give the group in question as a special case. (The form of the presentation of this example was suggested by Ágnes Szendrei.)

We start with infinite quadratic upper triangular matrices over a field F having finitely many off-diagonal non-zero entries. The rows and the columns are indexed by the integers. The place containing the entry indexed by 0, 0 will be called the middle of the matrix. The row and the column containing the middle will be called the middle row and middle column, respectively. These matrices form, obviously, a semigroup under the multiplication.

Now, we shall consider a subsemigroup. We can partite the matrices into nine blocks according to the signs of the indices:



Here, A and C are upper triangular matrices indexed by the negative and positive integers, respectively. x and y are column vectors, indexed by the negative integers, and $^{\mathrm{T}}$ stands for the transpose. The element m is the middle entry of the matrix. We shall consider only those matrices where $m=1$ and both A and C are the units of the corresponding matrix rings. Let (x), $(y)^{\mathrm{T}}$ and (Z) denote those matrices which we get when omitting all non-zero entries of the original matrix except those which belong to x, y^{T} and Z, respectively. Thus, the matrices in question are of the form $I+(x)+(y)^{\mathrm{T}}+(Z)$, where I denotes the identity matrix. Computing the product of matrices of this form, we get:

$$\big(I+(x)+(y)^{\mathrm{T}}+(Z)\big)\big(I+(u)+(v)^{\mathrm{T}}+(W)\big) = \big(I+(x+u)+(y+v)^{\mathrm{T}}+(Z+W+x\otimes v)\big).$$

Hence, these matrices form a semigroup G, and I is the unit element of this semigroup. An easy computation shows that $(I+(x)+(y)^{\mathrm{T}}+(Z))^{-1}=I+(-x)+(-y)^{\mathrm{T}}+$ $+(-Z+x\otimes y)$. Since this element belongs to G, G is a group.

It is easy to see, that the elements of the form $I+(x)$, resp. $I+(y)^{\mathrm{T}}$, form a subgroup H, resp. K. In virtue of the equations $\big(I+(x)\big)\big(I+(y)^{\mathrm{T}}\big)\big(I+(-x)\big)\big(I+(-y)^{\mathrm{T}}\big)=$ $=I+(x\otimes y)$ and $I+(x)+(y)^{\mathrm{T}}+\sum\{x_i\otimes y_i|1\leqq i\leqq n\}=\big(I+(y)^{\mathrm{T}}\big)\big(I+(x_1\otimes y_1)\big)\cdot\ldots$ $\ldots\cdot\big(I+(x_n\otimes y_n)\big)\big(I+(x)\big)$ we get that the subgroup generated by $H\cup K$ contains all elements of the form $I+(x)+(y)^{\mathrm{T}}+(Z)$, where Z is a sum of tensor products. Since any matrix having a single non-zero entry is the tensor product of two suitable vectors and the matrix ring in question has elements with finitely many non-zero off-diagonal entries only, the subgroups H and K generate G.

The elements of $H\cup K$ are, clearly, of the form $I+(x)+(y)^{\mathrm{T}}$. We have $\Pi\{(I+(x_i)+(y_i)^{\mathrm{T}})|1\leqq i\leqq n\}=I+(x)+(y)^{\mathrm{T}}+(Z)$ with $x=\sum x_i$, $y=\sum y_i$ and $Z=\sum\{x_i\otimes y_j|1\leqq i<j\leqq n\}$ yielding that the rank of Z is at most $\binom{n}{2}$. Since G has elements $I+(x)+(y)^{\mathrm{T}}+(Z)$ such that the rank of Z is greater than a given integer, no power of $H\cup K$ is equal to G.

The only thing left is to choose the field F so that the exponent of G be equal to three. This is fulfilled whenever F has characteristic three, for $(I+(x)+(y)^{\mathrm{T}}+(Z))^3=$ $=(I+3(x)+3(y)^{\mathrm{T}}+3(Z+x\otimes y))$.

Problems. 1. Describe all the Mal'cev graphs with diameter 1; or at least those all factors of which have diameter 1, as well.

2. Does there exist a Mal'cev graph with diameter 1 having a factor with unbounded diameter in some color such that each component in this color has finite diameter?

3. Does there exist a Mal'cev graph with diameter 1 having a factor in which each color has finite diameter but the supremum of the diameters is not finite?

Reference

[1] J. T. Baldwin and J. Berman, Definable principal congruence relations: Kith and kin, *Acta Sci. Math.*, 44 (1982), 255—270.

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