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Techniques of Finsler geometry in the theory of vector bundles

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A topical problem in geometry is the study of the differential geometry of vector bundles. For this study the classical methods are not convenient enough, because of the very complicated analytical expressions of some important geometrical objects defined on the total space of a vector bundle.

Using the ideas from Finsler geometry [10, 11, 12, 13] we can considerably simplify the theory. For this, the notion of Finsler connection on the total space E of a vector bundle $\xi=(E, \pi, M)$ is fundamental. To define it firstly we define the notion of the nonlinear connection N on E, then we use this concept to obtain the algebra of Finsler tensor fields on E. A Finsler connection on the space E is a linear connection ∇ on E which preserves by parallelism the horizontal distribution N and the vertical distribution E^V of ξ. ∇ has four local coefficients which have very simple transformation laws to a change of canonical coordinates on E. Also, its torsion and its curvature have a small number of nontrivial components which are the Finsler tensor fields.

By applying this theory to the Riemannian structure G on E we get a Finsler canonical connection which has a simple form.

In this way the geometrical theory of ξ can be constructed without difficulty. This method was used in our talk "Vector bundles Finsler geometry" presented at the second National Seminar on Finsler spaces [13], at the University of Braşov, Romania, February 15, 1982.

1. Vector bundles

Let $\xi=(E, \pi, M)$ be a vector bundle of the class C^∞. We suppose that the total space E has $n+m$ dimensions, the base M has n dimensions and the local fibre $E_p=\pi^{-1}(p)$, $p\in M$, is a real vector space of dimension n.

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If (U_α, Φ_α) is a vectorial chart of ξ, determined by the chart $(U_\alpha, \varphi_\alpha)$ of the base manifold M, then $\Phi_\alpha\colon \pi^{-1}(U_\alpha)\to U_\alpha\times\mathbf{R}^m$ is a diffeomorphism with the property $\mathrm{pr}_1\circ\Phi_\alpha=\pi$ and $\Phi_{\alpha,p}=\Phi_{\alpha|E_p}\colon E_p\to\mathbf{R}^m$ is an isomorphism of vector spaces.

The mappings $g_{\beta\alpha}\colon U_\beta\cap U_\alpha\to Gl(m, R)$, given by $g_{\beta\alpha}(p)=\Phi_{\beta,p}\circ\Phi_{\alpha,p}^{-1}$, $p\in U_\beta\cap\cap U_\alpha$, are the transition functions of ξ. To the vector chart (U_α, Φ_α) these corresponds a chart on the total space E, $(\pi^{-1}(U_\alpha), \tilde{\Phi}_\alpha)$, where $\tilde{\Phi}_\alpha$ is the diffeomorphism $\tilde{\Phi}_\alpha=(\varphi_\alpha\times 1_{\mathbf{R}^m})\circ\Phi_\alpha$. Therefore, the coordinates of the point $u\in\pi^{-1}(U_\alpha)\subset E$ in this chart, for $u=\Phi_\alpha^{-1}(p, y)$, $(p, y)\in U_\alpha\times\mathbf{R}^m$, are

$$\tilde{\Phi}_\alpha(u) = (\varphi_\alpha\times 1_{\mathbf{R}^m})\circ\Phi_\alpha\circ\Phi_\alpha^{-1}(p, y) = (x, y) = (x^1, \dots, x^n, y^1, \dots, y^m)$$

and they are called the canonical coordinates of the point u determinated by the coordinates (x^i) of the point $p=\pi(u)$. Everywhere, the indices $i, j, k, l, \dots$; $i', j', k', l', \dots; i'', j'', k'', l'', \dots$ take the values $1, 2, \dots, n$ and $a, b, c, d, \dots$; $a', b', c', d', \dots; a'', b'', c'', d'', \dots$ take the values $1, 2, \dots, m$.

The transformations of the canonical coordinates $(x, y)\to(x', y')$ of the points of E, are given by:

$$(x', y') = \tilde{\Phi}_\beta\circ\tilde{\Phi}_\alpha^{-1}(x, y) = \big((\varphi_\beta\circ\varphi_\alpha^{-1})(x), g_{\beta\alpha}(p)y\big).$$

We write these transformations in the form:

$$x^{i'} = x^{i'}(x^1, \dots, x^n), \quad y^{a'} = M_a^{a'}(x)y^a, \quad \det\big(M_a^{a'}(x)\big) \neq 0 \tag{1.1}$$

and the inverse transformations:

$$x^i = x^i(x^{1'}, \dots, x^{n'}), \quad y^a = \tilde{M}_{a'}^a(x')y^{a'}, \quad \det\big(\tilde{M}_{a'}^a(x')\big) \neq 0.$$

The map $\pi\colon E\to M$ induces the π^T-morphism of the corresponding tangent bundles $\pi^T\colon T(E)\to T(M)$. Then $VE=\operatorname{Ker}\pi^T$ is a subbundle of $T(E)$ called the *vertical bundle*. VE defines a distribution

$$E^V\colon u\in E \to E_u^V,$$

where E_u^V is the fibre of VE in the point $u\in E$. E^V is called the vertical distribution of ξ. On the open set $\pi^{-1}(U_\alpha)$, $\dfrac{\partial}{\partial y^a}$, $a=1, \dots, m$, is a local basis of the vertical distribution E^V. Therefore E^V is integrable.

Definition 1.1. A non-linear connection on the total space E of ξ is a differentiable distribution $N\colon u\in E\to N_u\subset E_u$, with the property

$$E_u = N_u\oplus E_u^V, \tag{1.2}$$

where E_u is the tangent space in the point u to the manifold E.

It follows:

Proposition 1.1. *If the base M of the vector bundle ξ is paracompact, then, on E, there exist the non-linear connections.*

On $\pi^{-1}(U_\alpha)$ the distribution N has a local basis of the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a(x, y)\frac{\partial}{\partial y^a}; \tag{1.3}$$

$N_i^a(x, y)$ are called the coefficients of the non-linear connection N. Also, N can be locally determined by the Pfaff system $\delta y^a = 0$, where

$$\delta y^a = dy^a + N_i^a(x, y)\, dx^i. \tag{1.4}$$

For every vector field X on E there exists a unique decomposition

$$X = X^H + X^V, \quad X_u^H \in N_u, \quad X_u^V \in E_u^V, \quad \forall u \in E. \tag{1.5}$$

X^H is called the horizontal part and X^V the vertical part of X.

If ω is a 1-form field on E, then we can define the 1-form field ω^H on E by the condition:

$$\omega^H(X) = \omega(X^H), \quad \forall X \in \mathscr{X}(E);$$

ω^H is called the horizontal component of ω. Let $\omega^V = \omega - \omega^H$. Then ω has a unique decomposition

$$\omega = \omega^H + \omega^V; \tag{1.6}$$

ω^V is called the vertical component of ω. We have $\omega^H(X^V) = \omega^V(X^H) = 0$, $\forall X \in \mathscr{X}(E)$.

Let us observe that $\dfrac{\delta}{\delta x^i}$, $i = 1, \dots, n$, being n horizontal independent fields and $\partial/\partial y^a$, $a = 1, \dots, m$, being m vertical independent fields, $(\delta/\delta x^i, \partial/\partial y^a)$ provides a local basis of the module of the vector fields $\mathscr{X}(E)$ and $(dx^i, \delta y^a)$ is a local basis of the module of 1-form fields on E. These bases are dual:

$$\left\langle \frac{\delta}{\delta x^i}, dx^j \right\rangle = \delta_i^j, \ \left\langle \frac{\delta}{\delta x^i}, \delta y^a \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, dx^j \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, \delta y^b \right\rangle = \delta_a^b. \tag{1.7}$$

2. Algebra of Finsler tensor fields, Finsler connections

Definition 2.1. A tensor field t on the total space E of the vector bundle ξ is called a Finsler tensor field of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ if it has the property

$$\begin{aligned} &t(\underset{1}{\omega}, \dots, \underset{p}{\omega}, \underset{1}{X}, \dots, \underset{q}{X}, \underset{p+1}{\omega}, \dots, \underset{p+r}{\omega}, \underset{q+1}{X}, \dots, \underset{q+s}{X}) = \\ &= t(\underset{1}{\omega^H}, \dots, \underset{p}{\omega^H}, \underset{1}{X^H}, \dots, \underset{q}{X^H}, \underset{p+1}{\omega^V}, \dots, \underset{p+r}{\omega^V}, \underset{q+1}{X^V}, \dots, \underset{q+s}{X^V}), \\ &\qquad \forall \underset{\alpha}{\omega} \in \mathscr{X}^*(E), \quad \forall \underset{\beta}{X} \in \mathscr{X}(E). \end{aligned} \tag{2.1}$$

Proposition 2.1. *A Finsler tensor field of the type* $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ *on E has the following local form:*

$$t = t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y)\frac{\delta}{\delta x^{i_1}}\otimes\dots\otimes\frac{\delta}{\delta x^{i_p}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\otimes \tag{2.2}$$

$$\otimes\frac{\partial}{\partial y^{a_1}}\otimes\dots\otimes\frac{\partial}{\partial y^{a_r}}\otimes\delta y^{b_1}\otimes\dots\otimes\delta y^{b_s}.$$

The coordinate transformation (1.1) changes the coefficients $t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y)$ according to the law:

$$t^{i'_1\dots i'_p a'_1\dots a'_r}_{j'_1\dots j'_q b'_1\dots b'_s}(x', y') = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\dots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\dots\frac{\partial x^{j_q}}{\partial x^{j'_q}}M^{a'_1}_{a_1}\dots M^{a'_r}_{a_r}\tilde{M}^{b_1}_{b'_1}\dots\tilde{M}^{b_s}_{b'_s}t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y).$$

Theorem 2.1. *If w is a tensor field on E of the type* (p, q) *then it determines* 2^{p+q} *Finsler tensor fields on E of the type* $\begin{pmatrix} p-r & r \\ q-s & s \end{pmatrix}$ $(r=0, 1, \dots, p; s=0, 1, \dots, q)$.

Proof. The sum

$$w(\underset{1}{\omega^H}+\underset{1}{\omega^V}, \dots, \underset{p}{\omega^H}+\underset{p}{\omega^V}, \underset{1}{X^H}+\underset{1}{X^V}, \dots, \underset{q}{X^H}+\underset{q}{X^V})$$

has 2^{p+q} terms, each of them being a Finsler tensor field of the type mentioned.

The vector field X^H is a Finsler tensor field of the type $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the vector field X^V is a Finsler tensor field of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and ω^H, ω^V are Finsler tensor fields of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

A remarkable local vector field is $y = y^a\frac{\partial}{\partial y^a}$. It is called the intrinsic vector field of ξ.

If $\underset{F}{\mathcal{T}}{}^{pr}_{qs}(E)$ is the $\mathcal{F}(E)$-module of the Finsler tensor fields of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then the $\mathcal{F}(E)$-module

$$\underset{F}{\mathcal{T}}(E) = \bigoplus_{p,q,r,s=0,1,\dots}\underset{F}{\mathcal{T}}{}^{pr}_{qs}(E)$$

and the product tensor is a graded algebra, called the algebra of Finsler tensor fields on E.

Definition 2.2. A Finsler connection on E is a linear connection ∇ on E with the property that the horizontal linear spaces N_u, $u\in E$, of the distribution N are parallel with respect to ∇ and similarly, the vertical linear spaces E^V_u, $u\in E$, are parallel with respect to ∇.

In what follows we shall prove the existence of a Finsler connection on E. We observe that a linear connection ∇ on E is a Finsler connection on E if and only if

(2.3) $$(\nabla_X Y^H)^V = 0, \quad (\nabla_X Y^V)^H = 0, \quad \forall X, Y \in \mathscr{X}(E).$$

Then, we have:

Theorem 2.2. *The following statements are equivalent:*
(a) *∇ is a Finsler connection on E,*
(b) $\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V, \quad \forall X, Y \in \mathscr{X}(E)$,
(c) $\nabla_X \omega = (\nabla_X \omega^H)^H + (\nabla_X \omega^V)^V, \quad \forall \omega \in \mathscr{X}^*(E), \forall X \in \mathscr{X}(E)$.

For a Finsler connection ∇ we put:

(2.4) $$\nabla^H_X Y = \nabla_{X^H} Y, \quad \nabla^V_X Y = \nabla_{X^V} Y, \quad \forall X, Y \in \mathscr{X}(E).$$

The following theorem is easy to prove:

Theorem 2.3. *For any Finsler connection ∇ on E, ∇^H and ∇^V given by* (2.4) *are the covariant derivatives in the algebra $\underset{F}{\mathscr{T}}(E)$.*

∇^H is called the h-covariant derivative and ∇^V is called the v-covariant derivative of the Finsler connection ∇.

Proposition 2.2. *We have:*

(1) $\nabla^H_X f = X^H f, \quad (\nabla^H_X Y^H)^V = 0, \quad (\nabla^H_X Y^V)^H = 0,$

(2) $\nabla^H_X Y = (\nabla^H_X Y^H)^H + (\nabla^H_X Y^V)^V,$

(3) $\nabla^V_X f = X^V f, \quad (\nabla^V_X Y^H)^V = 0, \quad (\nabla^V_X Y^V)^H = 0,$

(4) $\nabla^V_X Y = (\nabla^V_X Y^H)^H + (\nabla^V_X Y^V)^V.$

We have analogous formulas for $\nabla_X \omega$, too.

Theorem 2.4. *If ∇^H and ∇^V are two covariant derivatives in the algebra of Finsler tensor fields $\underset{F}{\mathscr{T}}(E)$, having the properties* (1) *and* (3) *from Proposition* 2.2, *then there exists an unique Finsler connection ∇ on E for which ∇^H is the h-covariant derivative and ∇^V is the v-covariant derivative of ∇.*

If t is a Finsler tensor field of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then for any Finsler connection ∇ the following formulas hold:

$$\nabla_X t = \nabla^H_X t + \nabla^V_X t,$$

$$(\nabla^H_X t)(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) = X^H t(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) - t(\nabla^H_X \underset{1}{\omega}, \ldots, \underset{q+s}{X}) - \ldots - t(\underset{1}{\omega}, \ldots, \nabla^H_X \underset{q+s}{X}),$$

$$(\nabla^V_X t)(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) = X^V t(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) - t(\nabla^V_X \underset{1}{\omega}, \ldots, \underset{q+s}{X}) - \ldots - t(\underset{1}{\omega}, \ldots, \nabla^V_X \underset{q+s}{X}).$$

In the canonical coordinates (x^i, y^a) there exists a well-determined set of differentiable functions on $\pi^{-1}(U)$, $(F^i_{jk}(x,y), F^a_{bk}(x,y), C^i_{ja}(x,y), C^a_{bc}(x,y))$ such that

$$(2.5)\qquad \begin{cases} \nabla^H_{\delta/\delta x^k}\dfrac{\delta}{\delta x^j} = F^i_{jk}(x,y)\dfrac{\delta}{\delta x^i}, & \nabla^H_{\delta/\delta x^k}\dfrac{\partial}{\partial y^b} = F^a_{bk}(x,y)\dfrac{\partial}{\partial y^a},\\ \nabla^V_{\partial/\partial y^c}\dfrac{\delta}{\delta x^j} = C^i_{jc}(x,y)\dfrac{\delta}{\delta x^i}, & \nabla^V_{\partial/\partial y^c}\dfrac{\partial}{\partial y^b} = C^a_{bc}(x,y)\dfrac{\partial}{\partial y^a}.\end{cases}$$

Let us consider, for simplicity, the Finsler tensor field

$$K = K^{ia}_{jb}(x,y)\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b.$$

Then, if we put for $X^H = X^k\dfrac{\delta}{\delta x^k}$, $X^V = X^c\dfrac{\partial}{\partial y^c}$,

$$\nabla^H_X K = X^k K^{ia}_{jb|k}\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b,\qquad \nabla^V_X K = X^c K^{ia}_{jb}|_c\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b,$$

we have:

$$(2.6)\qquad \begin{aligned} K^{ia}_{jb|k} &= \frac{\delta K^{ia}_{jb}}{\delta x^k} + F^i_{hk}K^{ha}_{jb} - F^h_{jk}K^{ia}_{hb} + F^a_{ck}K^{ic}_{jb} - F^c_{bk}K^{ia}_{jc},\\ K^{ia}_{jb}|_c &= \frac{\partial K^{ia}_{jb}}{\partial y^c} + C^i_{hc}K^{ha}_{jb} - C^h_{jc}K^{ia}_{hb} + C^a_{dc}K^{id}_{jb} - C^d_{bc}K^{ia}_{jd},\end{aligned}$$

which are the local expressions of the h- and v-covariant derivatives.

Proposition 2.3. *To a transformation of the canonical coordinates* (1.1) *the coefficients* (F^i_{jk}, F^a_{bk}) *from* (2.5) *have the following transformation laws:*

$$F^{i'}_{j'k'}(x'\,y') = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}F^i_{jk}(x,y) + \frac{\partial x^{i'}}{\partial x^h}\frac{\partial^2 x^h}{\partial x^{j'}\partial x^{k'}},$$

$$F^{a'}_{b'j'}(x',y') = M^{a'}_a\tilde{M}^b_{b'}\frac{\partial x^j}{\partial x^{j'}}F^a_{bj}(x,y) + M^{a'}_c\frac{\partial\tilde{M}^c_{b'}}{\partial x^{j'}},$$

and the coefficients (C^i_{jc}, C^a_{bc}) *from* (2.5) *are the Finsler tensor fields of the type* $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $\begin{pmatrix}0&1\\0&2\end{pmatrix}$, *respectively.*

Remark. If $N^a_i(x,y)$ are the coefficients of the non-linear connection N, then $\dfrac{\partial N^a_k}{\partial y^b}$ have the same transformation law as $F^a_{bk}(x,y)$.

These considerations allow to prove

Theorem 2.5. *If $\xi=(E, \pi, M)$ is a vector bundle with the base M paracompact, and N is a fixed non-linear connection on the total space E, then there exist Finsler connections ∇ on E, which preserve by parallelism the distributions N and E^V.*

The Finsler connection ∇ with the coefficients $F^a_{bj}=\dfrac{\partial N^a_j}{\partial y^b}$, $C^i_{jb}=0$, $C^a_{bc}=0$ and $F^i_{jk}(x,y)$ arbitrary, is called a Berwald connection. Its simplicity made it very malleable in applications.

3. Torsion and curvature of Finsler connections

We consider again a non-linear connection N on the total space E of a vector bundle $\xi=(E, \pi, M)$ and let ∇ be a Finsler connection on E, which preserves by parallelism the distributions N and E^V.

Proposition 3.1. *The torsion tensor field T of the Finsler connection ∇ is characterized by five Finsler tensor fields:*

$$[T(X^H, Y^H)]^H, \quad [T(X^H, Y^H)]^V, \quad [T(X^H, Y^V)]^H, \quad [T(X^H, Y^V)]^V, \quad [T(X^V, Y^V)]^V. \tag{3.1}$$

If T^i_{jk}, T^a_{jk}, P^i_{jb}, P^a_{jb} and S^a_{bc} are their local components $\left(\text{where } T^i_{jk}\dfrac{\delta}{\delta x^i}= =\left[T\left(\dfrac{\delta}{\delta x^k}, \dfrac{\delta}{\delta x^j}\right)\right]^H, \text{ etc.}\right)$, then we have

$$\begin{aligned} T^i_{jk} &= F^i_{jk}-F^i_{kj}, \quad T^a_{jk} = R^a_{jk} = \frac{\delta N^a_j}{\delta x^k}-\frac{\delta N^a_k}{\delta x^j}, \\ P^i_{jb} &= C^i_{jb}, \quad P^a_{jb} = \frac{\partial N^a_j}{\partial y^b}-F^a_{bj}, \quad S^a_{bc} = C^a_{bc}-C^a_{cb}. \end{aligned} \tag{3.2}$$

It follows that the torsions P^i_{jb}, P^a_{jb}, S^a_{bc} of a Berwald connection vanish.

Proposition 3.2. *The Finsler tensor field $[T(X^H, Y^H)]^V$ vanishes if and only if the distribution N is integrable.*

The curvature tensor field R of a Finsler connection ∇ on E has only six non-trivial Finsler components.

Proposition 3.3. *The curvature tensor field R of a Finsler connection ∇ on the total space E of a vector bundle ξ has the property*

$$[R(X, Y)Z^H]^V = [R(X, Y)Z^V]^H = 0, \quad \forall X, Y, Z \in \mathscr{X}(E).$$

Then, we have

Theorem 3.1. *The curvature tensor field R of a Finsler connection ∇ on the total space E of a vector bundle ξ is characterized by the following six Finsler tensor fields:*

$$(3.3)\quad \begin{cases} R(X^H, Y^H)Z^H = \nabla_X^H \nabla_Y^H Z^H - \nabla_Y^H \nabla_X^H Z^H - \nabla^H_{[X^H, Y^H]} Z^H - \nabla^V_{[X^H, Y^H]} Z^H, \\ R(X^H, Y^H)Z^V = \nabla_X^H \nabla_Y^H Z^V - \nabla_Y^H \nabla_X^H Z^V - \nabla^H_{[X^H, Y^H]} Z^V - \nabla^V_{[X^H, Y^H]} Z^V, \end{cases}$$

$$(3.3)'\quad \begin{cases} R(X^V, Y^H)Z^H = \nabla_X^V \nabla_Y^H Z^H - \nabla_Y^H \nabla_X^V Z^H - \nabla^H_{[X^V, Y^H]} Z^H - \nabla^V_{[X^V, Y^H]} Z^H, \\ R(X^V, Y^H)Z^V = \nabla_X^V \nabla_Y^H Z^V - \nabla_Y^H \nabla_X^V Z^V - \nabla^H_{[X^V, Y^H]} Z^V - \nabla^V_{[X^V, Y^H]} Z^V, \end{cases}$$

$$(3.3)''\quad \begin{cases} R(X^V, Y^V)Z^H = \nabla_X^V \nabla_Y^V Z^H - \nabla_Y^V \nabla_X^V Z^H - \nabla^V_{[X^V, Y^V]} Z^H, \\ R(X^V, Y^V)Z^V = \nabla_X^V \nabla_Y^V Z^V - \nabla_Y^V \nabla_X^V Z^V - \nabla^V_{[X^V, Y^V]} Z^V. \end{cases}$$

Let $R^i_{jkl}, R^a_{bkl}, P^i_{jkc}, P^a_{bkc}, S^i_{jbc}, S^a_{bcd}$ be the local components of the Finsler tensor fields (3.3), (3.3)', (3.3)'', respectively $\left\{R\left(\frac{\delta}{\delta x^l}, \frac{\delta}{\delta x^k}\right)\frac{\delta}{\delta x^j} = R^i_{jkl}\frac{\delta}{\delta x^i}, \text{ etc.}\right\}$. Then, we have:

Theorem 3.2. *The curvature tensor field R of a Finsler connection ∇ with the coefficients $(F^i_{jk}, F^a_{bk}, C^i_{jc}, C^a_{bc})$ is characterized by the Finsler tensor fields* (3.3), (3.3)', (3.3)'', *whose components are given by*

$$(3.4)\quad \begin{cases} R^i_{jkl} = \dfrac{\delta F^i_{jk}}{\delta x^l} - \dfrac{\delta F^i_{jl}}{\delta x^k} + F^h_{jk} F^i_{hl} - F^h_{jl} F^i_{hk} + C^i_{ja} R^a_{kl}, \\ R^a_{bkl} = \dfrac{\delta F^a_{bk}}{\delta x^l} - \dfrac{\delta F^a_{bl}}{\delta x^k} + F^c_{bk} F^a_{cl} - F^c_{bl} F^a_{ck} + C^a_{bc} R^c_{kl}, \end{cases}$$

$$(3.4)'\quad \begin{cases} P^i_{jkc} = \dfrac{\partial F^i_{jk}}{\partial y^c} - C^i_{jc|k} + C^i_{jb} P^b_{kc}, \\ P^a_{bkc} = \dfrac{\partial F^a_{bk}}{\partial y^c} - C^a_{bc|k} + C^a_{bd} P^d_{kc}, \end{cases}$$

$$(3.4)''\quad \begin{cases} S^i_{jbc} = \dfrac{\partial C^i_{jb}}{\partial y^c} - \dfrac{\partial C^i_{jc}}{\partial y^b} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb}, \\ S^a_{bcd} = \dfrac{\partial C^a_{bc}}{\partial y^d} - \dfrac{\partial C^a_{bd}}{\partial y^c} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}. \end{cases}$$

Observe the simplicity of these expressions, as compared to the components of the curvature tensor field R written in the natural frame $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$, [4].

The Ricci and Bianchi identities can be written down without difficulty.

4. Riemannian structures; Finsler connections compatible with a metrical structure

The efficiency of the techniques of Finsler geometry in the study of vector bundles is visible. In particular, its applications to the theory of several geometric structures on the total space of a vector bundles are useful.

We study the Riemannian structure G on the total space E of a vector bundle $\xi=(E, \pi, M)$. If E_u^V is the vertical space at the point u of E, then the vectors orthogonal to E_u^V, with respect to G, uniquely determine the vector subspace N_u complementary to E_u^V. That is, $E_u=N_u\oplus E_u^V$, $u\in E$, and the map $N\colon u\to N_u$ defines, in a geometrical way, a non-linear connection N on the total space E.

Proposition 4.1. *If G is a Riemannian structure on E, then there exists an unique non-linear connection N on E with the property*

$$G(X,Y)=0, \quad \forall X\in N, \quad \forall Y\in E^v. \tag{4.1}$$

Proposition 4.2. *For a Riemannian structure G on E, there exist an unique symmetric Finsler tensor field G^H of the type $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ non-degenerate on the fibres of the horizontal bundle HE, and an unique symmetric Finsler tensor field G^V of the type $\begin{pmatrix}0&0\\0&2\end{pmatrix}$ non-degenerate on the fibres of the vertical bundle VE, such that*

$$G=G^H+G^V. \tag{4.2}$$

Indeed, from (4.1) we get $G(X,Y)=G(X^H,Y^H)+G(X^V,Y^V)$. Putting $G^H(X,Y)=G(X^H,Y^H)$ and $G^V(X,Y)=G(X^V,Y^V)$ one obtains two Finsler tensor fields with the properties mentioned above.

A Finsler connection ∇ on E, which preserves by parallelism the distributions N and E^V, is called compatible with the Riemannian structure G or is called a metrical Finsler connection if $\nabla_X G=0$, $\forall X\in\mathfrak{X}(E)$.

Proposition 4.3. *A Finsler connection ∇ on the total space E is a metrical connection, with respect to the Riemannian structure G, if and only if*

$$\nabla_X^H G^H=0, \quad \nabla_X^H G^V=0, \quad \nabla_X^V G^H=0, \quad \nabla_X^V G^V=0. \tag{4.3}$$

Theorem 4.1. *If G is a Riemannian structure on the total space E of a vector bundle $\xi=(E,\pi,M)$, then the following Finsler connection is compatible with the structure G:*

$$2G^H(\nabla_X^H Y^H, Z^H)=X^H G^H(Y^H,Z^H)+Y^H G^H(Z^H,X^H)-Z^H G^H(X^H,Y^H)-$$
$$-G^H(X^H;[Y^H,Z^H]^H)+G^H(Y^H,[Z^H,X^H]^H)+G^H(Z^H,[X^H,Y^H]^H),$$

$$\begin{aligned}(4.4)\quad &\nabla^H_X Y^V = \mathring{\nabla}^H_X Y^V + B(Y^V, X^H), \quad G^V(B(Y^V, X^H), Z^V) = (1/2)(\mathring{\nabla}^H_X G^V)(Y^V, Z^V),\\ &\nabla^V_X Y^H = \mathring{\nabla}^V_X Y^H + D(Y^H, X^V), \quad G^H(D(Y^H, X^V), Z^H) = (1/2)(\mathring{\nabla}^V_X G^H(Y^H, Z^H),\\ &2G^V(\nabla^V_X Y^V, Z^V) = X^V G^V(Y^V, Z^V) + Y^V G^V(Z^V, X^V) - Z^V G^V(X^V, Y^V) -\\ &\quad - G^V(X^V, [Y^V, Z^V]^V) + G^V(Y^V, [Z^V, X^V]^V) + G^V(Z^V, [X^V, Y^V]^V),\end{aligned}$$

where $\mathring{\nabla}$ *is a fixed Finsler connection, which preserves by parallelism the distributions* N *and* E^V.

Proof. We know that there is a Finsler connection $\mathring{\nabla}$ which preserves by parallelism the distributions N and E^V. In the condition $[T(X^H, Y^H)]^H = 0$, by the classical method, the first equality in (4.4) gives, uniquely, $\nabla^H_X Y^H$ and the second one $\nabla^H_X Y^V$. It is easy to see that ∇^H, determined in this way, is a h-covariant derivative in the algebra $\mathscr{T}_F(E)$ and that we have $\nabla^H_X G^H = 0$, $\nabla^H_X G^V = 0$. Analogously, the third equation in (4.4) gives uniquely $\nabla^V_X Y^H$, and the last one, in condition $[T(X^V, Y^V)]^V = 0$, allows to determine uniquely $\nabla^V_X Y^V$. This ∇^V is a v-covariant derivative in the algebra $\mathscr{T}_F(E)$ and it has the properties $\nabla^V_X G^H = 0$, $\nabla^V_X G^V = 0$. Therefore $\nabla = \nabla^H + \nabla^V$ is a Finsler connection on E compatible with the Riemannian structure G.

In canonical coordinates, let $g_{ij} = G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)$, $g_{ab} = G\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$. The equations (4.4) give the following coefficients of the metrical Finsler connection ∇:

$$\begin{aligned}(4.5)\quad &F^i_{jk}(x,y) = \frac{1}{2} g^{ih}\left(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right), \quad F^a_{bk}(x,y) = \mathring{F}^a_{bk}(x,y) + \frac{1}{2} g^{ac} g_{bc\mathring{|}k},\\ &C^i_{jb}(x,y) = \mathring{C}^i_{jb}(x,y) + \frac{1}{2} g^{ih} g_{jh}\mathring{|}_b, \quad C^a_{bc}(x,y) = \frac{1}{2} g^{ad}\left(\frac{\partial g_{db}}{\partial y^c} + \frac{\partial g_{dc}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^d}\right),\end{aligned}$$

where $(\mathring{F}^i_{jk}, \mathring{F}^a_{bk}, \mathring{C}^i_{jb}, \mathring{C}^a_{bc})$ are the coefficients of a fixed Finsler connection $\mathring{\nabla}$, and $\mathring{|}$, $\mathring{|}$ are the h- and v-covariant derivatives with respect to $\mathring{\nabla}$.

Observing that the distribution N is geometrically determined by the structure G we get that $F^i_{jk}(x,y)$ and $C^a_{bc}(x,y)$ are well-determined by G. Then, considering as a fixed Finsler connection $\mathring{\nabla}$ the Finsler connection with coefficients

$$(4.6)\quad \mathring{F}^i_{jk}(x,y) = F^i_{jk}(x,y),\ \mathring{F}^a_{bk}(x,y) = \frac{\partial N^a_k}{\partial y^b}, \quad \mathring{C}^i_{jb}(x,y) = 0, \quad \mathring{C}^a_{bc} = C^a_{bc},$$

we have

Theorem 4.2. *The Finsler connection* (4.5), (4.6) *is metrical and depends only on the Riemannian structure* G.

This connection can be called the canonical metrical Finsler connection. We get without difficulty:

Theorem 4.3. *The Riemann—Christoffel connection of a Riemannian structure G on the total space E of a vector bundle ξ coincides with the canonical metrical Finsler connection of G if and only if*

(1) *the horizontal distribution N is integrable,*

(2) *the metrical tensor field G is constant on the fibres of vertical subbundle VE,*

(3) $[T(X^H, Y^V)]^V = 0$.

References

[1] M. H. Akbar-Zadeh, Les espaces de Finsler et certains de leurs généralisations, *Ann. Sci. Ecole Norm. Sup.*, **80** (1963), 1—79.

[2] M. Anastasiei, Finsler geometric objects and their Lie derivative, in: *Proceedings of the National Seminar on Finsler spaces* (Braşov, 1980); pp. 11—26.

[3] G. Atanasiu, Structures et connexions Finsler presque complexes et presque hermitiennes, in: *Proceedings of the National Seminar on Finsler spaces* (Braşov, 1980); pp. 41—55.

[4] P. T. V. Duc, Sur la géométrie différentielle des fibres vectoriels, *Kodai Math. Sem. Rep.*, **26** (1975), 349—408.

[5] J. Grifone, Structure presque tangente et connexions. I, *Ann. Inst. Fourier (Grenoble)*, **22** (1972), 287—334.

[6] M. Haimovici, Le parallelisme dans les espaces de Finsler et la différentiation invariante de M. Levi-Cività, *Ann. Sci. Univ. Jassy, I*, **24** (1938), 214—218.

[7] M. Hashiguchi, On conformal transformations of Finsler metrics, *J. Math. Kyoto Univ.*, **16** (1976), 25—50.

[8] Fr. Klepp, Finsler geometry on vector bundles, *An. Univ. Iaşi*, supl. t. XXVII, s. Ia, 1981; pp. 37—42.

[9] Y. Ichijyo Almost complex structures of tangent bundles and Finsler metrics, *J. Math. Kyoto Univ.* **6** (1967), 419—452.

[10] M. Matsumoto, The theory of Finsler connections, in: *Publ. of the Study Group of Geometry*, **5**, Okayama Univ. (1970).

[11] M. Matsumoto, *Foundation of Finsler Geometry and Special Finsler Spaces*, Kyoto Univ. (1982).

[12] R. Miron, Introduction to the theory of Finsler spaces, in: *Proceedings of the National Seminar on Finsler spaces* (Braşov, 1980); pp. 131—184.

[13] R. Miron, Vector bundles Finsler geometry, in: *Proceedings of the Second National Seminar on Finsler spaces* (Braşov, 1982); pp. 147—188.

[14] R. Miron and M. Hashiguchi, Metrical Finsler connections, *Rep. Fac. Sci. Kagoshima Univ.*, **12** (1979), 21—35.

[15] D. Opris, Fibres vectoriels de Finsler et connexions associées, in: *Proceedings of the National Seminar on Finsler spaces* (Braşov, 1980); pp. 185—195.

[16] V. Oproiu, Some properties of the tangent bundles related to the Finsler geometry, in: *Proceedings of the National Seminar on Finsler spaces* (Braşov, 1980); pp. 195—209.

[17] Z. I. Szabó, Positive definite Berwald spaces, *Tensor* (N. S.) **35** (1981), 25—39.

[18] K. Yano and S. Ishihara, *Tangent and cotangent bundles*, M. Dekker (New York, 1973).

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